INDECOMPOSABLE 2-GROUPS WITH ALL AUTOMORPHISMS CENTRAL

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This paper provides examples of indecomposable 2-groups with
non-abelian automorphism groups in which every automorphism is
central.

The papers by Malone [4] and the author [1] give examples of non-
abelian p-groups with non-abelian automorphism groups in which every
automorphism is central, and provide background material on the
significance of these results. The groups given in these examples,
however, all have an abelian direct factor and Malone questions in [4]
whether there are indecomposable p-groups with the same property. When
$p = 2$, the groups numbered 91, 92 and 99 in the Hall and Senior tables
[3] are indecomposable and the only groups in the tables with all their
automorphisms central. But in each case the full automorphism group is
abelian (see [1]). However, this paper shows that each of these groups can
be generalized to give an indecomposable 2-group with non-abelian auto-
morphism group in which every automorphism is central. The notation used
is standard and that of Gorenstein [2].

Consider the following groups:

$$G_1 = \langle a,\, b,\, c,\, d \mid a^{2^m} = b^{2^m} = c^2 = d^2 = 1,\, [a,\, c] = d,\, [b,\, c] = a^{2^{m-1}},$$
$$[c,\, d] = 1,\, \langle a,\, b,\, d \rangle \text{ abelian},\, m \geq 2 \rangle\, ;$$

$$G_2 = \langle a,\, b,\, c \mid a^{2^n} = b^{2^n} = c^{2^n} = 1,\, [a,\, b] = c^{2^{n-1}},\, [b,\, c] = b^{2^{n-1}},$$
$$[a,\, c] = 1,\, n \geq 2 \rangle\, ;$$

$$G_3 = \langle a,\, b,\, c \mid a^{2^r} = b^4 = c^{2^s} = 1,\, [a,\, b] = a^{2^{r-1}},\, [b,\, c] = b^2,$$
$$[a,\, c] = 1,\, r > s,\, r \geq 3,\, s \geq 1 \rangle\, .$$

When $m = 2$, $n = 2$, $r = 3$ and $s = 1$, the groups G_1, G_2, G_3 become the groups 91, 92, 99 respectively of the Hall and Senior tables, and as noted above these groups have abelian automorphism groups. When $s = 1$, G_3 becomes the group discussed by Struik [5], and this group, as the main result of [5] shows, has an abelian automorphism group. Thus, the proposition below, besides giving the groups promised in the abstract, also provides a unified proof that these automorphism groups are abelian.

We begin with a lemma, containing some obvious facts about G_1, G_2 and G_3 .

LEMMA 1. *(i)* $Z(G_1) = \langle a^2,\, b^2,\, d \rangle$; $Z(G_2) = \langle a^2,\, b^2,\, c^2 \rangle = Z(G_3)$.

(ii) *For* $1 \leq i \leq 3$ *, each* G_i *has an abelian subgroup* A_i *of index* 2 *, where* $A_1 = \langle a,\, b,\, d \rangle$; $A_2 = \langle a,\, b^2,\, c \rangle = A_3$.

(iii) $G_1' = \langle a^{2^{m-1}},\, d \rangle$; $G_2' = \langle b^{2^{n-1}},\, c^{2^{n-1}} \rangle$; $G_3' = \langle a^{2^{r-1}},\, b^2 \rangle$.

Now writing G for G_i and A for A_i $(1 \leq i \leq 3)$ we summarize some properties of G .

LEMMA 2. *(i)* $Z(G) = \Phi(G)$.

(ii) $G' \leq Z(G)$.

(iii) A *is a characteristic subgroup of* G .

Proof. *(i)* follows since G/G' has rank 3 and $G/Z(G)$ is an

elementary abelian group of order 8 .

   (ii) is obvious.

   (iii) A is characteristic since it is the only abelian subgroup of
index 2 in G . For if B were another such subgroup, then $G = AB$
implies $|G : A| = |B : A \cap B|$ and so $|G : A \cap B| = 4$. But
$A \cap B \leq Z(G)$ and yet $|G : Z(G)| = 8$, a contradiction.

   PROPOSITION. $G = G_i$ $(1 \leq i \leq 3)$ *is an indecomposable 2-group with
all automorphisms central. Further, the automorphism groups of* G_1 , G_2 , G_3
are abelian when $m = 2$, $n = 2$, $s = 1$ *respectively, and non-abelian
otherwise.*

   Proof. $G = \langle a, b, c \rangle$ cannot be the direct product of two non-abelian
groups. But if G has an abelian direct factor H say, then
$H \leq Z(G) = \Phi(G)$, so H is trivial. Thus G is indecomposable.

   Let θ be an arbitrary automorphism of G which maps the generators
a, b, c of G onto $\bar{a}, \bar{b}, \bar{c}$ respectively. We show each of the elements
$a^{-1}\bar{a}, b^{-1}\bar{b}, c^{-1}\bar{c}$ lies in the centre of G . Thus $g^{-1}g^{\theta} \in Z(G)$ for any
$g \in G$, and so θ is a central automorphism.

   Firstly, note every element in G can be written uniquely in the form
$a^i b^j c^k z$, for some integers i, j, k with $0 \leq i, j, k \leq 1$ and some
$z \in Z(G)$. Since $Z(G)$ is characteristic and $a, b, c \in G \backslash Z(G)$, each of
$\bar{a}, \bar{b}, \bar{c}$ is of the form $a^i b^j c^k z$, where $(i, j, k) \neq (0, 0, 0)$. We now
examine the 3 cases. For convenience in this discussion i, j, k, l, t, u
are all integers with $0 \leq i, j, k, l, t, u \leq 1$, and $z_a, z_b, z_c \in Z(G)$.

   (1) $G = G_1$. Since $a, b \in A$, a characteristic subgroup,
$\bar{a} = a^i b^j z_a$, $\bar{b} = a^k b^l z_b$, where $(i, j) \neq (0, 0)$, $(k, l) \neq (0, 0)$.

However $\mho^{m-1}(A) \cap G' = \langle a^{2^{m-1}} \rangle$ is characteristic in G , so $j = 0$ and
$l = 1$. That is, $\bar{a} = az_a$, $\bar{b} = a^k b z_b$. Now $c \notin A$ so $\bar{c} = a^t b^u c z_c$.
But c has order 2 , so $t = u = 0$ and $z_c^2 = 1$. That is, $\bar{c} = cz_c$.
Finally, from a defining relation for G and using the fact that

$G' \leq Z(G)$:

$$a^{2^{m-1}} = \bar{a}^{2^{m-1}} = [\bar{b}, \bar{c}] = [a^k b, c] = [a^k, c][b, c] = d^k a^{2^{m-1}} \quad, \text{ so } \quad k = 0 \ .$$

Thus $\quad \bar{a} = az_a$, $\quad \bar{b} = bz_b$, $\quad \bar{c} = cz_c$, where $\quad z_c^2 = 1$.

(2) $\quad G = G_2$. As in (1) since $\quad a, c \in A \quad$ and $\quad \vartheta^{n-1}(A) \cap G' = \langle c^{2^{n-1}} \rangle$,

$\bar{a} = ac^j z_a$, $\quad \bar{c} = cz_c$. Now $\quad b \notin A$, so $\quad \bar{b} = a^t bc^u z_b$. But $\quad G'$ is

characteristic so $\quad t = 0$. That is, $\quad \bar{b} = bc^u z_b$. Finally, from the

defining relations for $\quad G \quad$ and using the fact that $\quad G' \leq Z(G)$:

$$b^{2^{n-1}} \left(c^{2^{n-1}}\right)^u = \bar{b}^{2^{n-1}} = [\bar{b}, \bar{c}] = [bc^u, c] = [b, c] = b^{2^{n-1}} \quad, \text{ so } \quad u = 0 \ ;$$

$$c^{2^{n-1}} = \bar{c}^{2^{n-1}} = [\bar{a}, \bar{b}] = [ac^j, b] = [a, b][c^j, b] = c^{2^{n-1}} \left(b^{2^{n-1}}\right)^j \ ,$$
$$\text{so } \quad j = 0 \ .$$

Thus $\quad \bar{a} = az_a$, $\quad \bar{b} = bz_b$, $\quad \bar{c} = cz_c$.

(3) $\quad G = G_3$. As in (1) since $\quad a, c \in A$, $\quad \bar{a} = a^i c^j z_a$, $\quad \bar{c} = a^k c^l z_c$,

where $\quad (i, j) \neq (0, 0)$, $\quad (k, l) \neq (0, 0)$. However, $\quad a \quad$ has order $\quad 2^r$

and $\quad c \quad$ has order $\quad 2^s \quad$ where $\quad r > s$, so $\quad i = 1$, $\quad k = 0 \quad$ and $\quad z_c^{2^s} = 1$.

That is $\quad \bar{a} = ac^j z_a$, $\quad \bar{c} = cz_c$. Now $\quad b \notin A \quad$ so $\quad \bar{b} = a^t bc^u z_b$. However $\quad b$

has order $\quad 4 \quad$ so $\quad t = 0$. That is $\quad \bar{b} = bc^u z_b$. Finally, from the defining

relations for $\quad G \quad$ and using the fact that $\quad G' \leq Z(G)$:

$$a^{2^{r-1}} = \bar{a}^{2^{r-1}} = [\bar{a}, \bar{b}] = [ac^j, bc^u] = [a, b][c^j, b] = a^{2^{r-1}} b^{2^j} \ ,$$
$$\text{so } \quad j = 0 \ ;$$

$$b^2 = [b, c] = [bc^u, c] = [\bar{b}, \bar{c}] = \bar{b}^2 = \begin{cases} b^2 z_b^2 & \text{if } u = 0 \ , \\ \\ c^2 z_b^2 & \text{if } u = 1 \ , \end{cases}$$

so $u = 0$ and $z_b^2 = 1$. Thus $\bar{a} = az_a$, $\bar{b} = bz_b$, $\bar{c} = cz_c$ where

$$z_b^2 = 1 \ , \quad z_c^{2^s} = 1 \ .$$

Thus in each case all automorphisms are central. Further, each of the possibilities listed above for $\bar{a}, \bar{b}, \bar{c}$ occur as the image of an automorphism of G . For $\bar{a}, \bar{b}, \bar{c}$ satisfy the same relations as a, b, c and $G = \langle a, b, c \rangle = \langle az_a, bz_b, cz_c, z_a, z_b, z_c \rangle = \langle \bar{a}, \bar{b}, \bar{c} \rangle$, since $Z(G) \le \Phi(G)$. In particular, suppose $m > 2$ in G_1 , $n > 2$ in G_2 and $s > 1$ in G_3 . Then let θ_i $(1 \le i \le 3)$ be the automorphisms which map a to ab^2 in G_1 , b to b^{-1} in G_2 , a to ac^2 in G_3 respectively, and let ψ_i $(1 \le i \le 3)$ be the automorphisms which map b to ba^2 in G_1 , c to cb^2 in G_2 , c to $ca^{2^{r-s}}$ in G_3 respectively (where all other generators are fixed by these automorphisms). Now an easy calculation shows $\theta_i \psi_i \ne \psi_i \theta_i$, for $1 \le i \le 3$, so Aut G_i is non-abelian.

Finally, let $m = 2$ in G_1 , let $n = 2$ in G_2 and let θ be an arbitrary (central) automorphism of $G = G_i$ $(1 \le i \le 2)$ as listed above. Then a^2, b^2, c^2 are fixed by θ since $Z(G)$ has exponent 2 , and $d \in G_1'$ is also fixed. Thus θ fixes $Z(G)$ and so $\theta^2 = 1$, that is Aut G is (elementary) abelian. If $s = 1$ in G_3 , then any (central) automorphism θ of G_3 as listed above is of form $a^\theta = a^{2j+1}g_a$ $\left(0 \le j \le 2^{r-1}-1\right)$, $b^\theta = bg_b$, $c^\theta = cg_c$, where $g_a, g_b, g_c \in G_3'$ and are fixed by any automorphism of G_3 . Clearly any two automorphisms of this form commute so Aut G is abelian.

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