A FIRST APPROXIMATION TO $\{X, Y\}$

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1. Introduction. If $\mathfrak{C}$ and $\mathfrak{C}'$ are classes of finite abelian groups, we write $\mathfrak{C} + \mathfrak{C}'$ for the smallest class containing the groups of $\mathfrak{C}$ and of $\mathfrak{C}'$. For any positive number r, $\mathfrak{C} < r$ is the smallest class of abelian groups which contains the groups Z_p for all primes p less than r.

Our aim in this paper is to prove the following theorem.

THEOREM. *If $\mathfrak{C}$ is a class of finite abelian groups and*

(i) $\pi_i(Y) \in \mathfrak{C}$ *for* $i < n$,

(ii) $H_*(X; Z)$ *is finitely generated,*

(iii) $H^i(X; Z) \in \mathfrak{C}$ *for* $i > n + k$,

then

$$\{X, Y\} \approx \sum_{i=0}^{k} H^{n+i}(X;\ H_{n+i}(Y; Z)) \quad \mathrm{mod}(\mathfrak{C} + \mathfrak{C} < \tfrac{1}{2}(k + 4)).$$

This statement contains many of the classical results of homotopy theory: the Hurewicz and Hopf theorems, Serre's (mod $\mathfrak{C}$) version of these theorems, and Eilenberg's classification theorem. In fact, these are all contained in the case $k = 0$. Perhaps the most interesting case is when $\mathfrak{C}$ is the trivial class. Then in a way which will be made precise, the result is "best possible". Vaguely speaking, the Theorem solves the problem of computing $\{X, Y\}$ up to a certain "indeterminacy", the "indeterminacy" being p-primary components corresponding to small primes. It will be shown that if the answer is to be given as a "function" of only the homology of X and Y (as graded groups), then no theorem of smaller "indeterminacy" can be true.

Two definitions are introduced: those of maps being homotopic modulo a class and of spaces being weakly equivalent modulo a class; and the relationship between these ideas is studied. The proof of Theorem A then relies on (**4**, Theorem 6) and on some results of Cartan and Serre.

2. On weak homotopy equivalence (mod $\mathfrak{C}$).

Definition. A map f: $X \to Y$ will be called a weak homotopy equivalence (mod $\mathfrak{C}$) if the induced homomorphism f_*: $H_i(X) \to H_i(Y)$ is an isomorphism (mod $\mathfrak{C}$) for all i. (Integer coefficients are suppressed.)

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Definition. Two spaces, X and Y, are of the same weak homotopy type (mod $\mathfrak{C}$) if there exists a third space Z and weak homotopy equivalences (mod $\mathfrak{C}$) $d_1: Z \to X$, $d_2: Z \to Y$. This will be written $X \sim_{\mathfrak{C}} Y$.

The first problem is to show that $\sim_{\mathfrak{C}}$ is an equivalence relation (for simply connected spaces). This will be proved in a series of lemmas. This proof is analogous to the proof that (mod $\mathfrak{C}$) isomorphism is an equivalence relation in the category of abelian groups.

LEMMA 1. *Given abelian groups and homomorphisms*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

if any two of f, g, gf are isomorphisms (mod $\mathfrak{C}$), *then so is the third.*

Proof. The sequence

$$0 \to \ker f \to \ker gf \to \ker g \to \operatorname{coker} f \to \operatorname{coker} gf \to \operatorname{coker} g \to 0$$

is exact.

LEMMA 2. *Given spaces and maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

if any two of f, g, gf are weak homotopy equivalences (mod $\mathfrak{C}$), *then so is the third.*

Proof. Apply Lemma 1 to the induced homology homomorphisms.

LEMMA 3. *If $X_1 \sim_{\mathfrak{C}} X_2$, then there exists a space W and maps w_i: $X_i \to W$ ($i = 1, 2$) which are weak homotopy equivalences whe(s)* (mod $\mathfrak{C}$).

Proof. We are given a space Z and maps d_i: $Z \to X_i$ ($i = 1, 2$) which are whes (mod $\mathfrak{C}$). Let W be the disjoint union of X_1 and X_2, where $x_1 \in X_1$ and $x_2 \in X_2$ are identified if $x_1 = d_1(z)$ and $x_2 = d_2(z)$ for some $z \in Z$. That is,

$$W = X_1 \bigcup_{d_1(z)=d_2(z)} X_2.$$

Let w_i: $X_i \to W$ be the natural projections. Then $w_1 d_1 = w_2 d_2$, and $Cw_i d_i = Cd_1 \vee Cd_2$. Now

$$H_j(Cd_1 \vee Cd_2) = H_j(Cd_1) + H_j(Cd_2) \in \mathfrak{C}.$$

Therefore, the $w_i d_i$ are whes (mod $\mathfrak{C}$), and consequently, by Lemma 2 the maps w_i: $X_i \to W$ are whes (mod $\mathfrak{C}$).

LEMMA 4. *Suppose that X_1, X_2, and W are simply connected and the maps m_i: $X_i \to W$ ($i = 1, 2$) are whes* (mod $\mathfrak{C}$). *Then $X_1 \sim_{\mathfrak{C}} X_2$.*

Proof. We can construct spaces $\bar{X}_i$, fibrings p_i: $\bar{X}_i \to W$ (with fibres F_i) and homotopy equivalences h_i: $\bar{X}_i \to X_i$ such that $m_i h_i = p_i$. Then the p_i must be whes (mod $\mathfrak{C}$) (Lemma 2).

Let p: $Z \to W$ be the Whitney sum of the fibrings p_1: $\bar{X}_1 \to W$ and p_2: $\bar{X}_2 \to W$. The fibre of p is $F_1 \times F_2$. Since

$$\pi_j(F_1 \times F_2) = \pi_j(F_1) \oplus \pi_j(F_2) \in \mathfrak{C},$$

we have $H_j(F_1 \times F_2) \in \mathfrak{C}$ for all j. Therefore, the (mod $\mathfrak{C}$) Serre sequence is infinitely long and p_*: $H_j(Z) \to H_j(W)$ is a (mod $\mathfrak{C}$) isomorphism for all j.

Since Z is the Whitney sum of fibrings, we have the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "f_2"] \arrow[d, "f_1"'] \arrow[dr, dashed, "p"] & \bar{X}_2 \arrow[d, "p_2"] \\ X_1 \arrow[r, "p_1"'] & W \end{tikzcd}$$

where p, p_1, and p_2 are whes (mod $\mathfrak{C}$). Consequently, so are f_1 and f_2 (Lemma 2) and also $m_i f_i$: $Z \to \bar{X}_i \to X_i$ ($i = 1, 2$). That is, $X_1 \sim_{\mathfrak{C}} X_2$.

THEOREM 1. *$\sim_{\mathfrak{C}}$ is an equivalence relation for simply connected spaces.*

Proof. Only transitivity presents a problem. Suppose that $X_1 \sim_{\mathfrak{C}} X_2 \sim_{\mathfrak{C}} X_3$. By definition, there exist spaces Z_1 and Z_2 and maps

$$\begin{tikzcd} & X_1 \\ Z_1 \arrow[ur, "r"] \arrow[dr, "s"] & \\ & X_2 \\ Z_2 \arrow[ur, "t"] \arrow[dr, "u"] & \\ & X_3 \end{tikzcd}$$

which are whes (mod $\mathfrak{C}$). Consider

$$\begin{tikzcd} Z_1 \arrow[dr, "s"] & \\ & X_2 \\ Z_2 \arrow[ur, "t"] & \end{tikzcd}$$

By Lemma 4, there exists a space $\bar{Z}$ and maps m_i: $\bar{Z} \to Z_i$ ($i = 1, 2$) which are whes (mod $\mathfrak{C}$). Therefore, rm_1: $\bar{Z} \to X_1$ and um_2: $\bar{Z} \to X_3$ are whes (mod $\mathfrak{C}$) (Lemma 2). That is, $X_1 \sim_{\mathfrak{C}} X_3$.

For the remainder of this paper we will restrict the discussion to path-connected simply connected spaces. In (**4**, Theorem 6) it is implied that if $X \sim_{\mathfrak{C}} Y$, then the groups $H_i(\ )$, $\pi_i(\ )$, and $[L, \Omega^2\ ]$ (for a finite complex L) $[\ , \Omega^2 P]$ (when $\pi_*(P)$ is finitely generated) are isomorphic to one another (mod $\mathfrak{C}$).

3. On a decomposition (mod $\mathfrak{C}$).

Definition. Suppose that $[X, Y]$ is a group. Then $f \sim_{\mathfrak{C}} g$ if $([f] - [g]) \in \mathfrak{C}$, that is, if the cyclic group generated by $([f] - [g])$ is in $\mathfrak{C}$.

Let ϕ_q: $\Omega X \to \Omega X$ be the composition

$$\Omega X \xrightarrow{d} \Omega X \times \overset{q}{\ldots} \times \Omega X \xrightarrow{m} \Omega X,$$

where d is the diagonal map and m is the (homotopy associative) loop multiplication.

LEMMA 5. *If $Z_q \in \mathfrak{C}$, then ϕ_q is a whe* (mod $\mathfrak{C}$).

Proof. ϕ_{q*}: $\pi_i(\Omega X) \to \pi_i(\Omega X)$ has kernel $\mathrm{Tor}(\pi_i(\Omega X), Z_q)$ and cokernel $\pi_i(\Omega X) \otimes Z_q$. Therefore, it is a (mod $\mathfrak{C}$) isomorphism for all i.

Let p: $E \to \Omega^2 B$ be the acyclic fibring over $\Omega^2 B$. Let f: $Y \to \Omega^2 B$ be a map. Then we have a commutative diagram

$$\begin{array}{ccc} E_f & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & \Omega^2 B \end{array}$$

where $\bar{p}$: $E_f \to Y$ is the induced fibring with fibre $\Omega^3 B$. E_f may also be thought of as the fibre when we change f: $Y \to \Omega^2 B$ into a fibre map. We will write E_f without explicit reference to the acyclic fibring p: $E \to \Omega^2 B$.

LEMMA 6. *If $Z_q \in \mathfrak{C}$ and f: $Y \to \Omega^2 B$ is a map, then $E_f \sim_{\mathfrak{C}} E_{qf}$.*

Proof. Let ϕ_q: $\Omega^2 B \to \Omega^2 B$ be the multiplication by q defined above. Then the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & \Omega^2 B \\ & {\scriptstyle qf}\searrow & \downarrow{\scriptstyle \phi_q} \\ & & \Omega^2 B \end{array}$$

is homotopy commutative. Change qf (without changing notation (or altering homotopy class)) to make the diagram commutative. Then we will have a commutative diagram

$$\begin{array}{ccccc} E_f & \longrightarrow & Y & \xrightarrow{f} & \Omega^2 B \\ \downarrow{\scriptstyle h} & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \phi_q} \\ E_{qf} & \longrightarrow & Y & \xrightarrow{qf} & \Omega^2 B \end{array}$$

and a commutative diagram of induced homomorphisms

$$\begin{array}{ccccccccc}
\pi_{i+1}(Y) & \xrightarrow{f_*} & \pi_{i+1}(\Omega^2 B) & \rightarrow & \pi_i(E_f) & \rightarrow & \pi_i(Y) & \xrightarrow{f_*} & \pi_i(\Omega^2 B) \\
\| & & \downarrow \phi_{q_*} & & h_* \downarrow & & \| & & \downarrow \phi_{q_*} \\
\pi_{i+1}(Y) & \xrightarrow{(qf)_*} & \pi_{i+1}(\Omega^2 B) & \rightarrow & \pi_i(E_{qf}) & \rightarrow & \pi_i(Y) & \xrightarrow{(qf)_*} & \pi_i(\Omega^2 B)
\end{array}$$

Here the two rows are exact and the four outside vertical homomorphisms are (mod $\mathfrak{C}$) isomorphisms. Therefore,

$$h_*: \pi_i(E_f) \rightarrow \pi_i(E_{qf})$$

is a (mod $\mathfrak{C}$) isomorphism for all i. That is, m: $E_f \rightarrow E_{qf}$ is a whe (mod $\mathfrak{C}$).

LEMMA 7. *Suppose that* $f \sim_{\mathfrak{C}} g$: $Y \rightarrow \Omega^2 B$, *then* $E_f \sim_{\mathfrak{C}} E_g$.

Proof. $f \sim_{\mathfrak{C}} g$. Therefore, $q(f - g) \sim 0$ for some q such that $Z_q \in \mathfrak{C}$. That is, $qf \sim qg$. Using standard homotopy theory, we obtain that E_{qf} and E_{qg} are of the same homotopy type. Lemma 2 implies that $E_{qf} \sim_{\mathfrak{C}} E_f$ and $E_{qg} \sim_{\mathfrak{C}} E_g$. Therefore, $E_f \sim_{\mathfrak{C}} E_g$.

LEMMA 8. *Suppose that* $Y_1 \sim_{\mathfrak{C}} Y_2$, $[Y_1, \Omega^2 B] \in \mathfrak{C}$, *and* $\pi_*(B)$ *is finitely generated; then for any map* f: $Y_2 \rightarrow \Omega^2 B$, *we have*

$$E_f \sim_{\mathfrak{C}} Y_2 \times \Omega^3 B.$$

Proof. Since $Y_1 \sim_{\mathfrak{C}} Y_2$ and since $\pi_*(B)$ is finitely generated, we have

$$[Y_1, \Omega^2 B] \approx [Y_2, \Omega^2 B] \quad (\text{mod } \mathfrak{C}).$$

Therefore, $[Y_2, \Omega^2 B] \in \mathfrak{C}$ and

$$f \sim_{\mathfrak{C}} T: Y_2 \rightarrow \Omega^2 B,$$

where T is the trivial map. Lemma 3 implies that $E_f \sim_{\mathfrak{C}} E_T = Y_2 \times \Omega^3 B$.

We now recall some results due to Serre and Cartan (**12**; **5**). If A and D are cyclic groups of prime power or infinite order and $n > k > 1$, then

$$H^{n+k}(K(A, n); D) \in \mathfrak{C} < \tfrac{1}{2}(k + 3).$$

Furthermore,

$$H^{n+1}(K(A, n); Z) = 0.$$

LEMMA 9. *Let* G *and* H *be finitely generated abelian groups. Suppose that* $n > k > 1$, *then*

$$H^{n+k}(K(G, n); H) \in \mathfrak{C} < \tfrac{1}{2}(k + 3).$$

Proof. Write G as a direct sum $A \oplus B$, where A is cyclic of prime power or infinite order (and B may be trivial). Then

$$K(G, n) = K(A, n) \times K(B, n)$$

and

$$H^{n+k}(K(G,n);H) = H^{n+k}(K(A,n)\times K(B,n);H)$$
$$= \sum_{i+j=n+k} H^i(K(A,n);H^j(K(B,n);H)).$$

For each j, $H^j(K(B,n);H)$ is finitely generated and we may write

$$H^j(K(B,n);H) = \sum_d D_d,$$

where D_d is a cyclic group of infinite or prime power order. Then

$$H^i(K(A,n);H^j(K(B,n);H)) = \sum_d H^i(K(A,n);D_d)$$

and by the remark above, each summand is in $\mathfrak{C} < \frac{1}{2}(k+3)$. Therefore,

$$H^i(K(A,n);H^j(K(B,n);H)) \in \mathfrak{C} < \tfrac{1}{2}(k+3)$$

for each pair i, j. This proves our lemma.

THEOREM 2. *Suppose that $n > k+1$ and that $\pi_i(Y) \in \mathfrak{C}$ for $i < n$ and for $i > n+k$. Then*

$$Y \sim_{\mathfrak{C}'} \prod_{i=0}^{k} K(\pi_{n+i}(Y), n+i),$$

where

$$\mathfrak{C}' = \mathfrak{C} + \mathfrak{C} < \tfrac{1}{2}(k+4).$$

Proof. Let $p\colon Y \to Y^{n-1}$ be a projection of Y onto the space Y^{n-1} made up of the first $n-1$ homotopy groups of Y. Let F be the fibre. Then $i\colon F \to Y$ is a whe (mod $\mathfrak{C}$). Let $p'\colon F \to F^{n+k}$ be the projection of F onto the space F^{n+k} made up of its first $n+k$ homotopy groups. Then p' is a whe (mod $\mathfrak{C}$); that is,

$$Y \sim_{\mathfrak{C}} F^{n+k} \quad \text{and} \quad \pi_i(F^{n+k}) = 0 \qquad \text{if } i < n \text{ or } i > n+k.$$

Take a Postnikov system for F^{n+k}:

$$\begin{array}{c} F^{n+k} \to F^{n+k-1} \to \ldots \to F^{n+j} \to F^{n+j-1} \to \ldots \to K(\pi_n(Y), n) \\ \downarrow f_j \\ K(\pi_{n+j}(Y), n+j+1) \end{array}$$

where $F^{n+j} = E_{f_j}$. The proof is by induction on j.

Suppose that F^{n+j-1} and $\prod_{i=0}^{j-1} K(\pi_{n+i}(Y),\ n+i)$ are equivalent (mod $\mathfrak{C} < \frac{1}{2}(k+4)$). Then

$$H^{n+j+1}(F^{n+j-1};\ \pi_{n+j}(Y))$$
$$\approx H^{n+j+1}\left(\prod_{i=0}^{j-1} K(\pi_{n+i}(Y), n+i);\ \pi_{n+j}(Y)\right) \quad (\text{mod } \mathfrak{C} < \tfrac{1}{2}(k+4))$$
$$\approx \sum_{i=0}^{j-1} H^{n+j+1}(K(\pi_{n+i}(Y), n+i);\ \pi_{n+j}(Y))$$

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(since $j + 1 < k + 1 < n$). Now, since $n + j + 1 - (n + i) < k + 1$. Lemma 9 implies that each summand is in $\mathfrak{C} < \frac{1}{2}((k + 1) + 3)$. That is, $H^{n+j+1}(F^{n+j-1}; \pi_{n+j}(Y))$ is in $\mathfrak{C} < \frac{1}{2}(k + 4)$. Therefore, by Lemma 8,

$$F^{n+j} = E_{f_j} \sim F^{n+j-1} \times K(\pi_{n+j}(Y), n + j) \quad (\text{mod } \mathfrak{C} < \tfrac{1}{2}(k + 4))$$
$$\sim \prod_{i=0}^{j} K(\pi_{n+i}(Y), n + i) \quad (\text{mod } \mathfrak{C} < \tfrac{1}{2}(k + 4)).$$

This completes the induction and we have

$$F^{n+k} \sim \prod_{i=0}^{k} K(\pi_{n+i}(Y), n + i) \quad (\text{mod } \mathfrak{C} < \tfrac{1}{2}(k + 4)).$$

Since $Y \sim_{\mathfrak{C}} F^{n+k}$, this proves our theorem.

Lemma 5 and the remark preceding it imply that when $n > k > 0$, we have

$$H_{n+k}(K(G, n); Z) \in \mathfrak{C} < \tfrac{1}{2}(k + 3)$$

for any finitely generated abelian group G. Using the Künneth formulas we see that

$$H_{n+j}\left(\prod_{i=0}^{k} K(G_i, n + i); Z\right) \approx G_j \quad (\text{mod } \mathfrak{C} < \tfrac{1}{2}(k + 3))$$

provided that $n > k > j$. We can now prove the following lemma.

LEMMA 10. *If $\pi_i(Y) \in \mathfrak{C}$ for $i < n$ and for $i > n + k$, $n > k > j$, then $H_{n+j}(Y; Z) \approx_{\mathfrak{C}'} \pi_{n+j}(Y)$, where $\mathfrak{C}' = \mathfrak{C} + \mathfrak{C} < \frac{1}{2}(k + 4)$.*

Proof. $Y \sim_{\mathfrak{C}'} \prod_{i=0}^{k} K(\pi_{n+i}(Y), n + i)$, and hence

$$H_{n+j}(Y) \approx_{\mathfrak{C}'} H_{n+j}\left(\prod_{i=0}^{k} K(\pi_{n+i}(Y), n + i)\right) \approx_{\mathfrak{C}'} \pi_{n+j}(Y).$$

In fact, the Hurewicz homomorphism induces a (mod $\mathfrak{C}'$) isomorphism; however, this fact will not be proved or used here.

LEMMA 11. *If $G_1 \approx_{\mathfrak{C}} G_2$, then $H^n(X; G_1) \approx_{\mathfrak{C}} H^n(X; G_2)$ for any n.*

Proof. There exists a group G and (mod $\mathfrak{C}$) isomorphisms

$$f_j\colon\ G \to G_j \qquad (j = 1, 2).$$

Now take the Eilenberg-MacLane spaces $K(G, n)$, $K(G_1, n)$, and $K(G_2, n)$, and the induced maps $\bar{f}_j$: $K(G, n) \to K(G_j, n)$. These are clearly whes (mod $\mathfrak{C}$). Therefore,

$$K(G_1, n) \sim_{\mathfrak{C}} K(G_2, n) \quad \text{and} \quad [X, K(G_1, n)] \approx_{\mathfrak{C}} [X, K(G_2, n)].$$

Thus, $H^n(X; G_1) \approx_{\mathfrak{C}} H^n(X; G_2)$.

THEOREM. *Suppose that*

(i) $\pi_i(Y) \in \mathfrak{C}$ *for* $i < n$,

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(ii) $H^i(X) \in \mathfrak{C}$ *for* $i > n + k$,
(iii) $H_*(X)$ *is finitely generated.*
Then

$$\{X,\ Y\} \approx_{\mathfrak{C}'} \sum_{i=0}^{k} H^{n+i}(X; H_{n+i}(Y; Z)),$$

where

$$\mathfrak{C}' = \mathfrak{C} + \mathfrak{C} < \tfrac{1}{2}(k + 4).$$

Proof. Take d large. Then $\{X,\ Y\} = [S^dX, S^dY]$. $H_{d+i}(S^dY) = H_i(Y) \in \mathfrak{C}$ for $0 < i < n$. Furthermore, S^dY is simply connected. Therefore, $\pi_{d+i}(S^dY) \in \mathfrak{C}$ for $0 < i < n$. Clearly, $\pi_j(S^dY) = 0$ when $j < d$.

Let B^{n+k+d} be the space in the Postnikov system for S^dY made up of the first $n + k + d$ homotopy groups of S^dY. Then we have a fibring p: $S^dY \to B^{n+k+1}$ with fibre F and $\pi_j(F) = 0$ for $j < n + k + d$. The sequence

$$[S^dX, F] \to [S^dX, Y] \xrightarrow[p_*]{} [S^dX, B^{n+k+d}] \to [S^{d-1}X, F]$$

is exact and by (**4**, Lemma 14), the two outside groups are in $\mathfrak{C}$. Therefore, p_* is a (mod $\mathfrak{C}$) isomorphism. This yields

$$\pi_i(B^{n+k+d}) \in \mathfrak{C} \quad \text{when } i < n + d$$

and

$$\pi_j(B^{n+k+d}) = 0 \quad \text{when } j > n + k + d.$$

By Theorem 2,

$$\begin{aligned} B^{n+k+d} &\sim \prod_{i=0}^{k} K(\pi_{n+d+i}(B^{n+k+d}), n + d + i) \quad (\text{mod } \mathfrak{C} < \tfrac{1}{2}(k + 4)) \\ &= \prod_{i=0}^{k} K(\pi_{n+d+i}(S^dY), n + d + i). \end{aligned}$$

Therefore, $[S^dX, B^{n+k+d}]$ is isomorphic (mod $(\mathfrak{C} + \mathfrak{C} < \frac{1}{2}(k + 4))$) to

$$\begin{aligned} \left[S^dX, \prod_{i=0}^{k} K(\pi_{n+d+i}(S^dY), n + d + i)\right] &= \sum_{i=0}^{k} H^{n+d+i}(S^dX;\ \pi_{n+d+i}(S^dY)) \\ &= \sum_{i=0}^{k} H^{n+i}(X;\ \pi_{n+d+i}(S^dY)) \\ &\approx \sum_{i=0}^{k} H^{n+i}(X; H_{n+d+i}(S^dY; Z)) \\ &\qquad (\text{mod } \mathfrak{C} < \tfrac{1}{2}(k + 4)) \\ &= \sum_{i=0}^{k} H^{n+i}(X; H_{n+i}(Y; Z)); \end{aligned}$$

cf. Lemmas 10 and 11. Combining these results, we obtain our assertion.

4. Some remarks. Let F be a rule which assigns an abelian group to an ordered pair of graded abelian groups in such a way that the isomorphism class of $F(A,\ B)$ depends only on the isomorphism classes of A and B (as graded groups). Suppose that the following statement is true:

"For any pair of spaces X and Y such that

(i) $H_*(X)$ is finitely generated,

(ii) $H^i(X; Z) = 0$ for $i > n + k$, and

(iii) $\pi_i(Y) = 0$ for $i < n$,

we have $\{X, Y\} \approx_{\mathfrak{C}} F(H_*(X), H_*(Y))$, where F and $\mathfrak{C}$ are fixed (though they may depend on k)."

Then, for any prime $p < \frac{1}{2}(k + 4)$, the group Z_p must be contained in $\mathfrak{C}$.

Proof. We may suppose that $n > k + 1$ so that we are already in the stable range. Let $f: S^n \to K(Z, n)$ represent a generator of $\pi_n(K(Z, n))$. Let $Y = K(Z, n)$ and $\bar{Y} = C_f \vee S^n$, as graded groups $H_*(Y) \approx H_*(\bar{Y})$, and $\pi_i(Y) = \pi_i(\bar{Y}) = 0$ for $i < n$. For a prime $p < \frac{1}{2}(k+4)$ take $X = S^{n+2p-3}$. Then $H^i(X) = 0$ for $i > n + k$. Hence, if the statement is true, we have

$$[X, Y] \approx_{\mathfrak{C}} F(H_*(X), H_*(Y)) \approx F(H_*(X), H_*(\bar{Y})) \approx_{\mathfrak{C}} [X, \bar{Y}].$$

However,

$$[X, Y] = H^n(S^{n+2p-3}; Z) = 0$$

and

$$[X, \bar{Y}] = [S^{n+2p-3}, S^n] \oplus [S^{n+2p-3}, C_f].$$

$[S^{n+2p-3}, S^n]$ contains a Z_p summand, and therefore $Z_p \approx 0 \pmod{\mathfrak{C}}$.

One can prove the following result similarly to Theorem 2.

Let $n > k$ and $H_i(X) \in \mathfrak{C}$ for $i < n$ and for $i > n + k$. Then

$$X \sim_{\mathfrak{C}'} \bigvee_{i=0}^{k} M(H_{n+i}(X), n + i),$$

where $\mathfrak{C}' = \mathfrak{C} < \frac{1}{2}(k + 4) + \mathfrak{C}$.

The proof is similar to that of Theorem 2. One uses Eckmann-Hilton decompositions instead of Postnikov systems and mapping cones instead of induced fibrations. The key statement is that

$$\pi_{n+j}(S^n) \in \mathfrak{C} < \tfrac{1}{2}(k + 4)$$

when $0 < j < k < n - 1$; see (**11**). This statement is equivalent to the one about the cohomology of Eilenberg-MacLane spaces used in Theorem 2.

This approach will lead to an alternative proof of the Theorem. The key new notion in both proofs is the study of maps that are homotopic modulo a class of abelian groups.

The phrase "weak homotopy equivalence" was used since it may be of some interest to study the following relation: if $[X, Y]$ and $[Y, X]$ are both groups, then X is of the same (mod $\mathfrak{C}$) homotopy type as Y if there exist maps $f: X \to Y$, $g: Y \to X$ such that $fg \sim_{\mathfrak{C}} I_Y$ and $gf \sim_{\mathfrak{C}} I_X$. f and g will then be whes (mod $\mathfrak{C}$). On the other hand, neither of the following statements is true:

(i) a whe (mod $\mathfrak{C}$) has a (mod $\mathfrak{C}$) inverse;

(ii) if $X \sim_{\mathfrak{C}} Y$ (and $[X, Y]$, $[Y, X]$ are groups), then X is of the same (mod $\mathfrak{C}$) homotopy type as Y.

However, this notion is not relevant to the work done here.

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