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DIFFERENTIAL INEQUALITIES AND A MARTY-TYPE CRITERION FOR QUASI-NORMALITY

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Abstract

We show that the family of all holomorphic functions f in a domain D satisfying

$$\frac{|f^{(k)}|}{1+|f|}(z) \leq C \quad \text{for all } z \in D$$

(where k is a natural number and $C > 0$) is quasi-normal. Furthermore, we give a general counterexample to show that for $\alpha > 1$ and $k \geq 2$ the condition

$$\frac{|f^{(k)}|}{1+|f|^{\alpha}}(z) \leq C \quad \text{for all } z \in D$$

does not imply quasi-normality.

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1. Introduction and statement of results

One of the key results in the theory of normal families of meromorphic functions is Marty's theorem [11], which says that a family $\mathcal{F}$ of meromorphic functions in a domain D in the complex plane $\mathbb{C}$ is normal (in the sense of Montel) if and only if the family $\{f^{\#} : f \in \mathcal{F}\}$ of the corresponding spherical derivatives $f^{\#} := |f'|/(1+|f|^2)$ is locally uniformly bounded in D.

A substantial (and best possible) improvement of the direction '$\Longleftarrow$' in Marty's theorem is due to Hinkkanen [7]: a family of meromorphic (respectively holomorphic) functions is already normal if the corresponding spherical derivatives are bounded on the preimages of a set consisting of five (respectively three) elements. (An analogous result for normal functions was earlier proved by Lappan [8].)

In several previous papers [1–6, 10], we studied the question of how normality (or quasi-normality) can be characterized in terms of the more general quantity

$$\frac{|f^{(k)}|}{1+|f|^{\alpha}}, \qquad \text{where } k \in \mathbb{N},\ \alpha > 0$$

rather than the spherical derivative $f^{\#}$.

Before summarizing the main results from these studies we would like to recall the definition of quasi-normality and also to introduce some notations.

A family $\mathcal{F}$ of meromorphic functions in a domain $D \subseteq \mathbb{C}$ is said to be *quasi-normal* if from each sequence $\{f_n\}_n$ in $\mathcal{F}$ one can extract a subsequence which converges locally uniformly (with respect to the spherical metric) on $D \setminus E$, where the set E (which may depend on $\{f_n\}_n$) has no accumulation point in D. If the exceptional set E can always be chosen to have at most q points, yet for some sequence there actually occur q such points, then we say that $\mathcal{F}$ is *quasi-normal of order* q.

We set $\Delta(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ for the open disk with center $z_0 \in \mathbb{C}$ and radius $r > 0$. By $\mathcal{H}(D)$ we denote the space of all holomorphic functions and by $\mathcal{M}(D)$ the space of all meromorphic functions in a domain D. We write P_f and Z_f for the set of poles, respectively for the set of zeros, of a meromorphic function f, and we use the notation '$f_n \overset{\chi}{\Longrightarrow} f$ (in D)' to indicate that the sequence $\{f_n\}_n$ converges to f locally uniformly in D (with respect to the spherical metric).

The Marty-type results known so far can be summarized as follows.

Theorem A. *Let k be a natural number, $\alpha > 0$ be a real number and $\mathcal{F}$ be a family of functions meromorphic in a domain D. Consider the family*

$$\mathcal{F}^*_{k,\alpha} := \left\{ \frac{|f^{(k)}|}{1+|f|^{\alpha}} : f \in \mathcal{F} \right\}.$$

Then the following holds.

(a) [6, 10] *If each $f \in \mathcal{F}$ has zeros only of multiplicity $\geq k$ and if $\mathcal{F}^*_{k,\alpha}$ is locally uniformly bounded in D, then $\mathcal{F}$ is normal.*

(b) *(Xu [16]) Assume that there are a value $w^* \in \mathbb{C}$ and a constant $M < \infty$ such that for each $f \in \mathcal{F}$ we have $|f'(z)| + \cdots + |f^{(k-1)}(z)| \leq M$ whenever $f(z) = w^*$ and that there exists a set $E \subset \overline{\mathbb{C}}$ consisting of $k+4$ elements such that for all $f \in \mathcal{F}$ and all $z \in D$,*

$$f(z) \in E \quad \Longrightarrow \quad \frac{|f^{(k)}|}{1+|f|^{k+1}}(z) \leq M.$$

Then $\mathcal{F}$ is normal.

If all functions in $\mathcal{F}$ are holomorphic, then this also holds if one merely assumes that E has at least three elements.

(c) [4] *If $\alpha > 1$ and if each $f \in \mathcal{F}$ has poles only of multiplicity $\geq k/(\alpha - 1)$, then the normality of $\mathcal{F}$ implies that $\mathcal{F}^*_{k,\alpha}$ is locally uniformly bounded.*

This does not hold in general for $0 < \alpha \leq 1$.

REMARKS.

(1) In (a) and (b) the assumption on the multiplicities of the zeros, respectively the (slightly weaker) condition on the existence of the value w^*, is essential. The condition $|f^{(k)}(z)|/1 + |f(z)|^\alpha \leq C$ itself does not imply normality. Indeed, each polynomial of degree at most $k - 1$ satisfies this condition, but those polynomials only form a quasi-normal, but not a normal family.

(2) It is worthwhile to mention two special cases of Theorem A(c).

- If $\alpha \geq k + 1$ and if $\mathcal{F}$ is normal, then the conclusion that $\mathcal{F}^*_{k,\alpha}$ is locally uniformly bounded holds without any further assumptions on the multiplicities of the poles. This had been proved already by Li and Xie [9].
- If all functions in $\mathcal{F}$ are holomorphic, then for any $\alpha > 1$ the normality of $\mathcal{F}$ implies that $\mathcal{F}^*_{k,\alpha}$ is locally uniformly bounded [6, Theorem 1(c)].

In this paper, we further study the differential inequality $|f^{(k)}(z)|/(1 + |f(z)|^\alpha) \leq C$, but this time without any additional assumptions on the multiplicities of the zeros of the functions under consideration. It turns out that for $\alpha = 1$ (and hence trivially for $\alpha < 1$), this differential inequality implies quasi-normality, but that this does not hold for $\alpha > 1$.

THEOREM 1.1. *Let $k \geq 2$ be a natural number, $C > 0$ and $D \subseteq \mathbb{C}$ a domain. Then the family*

$$\mathcal{F}_k := \left\{ f \in \mathcal{H}(D) : \frac{|f^{(k)}(z)|}{1 + |f(z)|} \leq C \right\}$$

is quasi-normal.

REMARKS.

(1) In Theorem 1.1, we restrict to holomorphic rather than meromorphic functions, since if a meromorphic function f has a pole at z_0, then $|f^{(k)}(z)|/(1 + |f(z)|) \leq C$ is clearly violated in a certain neighborhood of z_0.

(2) The result also holds for $k = 1$, and in this case we can even conclude that $\mathcal{F}$ is normal. However, this is just a trivial consequence of Hinkkanen's extension of Marty's theorem since the condition $|f'(z)|/(1 + |f(z)|) \leq C$ clearly implies that the derivatives f' (and hence the spherical derivatives $f^\#$) are uniformly bounded on the preimages of five finite values.

(3) In Theorem 1.1, for $k \geq 2$, the order of quasi-normality can be arbitrarily large. This is demonstrated by the sequence of the functions

$$f_n(z) := n(e^z - e^{\zeta z})$$

(where $\zeta := e^{2\pi i/k}$) on the strip $D := \{z \in \mathbb{C} : -1 < \mathrm{Re}((1 - \zeta)z) < 1\}$. Indeed, $f_n^{(k)} = f_n$, so the differential inequality from Theorem 1.1 trivially holds, but every subsequence of $\{f_n\}_n$ is not normal exactly at the infinitely many common zeros $z_j = (2\pi i j)/(1 - \zeta) \in D$ $(j \in \mathbb{Z})$ of the f_n, so $\{f_n\}_n$ is quasi-normal of infinite order.

(4) In the spirit of Bloch's heuristic principle, one might ask for a corresponding result for entire functions. However, since the exponential function (and, more generally, entire solutions of the linear differential equation $f^{(k)} = C \cdot f$) satisfy the condition $|f^{(k)}(z)|/(1 + |f(z)|) \le C$, there does not seem to be a natural analogue for entire functions.

(5) For $\alpha > 1$ and $k \ge 2$, the condition $|f^{(k)}(z)|/(1 + |f(z)|^{\alpha}) \le C$ does not imply quasi-normality. In Section 3 we will construct a general counterexample for arbitrary $k \ge 2$, $\alpha > 1$ and $C > 0$. (For $k = 2$ and $\alpha = 3$, we had given such a counterexample already in [6].)

In fact, it turns out that this condition does not even imply Q_{β}-normality for any ordinal number β. (For the exact definition of Q_{β}-normality, we refer to [12].) So, there is no chance to extend Theorem 1.1 to the case $\alpha > 1$ even if one replaces the concept of quasi-normality by a weaker concept.

The same counterexample also shows that Theorem 1.1 cannot be extended in the spirit of the afore-mentioned results of Hinkkanen and Xu (Theorem A(b)). More precisely, a condition like

$$f(z) \in E \quad \Longrightarrow \quad \frac{|f^{(k)}|}{1 + |f|}(z) \le C,$$

where E is any finite subset of $\mathbb{C}$, does not imply quasi-normality (and not even Q_{β}-normality). This is due to the fact that this condition is even weaker than $|f^{(k)}(z)|/(1 + |f(z)|^{\alpha}) \le C'$ for suitable $C' > 0$.

One crucial step in our proof of Theorem 1.1 consists in using the fact that also the reverse inequality $|f^{(k)}(z)|/(1 + |f(z)|) \ge C$ implies quasi-normality [5]. This is one of the main results from our studies [1, 2, 5, 10] on meromorphic functions satisfying differential inequalities of the form $|f^{(k)}(z)|/(1 + |f(z)|^{\alpha}) \ge C$. These investigations were inspired by the observation that there is a counterpart to Marty's theorem in the following sense: a family of meromorphic functions whose spherical derivatives are bounded away from zero has to be normal [3, 14]. For the sake of completeness, we summarize the main results from those studies.

Theorem B. *Let $k \ge 1$ and $j \ge 0$ be integers and $C > 0$ and $\alpha > 1$ be real numbers. Let $\mathcal{F}$ be a family of meromorphic functions in some domain D.*

(a) [2] *If*

$$\frac{|f^{(k)}|}{1 + |f|^{\alpha}}(z) \ge C \quad \text{for all } z \in D \text{ and all } f \in \mathcal{F},$$

then $\mathcal{F}$ is normal.

(b) [5, 10] *If*

$$\frac{|f^{(k)}|}{1 + |f|}(z) \ge C \quad \text{for all } z \in D \text{ and all } f \in \mathcal{F},$$

then $\mathcal{F}$ is quasi-normal, but in general not normal.

(c) [1] *If $k > j$ and*

$$\frac{|f^{(k)}|}{1+|f^{(j)}|^{\alpha}}(z) \geq C \quad \textit{for all } z \in D \textit{ and all } f \in \mathcal{F},$$

then $\mathcal{F}$ is quasi-normal in D. If all functions in $\mathcal{F}$ are holomorphic, $\mathcal{F}$ is quasi-normal of order at most $j-1$. (For $j=0$ and $j=1$, this means that it is normal.) This does not hold for $\alpha = 1$ if $j \geq 1$.

2. Proof of Theorem 1.1

We apply induction. As mentioned above, the quasi-normality (in fact, even normality) of $\mathcal{F}_1$ follows from Hinkkanen's generalization of Marty's theorem.

Let some $k \geq 2$ be given and assume that it is already known that (on arbitrary domains) each of the conditions

$$\frac{|f^{(j)}(z)|}{1+|f(z)|} \leq C, \quad \text{where } j \in \{1, \dots, k-1\}$$

implies quasi-normality.

Let $\{f_n\}_n$ be a sequence in $\mathcal{F}_k$ and z^* an arbitrary point in D. Suppose to the contrary that $\{f_n\}_n$ is not quasi-normal at z^*.

Case 1: There are an $m \in \{1, \dots, k-1\}$ and a subsequence $\{f_{n_\ell}\}_\ell$ such that both $\{f_{n_\ell}^{(m)}\}_\ell$ and $\{(f_{n_\ell}^{(m)})/(f_{n_\ell})\}_\ell$ are normal at z^*.

Then (after turning to an appropriate subsequence, which we again denote by $\{f_n\}_n$ rather than $\{f_{n_\ell}\}_\ell$), without loss of generality we may assume that in a certain disk $\Delta(z^*, r) =: U$ both sequences $\{f_n^{(m)}\}_n$ and $\{(f_n^{(m)})/(f_n)\}_n$ converge *uniformly* (with respect to the spherical metric) to limit functions $H \in \mathcal{H}(U) \cup \{\infty\}$ and $L \in \mathcal{M}(U) \cup \{\infty\}$, respectively.

Case 1.1: H is holomorphic.

For each n we choose p_n to be the $(m-1)$th Taylor polynomial of f_n at z^*, that is, p_n has degree at most $m-1$ and satisfies $p_n^{(j)}(z^*) = f_n^{(j)}(z^*)$ for $j = 0, \dots, m-1$. Then f_n has the representation

$$f_n(z) = p_n(z) + \int_{z^*}^{z} \int_{z^*}^{\zeta_1} \cdots \int_{z^*}^{\zeta_{m-1}} f_n^{(m)}(\zeta_m)\, d\zeta_m \cdots d\zeta_1.$$

Here for $n \to \infty$

$$\begin{aligned} &\int_{z^*}^{z} \int_{z^*}^{\zeta_1} \cdots \int_{z^*}^{\zeta_{m-1}} f_n^{(m)}(\zeta_m)\, d\zeta_m \cdots d\zeta_1 \\ &\overset{\chi}{\Longrightarrow} \int_{z^*}^{z} \int_{z^*}^{\zeta_1} \cdots \int_{z^*}^{\zeta_{m-1}} H(\zeta_m)\, d\zeta_m \cdots d\zeta_1 =: F(z), \end{aligned}$$

where F is holomorphic in U. Since the family of polynomials of degree at most $m-1$ is quasi-normal (cf. [13, Theorem A.5]), we obtain the quasi-normality of $\{f_n\}_n$ at z^*.

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Case 1.2: $L(z^*) \neq \infty$.

We choose $r_0 \in (0; r)$ such that $|L(z)| \leq |L(z^*)| + 1$ for all $z \in \Delta(z^*, r_0) =: U_0$.

Then for all $z \in U_0$ and all n large enough,

$$\frac{|f_n^{(m)}|}{1 + |f_n|}(z) \leq \frac{|f_n^{(m)}|}{|f_n|}(z) \leq |L(z)| + 1 \leq |L(z^*)| + 2,$$

so by the induction hypothesis we obtain the quasi-normality of $\{f_n\}_n$ at z^*.

Case 1.3: $H \equiv \infty$ and $L(z^*) = \infty$. (This comprises the cases that $L \equiv \infty$ and that L is meromorphic with a pole at z^*.)

We choose $r_0 \in (0; r)$ such that $|L(z)| \geq 3$ for all $z \in \Delta(z^*, r_0) =: U_0$. Then for sufficiently large n, say for $n \geq n_0$, and all $z \in U_0$,

$$\left|\frac{f_n^{(m)}}{f_n}(z)\right| \geq |L(z)| - 1 \geq 2 \quad \text{and} \quad |f_n^{(m)}(z)| \geq 2.$$

Now fix an $n \geq n_0$ and a $z \in U_0$. If $|f_n(z)| \leq 1$,

$$\frac{|f_n^{(m)}|}{1 + |f_n|}(z) \geq \frac{|f_n^{(m)}|}{2}(z) \geq 1.$$

If $|f_n(z)| \geq 1$,

$$\frac{|f_n^{(m)}|}{1 + |f_n|}(z) \geq \frac{|f_n^{(m)}|}{2|f_n|}(z) \geq 1.$$

Combining both cases, we conclude that

$$\frac{|f_n^{(m)}|}{1 + |f_n|}(z) \geq 1 \quad \text{for all } z \in U_0 \text{ and all } n \geq n_0,$$

so by Theorem B(b) we obtain the quasi-normality of $\{f_n\}_n$ at z^*.

Case 2: For each $j = 1, \ldots, k-1$ and each subsequence $\{f_{n_\ell}\}_\ell$, at least one of the sequences $\{f_{n_\ell}^{(j)}\}_\ell$ and $\{(f_{n_\ell}^{(j)})/(f_{n_\ell})\}_\ell$ is not normal at z^*.

Then, after turning to an appropriate subsequence, which we again denote by $\{f_n\}_n$, by Montel's theorem for all $j = 1, \ldots, k-1$ we find sequences $\{w_{j,n}\}_n$ such that $\lim_{n\to\infty} w_{j,n} = z^*$ and such that for each n we have $|f_n^{(j)}(w_{j,n})| \leq 1$ or $|(f_n^{(j)})/(f_n)(w_{j,n})| \leq 1$. Both cases can be unified by writing

$$|f_n^{(j)}(w_{j,n})| \leq 1 + |f_n(w_{j,n})| \quad \text{for all } j = 1, \ldots, k-1 \text{ and all } n. \tag{2.1}$$

Furthermore, since $\{f_n\}_n$ is not quasi-normal and hence not normal at z^*, we may also assume that there is a sequence $\{w_{0,n}\}_n$ such that $\lim_{n\to\infty} w_{0,n} = z^*$ and $|f_n(w_{0,n})| \leq 1$ for all n.

We choose $r > 0$ sufficiently small such that $\overline{\Delta(z^*, r)} \subseteq D$, $2r < 1$ and $(4r(1 + C))/(1 - 2r) \leq 1$. Then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $j = 0, \ldots, k-1$ we have $w_{j,n} \in \Delta(z^*, r)$.

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We use the notation

$$M(r,f) := \max_{|z-z^*|\le r} |f(z)| \quad \text{for } f \in \mathcal{H}(\overline{\Delta(z^*,r)})$$

and obtain for all $n \ge n_0$, all $j = 1,\ldots,k-1$ and all $z \in \overline{\Delta(z^*,r)}$,

$$\begin{aligned}
|f_n^{(j)}(z)| &= \left| f_n^{(j)}(w_{j,n}) + \int_{[w_{j,n};z]} f_n^{(j+1)}(\zeta)\,d\zeta \right| \\
&\le |f_n^{(j)}(w_{j,n})| + |z - w_{j,n}| \cdot \max_{\zeta\in[w_{j,n};z]} |f_n^{(j+1)}(\zeta)| \\
&\le 1 + |f_n(w_{j,n})| + 2r \cdot M(r, f_n^{(j+1)}),
\end{aligned}$$

where for the last estimate we have applied (2.1).

Since this holds for any $z \in \overline{\Delta(z^*,r)}$, we conclude that for all $n \ge n_0$ and all $j = 1,\ldots,k-1$,

$$M(r, f_n^{(j)}) \le 1 + M(r, f_n) + 2r \cdot M(r, f_n^{(j+1)}).$$

Similarly, in view of $|f_n(w_{0,n})| \le 1$, we also have

$$M(r, f_n) \le 1 + 2r \cdot M(r, f_n').$$

Induction yields

$$\begin{aligned}
M(r, f_n) &\le 1 + \sum_{j=1}^{k-1} (2r)^j \cdot (1 + M(r, f_n)) + (2r)^k \cdot M(r, f_n^{(k)}) \\
&\le \sum_{j=0}^{k-1} (2r)^j + \sum_{j=1}^{k-1} (2r)^j \cdot M(r, f_n) + (2r)^k \cdot C \cdot (1 + M(r, f_n)) \\
&\le C + \frac{1}{1-2r} + \frac{2r\cdot(1+C)}{1-2r} \cdot M(r, f_n) \\
&\le C + \frac{1}{1-2r} + \frac{1}{2} \cdot M(r, f_n).
\end{aligned}$$

Hence,

$$M(r, f_n) \le 2C + \frac{2}{1-2r}$$

for all $n \ge n_0$. Thus, $\{f_n\}_{n\ge n_0}$ is uniformly bounded in $\Delta(z^*,r)$ and hence normal at z^* by Montel's theorem.

This completes the proof of Theorem 1.1. □

3. A general counterexample

In this section, we will show that for $\alpha > 1$ and $k \ge 2$ the differential inequality $|f^{(k)}(z)|/(1 + |f(z)|^\alpha) \le C$ does not imply quasi-normality. In [6], we had already given

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a counterexample for the case $k = 2$ and $\alpha = 3$. We generalize this example to arbitrary $k \geq 2$, $\alpha > 1$ and $C > 0$.

For given $k_0 \geq 2$, $C > 0$ and $\alpha > 1$, we construct a sequence $\{f_n\}_n$ of holomorphic functions in $D := \Delta(0; 2)$ such that $|f_n^{(k_0)}(z)|/(1 + |f_n(z)|^\alpha) \leq C$ for all $z \in D$ and all n, but $\{f_n\}_n$ is not quasi-normal in D.

First, take $p, q \in \mathbb{N}$ such that $1 < (p/q) < \min\{\alpha; 2\}$. The real function $h(x) := (1 + x^{p/q})/(1 + x^\alpha)$ is continuous in $[0, \infty)$ with $\lim_{x\to\infty} h(x) = 0$ and hence there exists an $M > 0$ such that

$$\frac{1 + x^{p/q}}{1 + x^\alpha} \leq M \qquad \text{for all } x \geq 0. \tag{3.1}$$

Let $g_n(z) := z^n - 1$ for $n \geq 1$. The zeros of g_n are the nth roots of unity $z_\ell^{(n)} = e^{2\pi i\ell/n}$ ($\ell = 0, 1, \ldots, n-1$), and they are all simple, $g_n'(z_\ell^{(n)}) \neq 0$. We consider the functions

$$h_n := g_n \cdot e^{p_n},$$

where the p_n are polynomials yet to be determined. Then

$$h_n' = e^{p_n}(g_n' + g_n p_n')$$

and

$$h_n'' = e^{p_n}(2g_n' p_n' + g_n p_n'^2 + g_n'' + g_n p_n''). \tag{3.2}$$

Our aim is to choose the p_n in such a way that for $\ell = 0, \ldots, n-1$,

$$h_n''(z_\ell^{(n)}) = h_n^{(3)}(z_\ell^{(n)}) = \cdots = h_n^{(k_0+1)}(z_\ell^{(n)}) = 0. \tag{3.3}$$

We first deduce several constraints on the p_n that are sufficient for (3.3), and then—by an elementary result on Hermite interpolation—we will see that it is possible to satisfy these constraints with polynomials p_n of sufficiently large degree.

First, in order to get $h_n''(z_\ell^{(n)}) = 0$, in view of (3.2), we will require that

$$p_n'(z_\ell^{(n)}) = -\frac{g_n''(z_\ell^{(n)})}{2g_n'(z_\ell^{(n)})} \quad (\ell = 0, 1, \ldots, n-1). \tag{3.4}$$

In order to proceed we need the following lemma.

Lemma 3.1. *For every $k \geq 2$,*

$$h_n^{(k)} = e^{p_n}[kg_n' p_n^{(k-1)} + g_n\varphi_k(p_n', \ldots, p_n^{(k-1)}) + \psi_k(g_n', \ldots, g_n^{(k)}, p_n', \ldots, p_n^{(k-2)}) + g_n p_n^{(k)}],$$

where $\varphi_k \in \mathbb{C}[x_1, \ldots, x_{k-1}]$ and $\psi_k \in \mathbb{C}[y_1, \ldots, y_k, x_1, \ldots, x_{k-2}]$ are polynomials.

Proof. We prove the lemma by induction on k. The base case $k = 2$ follows from (3.2) with $\varphi_2(x_1) = x_1^2$ and $\psi_2(y_1, y_2) = y_2$. Assume that the lemma holds for some $k \geq 2$.

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Then differentiating gives

$$
\begin{aligned}
h_n^{(k+1)} &= e^{p_n}\Big[\underbrace{k g_n'' p_n^{(k-1)}} + \boldsymbol{k g_n' p_n^{(k)}} + \underbrace{g_n' \varphi_k(p_n', \ldots, p_n^{(k-1)})} \\
&\quad + \underline{g_n \sum_{m=1}^{k-1} \frac{\partial \varphi_k}{\partial x_m}(p_n', \ldots, p_n^{(k-1)}) \cdot p_n^{(m+1)}} \\
&\quad + \underbrace{\sum_{m=1}^{k} \frac{\partial \psi_k}{\partial y_m}(g_n', \ldots, g_n^{(k)}, p_n', \ldots, p_n^{(k-2)}) \cdot g_n^{(m+1)}} \\
&\quad + \underbrace{\sum_{m=1}^{k-2} \frac{\partial \psi_k}{\partial x_m}(g_n', \ldots, g_n^{(k)}, p_n', \ldots, p_n^{(k-2)}) \cdot p_n^{(m+1)}} + \boldsymbol{g_n' p_n^{(k)}} \\
&\quad + g_n p_n^{(k+1)} + \underbrace{k g_n' p_n' p_n^{(k-1)}} + \underline{g_n p_n' \varphi_k(p_n', \ldots, p_n^{(k-1)})} \\
&\quad + \underbrace{p_n' \psi_k(g_n', \ldots, g_n^{(k)}, p_n', \ldots, p_n^{(k-2)})} + \underline{g_n p_n' p_n^{(k)}}\Big]. \\
&= e^{p_n} \cdot \Big[\boldsymbol{(k+1) g_n' p_n^{(k)}} + \underline{g_n \varphi_{k+1}(p_n', \ldots, p_n^{(k)})} \\
&\quad + \underbrace{\psi_{k+1}(g_n', \ldots, g_n^{(k+1)}, p_n', \ldots, p_n^{(k-1)})} + g_n p_n^{(k+1)}\Big],
\end{aligned}
$$

where

$$
\varphi_{k+1}(x_1, \ldots, x_k) := \sum_{m=1}^{k-1} \frac{\partial \varphi_k}{\partial x_m}(x_1, \ldots, x_{k-1}) \cdot x_{m+1} + x_1 \varphi_k(x_1, \ldots, x_{k-1}) + x_1 x_k
$$

and

$$
\begin{aligned}
\psi_{k+1}(y_1, \ldots, y_{k+1}, x_1, \ldots, x_{k-1}) := k y_2 x_{k-1} &+ y_1 \varphi_k(x_1, \ldots, x_{k-1}) \\
&+ \sum_{m=1}^{k} \frac{\partial \psi_k}{\partial y_m}(y_1, \ldots, y_k, x_1, \ldots, x_{k-2}) \cdot y_{m+1} \\
&+ \sum_{m=1}^{k-2} \frac{\partial \psi_k}{\partial x_m}(y_1, \ldots, y_k, x_1, \ldots, x_{k-2}) \cdot x_{m+1} \\
&+ k y_1 x_1 x_{k-1} + x_1 \psi_k(y_1, \ldots, y_k, x_1, \ldots, x_{k-2})
\end{aligned}
$$

are indeed polynomials of the requested form.

Hence, the lemma holds for $k + 1$ as well. □

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Now we inductively determine the required values of $p_n^{(k)}(z_\ell^{(n)})$ for $k = 2, \dots, k_0$ and $\ell = 0, \dots, n-1$. For given $k \in \{2, \dots, k_0\}$, let us assume that we already know the values of $p_n'(z_\ell^{(n)}), \dots, p_n^{(k-1)}(z_\ell^{(n)})$ that ensure that $h_n''(z_\ell^{(n)}) = \cdots = h_n^{(k)}(z_\ell^{(n)}) = 0$ for all admissible ℓ. (Note that the required values of $p_n'(z_\ell^{(n)})$ have been found in (3.4).)

In order to find the values of $p_n^{(k)}(z_\ell^{(n)})$ (which ensure that $h_n^{(k+1)}(z_\ell^{(n)}) = 0$), we apply Lemma 3.1 with $k+1$ in place of k and obtain the condition

$$p_n^{(k)}(z_\ell^{(n)}) = -\frac{\psi_{k+1}(g_n', \dots, g_n^{(k+1)}, p_n', \dots, p_n^{(k-1)})}{(k+1)g_n'}(z_\ell^{(n)}). \tag{3.5}$$

(Observe that evaluating the right-hand side requires only the knowledge of values of $p_n', \dots, p_n^{(k-1)}$ that have been previously determined.)

It is well known (see, for example, [15, page 52]) that for every $n \geq 1$ the conditions (3.4) and (3.5) (for $k = 2, \dots, k_0$) can be achieved with a polynomial p_n of degree at most nk_0.

In this way,

$$h_n''(z_\ell^{(n)}) = \cdots = h_n^{(k_0+1)}(z_\ell^{(n)}) = 0.$$

In particular, each $z_\ell^{(n)}$ is a zero of $h_n^{(k_0)}$ of multiplicity ≥ 2.

Now the functions $(h_n^{(k_0)q})/h_n^p$ are entire: h_n^p is entire and its zeros $z_\ell^{(n)}$ ($\ell = 0, 1, \dots, n-1$) have multiplicity p, while $h_n^{(k_0)q}$ has zeros at $z_\ell^{(n)}$ of multiplicity at least $2q > p$. Thus, $c_n := \max_{z \in \overline{D}} \left|((h_n^{(k_0)})^q)/(h_n^p)(z)\right| < \infty$. Define now for every $n \geq 1$,

$$f_n := a_n \cdot h_n,$$

where $a_n > 0$ is a large enough constant such that both

$$a_n \geq \left(\frac{c_n \cdot M^q}{C^q}\right)^{1/(p-q)} \quad \text{that is,} \quad \frac{c_n}{a_n^{p-q}} \leq \left(\frac{C}{M}\right)^q \tag{3.6}$$

and $f_n \overset{\chi}{\Longrightarrow} \infty$ on $\mathbb{C} \setminus \partial\Delta(0;1)$; the latter can be achieved by choosing

$$a_n \geq \frac{n}{\min\left\{|h_n(z)| : |z| \leq 1 - \frac{1}{n} \text{ or } 1 + \frac{1}{n} \leq |z| \leq n\right\}}.$$

Then $\{f_n\}_n$ is not quasi-normal in D (as it is not normal at any point of $\partial\Delta(0;1)$), yet satisfies

$$\frac{|f_n^{(k_0)}(z)|}{1 + |f_n(z)|^\alpha} \leq C \quad \text{for all } z \in D.$$

Indeed, for all $z \in D$,

$$\begin{aligned}\left(\frac{|f_n^{(k_0)}|}{1 + |f_n|^{p/q}}\right)^q(z) &\leq \frac{|f_n^{(k_0)}|^q}{1 + |f_n|^p}(z) \leq \frac{|f_n^{(k_0)}|^q}{|f_n|^p}(z) = \frac{a_n^q \cdot |h_n^{(k_0)}|^q}{a_n^p \cdot |h_n|^p}(z) \\ &\leq \frac{c_n}{a_n^{p-q}} \leq \left(\frac{C}{M}\right)^q,\end{aligned}$$

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where the last inequality is just (3.6). Therefore,

$$\frac{|f_n^{(k_0)}|}{1+|f_n|^{p/q}}(z) \le \frac{C}{M} \quad \text{for all } z \in D,$$

and together with (3.1) we conclude that

$$\frac{|f_n^{(k_0)}|}{1+|f_n|^{\alpha}}(z) = \frac{|f_n^{(k_0)}|}{1+|f_n|^{p/q}}(z) \cdot \frac{1+|f_n|^{p/q}}{1+|f_n|^{\alpha}}(z) \le \frac{C}{M} \cdot M = C \quad \text{for all } z \in D,$$

as desired.

REMARK. Actually, we have shown something stronger: the condition $(|f^{(k)}(z)|)/(1+|f(z)|^{\alpha}) \le C$ does not even imply Q_β-normality for any ordinal number β since the constructed sequence $\{f_n\}_n$ and all of its subsequences are not normal at any point of the continuum $\partial\Delta(0;1)$.

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