MEASURES EQUIVALENT TO THE HAAR MEASURE

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1. Introduction. We call two measures equivalent if each is absolutely continuous with respect to the other (cf. [1]). Let G be a locally compact topological group and let μ be a non-negative Baire measure on G (i.e. μ is defined on all Baire sets, finite on compact sets and positive on open sets). We say that μ is stable if $\mu(E)=0$ implies $\mu(tE)=0$ for each $t \in G$. A. M. Macbeath made the conjecture that every stable non-trivial Baire measure is equivalent to the Haar measure. In this paper we prove the following slightly stronger result :

THEOREM. *Every stable non-trivial measure defined on Baire sets and finite on some open set is equivalent to the Haar measure.*

It is obvious that not every stable measure on Baire sets is equivalent to the Haar measure ; a counter-example is provided by an invariant Hausdorff measure in Euclidean space which is of lower dimension than the space itself.

Theorems B, C and Lemma 1 are due to A. M. Macbeath. He suggested to me the idea of constructing a Haar measure by means of a " Jacobian function ". We have used a similar method of proof in [2].

We assume now that μ is a stable non-trivial measure on Baire sets such that $\mu(U)<\infty$ for some open set U. Let us observe that $\mu(V) > 0$ for every open set V. For, if $\mu(V) = 0$, then $\mu(tV) = 0$ and, since every compact set C can be covered by a finite union of sets tV, we have $\mu(C) = 0$ and the measure vanishes contrary to definition. Replacing, if necessary, U by an open bounded subset, we may assume in the sequel that U itself is bounded.

If X is a topological space, we denote by $B(X)$ the class of all extended real-valued Baire functions $f(x)$ defined on X (extended real numbers include $+\infty$ and $-\infty$) and by $B_+(X)$ the subclass of non-negative functions. We denote by $\mathbf{N}$ a complete system of bounded neighbourhoods of the unity e of G. χ_E will be used to denote the characteristic function of the set E (i.e. χ_E vanishes outside E and is equal to 1 on E).

We have to show that if E is a Baire set such that $\mu(E) = 0$, then E has Haar measure zero, and conversely. Since each Baire set E is contained in a σ-compact† open subgroup G_0 of G (cf. [1], § 5, Theorem D, p. 24) and the Haar measure on G serves also as a Haar measure on G_0, and since moreover μ is stable on G_0, we may assume in the following that G itself is σ-compact.

2. Preliminary results.

THEOREM A. *The measure μ is equivalent to a Baire measure $\bar{\mu}$.*

Proof. Consider a set $T \subset G$ which is minimal with respect to the property

$$G = \bigcup \{tU : t \in T\}$$

(so that the family $\{tU\}$, $t \in T$, is a minimal covering of G by sets tU). We show that every compact set C intersects only a finite number of sets tU, $t \in T$. Suppose the contrary. The compact set $\overline{CU^{-1}U}$ can be covered by a subfamily $\{tU\}$, $t \in T^*$, where T^* is a finite subset of T.

† A σ-compact set is a countable union of compact sets.

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On the other hand, by assumption, we have that $C \cap tU \neq \emptyset$ for infinitely many $t \in T$. Hence there is a $t_0 \in T - T^*$ such that $C \cap t_0 U \neq \emptyset$. We observe that $t_0 U \subset CU^{-1}U$ and hence the family $\{tU\}$, $t \in T - \{t_0\}$, covers $G - CU^{-1}U$. It covers also $CU^{-1}U$ because $T^* \subset T - \{t_0\}$. This contradicts the minimal nature of T.

Since G is σ-compact, T is countable. We define

$$\bar{\mu}(E) = \sum \{\mu(t^{-1}E \cap U) : t \in T\}.$$

Since, for fixed t, $\mu(t^{-1}E \cap U)$ is a Baire measure on the set E, we see that $\bar{\mu}$ is a Baire measure on G. Since E is covered by the family $\{tU\}$, $t \in T$, we have that $\mu(E) = 0$ is equivalent to $\mu(E \cap tU) = 0$ for each $t \in T$. This last condition is equivalent to $\bar{\mu}(E) = 0$, by the stability of μ. Thus μ and $\bar{\mu}$ are equivalent. It is obvious that if E is compact, then $\mu(t^{-1}E \cap U) > 0$ holds only for a finite number of $t \in T$ and thus $\bar{\mu}(E)$ is finite. This completes the proof of the theorem.

By Theorem A, it is enough to prove our main result for $\bar{\mu}$ instead of for μ. Equivalently, we shall assume that μ is a Baire measure.

DEFINITION. *For given $t \in G$, a function $J_t(x) \in B_+(G)$ is called a μ-Jacobian if, for every $f \in B_+(G)$,*

$$\int f(t^{-1}x)\, d\mu(x) = \int f(x) J_t(x)\, d\mu(x). \qquad \text{................................(1)}$$

THEOREM B. *There exists, for every t, an everywhere positive μ-Jacobian $J_t(x)$.*

Proof. For a fixed t, define the measure μ_t on G by $\mu_t(E) = \mu(tE)$. Since μ is stable, $\mu(E) = 0$ is equivalent to $\mu_t(E) = 0$ and thus the measures μ and μ_t are equivalent. Applying the Radon-Nikodym theorem to the totally σ-finite measures μ, μ_t (cf. [1], § 31, Theorem B, p. 128), we see that there is an everywhere positive function $J_t(x) \in B(G)$ with the property that

$$\mu_t(E) = \int \chi_E(t^{-1}x)\, d\mu(x) = \int \chi_E(x) J_t(x)\, d\mu(x).$$

This equality holds for every Baire set E and thus it holds also with any function $f \in B_+(G)$ in place of χ_E.

THEOREM C. *Jacobians satisfy, for every s, t,*

$$J_{st}(x) = J_s(tx) J_t(x)$$

for almost all x.

Proof. Let E be a Baire set. We have, by (1), $\mu(stE) = \int_E J_{st}(x)\, d\mu(x)$. On the other hand, also by (1),

$$\mu(stE) = \int_{tE} J_s(x)\, d\mu(x) = \int \chi_E(t^{-1}x) J_s(x)\, d\mu(x)$$

$$= \int_E J_s(tx) J_t(x)\, d\mu(x).$$

Since E is arbitrary, the result follows by comparing the integrals.

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THEOREM D. *If* $\mu(E) = 0$, *then* $\mu(Et) = 0$.

Proof. The result follows by the stability of μ if we show that, if $\mu(E) > 0$, then $\mu(E^{-1}) > 0$. Then from $\mu(E) = 0$ we have $\mu(E^{-1}) = 0$; hence $\mu(t^{-1}E^{-1}) = 0$ and thus $\mu(Et) = 0$.

Suppose that $\mu(E) > 0$. Let us show that then

$$I = \int \mu(tE^{-1} \cap E)\, d\mu(t) > 0. \qquad (2)$$

We have

$$I = \int\int \chi_{E^{-1}}(t^{-1}x)\chi_E(x)\, d\mu(x)\, d\mu(t)$$

$$= \int_E d\mu(x) \int \chi_{E^{-1}}(t^{-1}x)\, d\mu(t). \qquad (3)$$

Applying (1) to the last integral, we have, by Theorem B, for each x,

$$\int \chi_{E^{-1}}(t^{-1}x)\, d\mu(t) = \int \chi_{E^{-1}}((x^{-1}t)^{-1})\, d\mu(t) = \int \chi_{E^{-1}}(t^{-1})J_x(t)\, d\mu(t) = \int_E J_x(t)\, d\mu(t) > 0. \qquad (4)$$

So we have (2), by (3) and (4). Now, by (2), there is a $t \in G$ such that $\mu(tE^{-1}) > 0$ and thus $\mu(E^{-1}) > 0$. This completes our proof.

3. Proof of the main theorem.

LEMMA 1. *Let m be the right invariant Haar measure. There exists a Baire measure ν on $G \times G$ and a positive function $J(t, x) \in B(G \times G)$ such that, for any Baire sets D, E,*

$$\nu(D \times E) = \int_D \mu(tE)\, dm(t) = \int_E d\mu(x) \int_D J(t, x)\, dm(t). \qquad (5)$$

If D is bounded, then the measure $\mu_D(E)$ defined by $\mu_D(E) = \nu(D \times E)$ is finite on compact sets. If $m(D) > 0$, then μ and μ_D are equivalent.

Proof. Consider the space $G \times G$ with the Baire measure $m \times \mu$. Define, for $M \subset G \times G$,

$$\nu(M) = \int \chi_M(t, t^{-1}x)\, d(m \times \mu)(t, x). \qquad (6)$$

For $M = D \times E$ we have, from (6), $\mu_D(E) = \int_D \mu(tE)\, dm(t)$. Thus, if D is bounded, then, for compact E, DE is bounded and $\mu_D(E) \leqslant m(D)\mu(DE) < \infty$. If $m(D) > 0$, then, by the stability of μ, μ_D and μ are equivalent.

To define $J(t, x)$, we show that ν and $m \times \mu$ are equivalent. Since μ is stable, the functions $P(t) = \int \chi_M(t, x)\, d\mu(x)$, $Q(t) = \int \chi_M(t, t^{-1}x)\, d\mu(x)$ are, for each fixed t, both zero or both positive. Therefore the measures $(m \times \mu)(M) = \int P(t)\, dm(t)$, $\nu(M) = \int Q(t)\, dm(t)$ are both zero or both positive.

Applying the Radon-Nikodym theorem to the measures ν and $m \times \mu$ on $G \times G$, we see that there is a positive function $J(t, x) \in B(G \times G)$ such that

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$$\nu(M) = \int_M J(t, x)\, d(m \times \mu)(t, x). \qquad \ldots\ldots(7)$$

In particular, if $M = D \times E$, we have (5), by (6) and (7). This completes our proof.

We shall use the phrase " for almost all " if the measure concerned is one of the equivalent measures μ, μ_D. It will be convenient to denote a μ_D-Jacobian $J_t(x)$ by $J_D(t, x)$. Given a function $f(x)$ and a set $C \subset G$, we shall say that f is bounded away from zero and infinity on C if there are finite positive constants c_1 and c_2 such that $c_1 < f(x) < c_2$ holds for all $x \in C$.

LEMMA 2. *Let $D \in \mathrm{N}$. The formula*

$$J_D(\tau, x) = I_D(\tau, x)/I_D(e, x),$$

where $I_D(\tau, x) = \int_D J(t\tau, x)\, dm(t)$, *defines a μ_D-Jacobian. For almost all $x \in G$, the function $J_D(\tau, x)$, regarded as a function of τ, is bounded away from zero and infinity on every compact set.*

Proof. We have, by (5),

$$\mu_D(\tau E) = \int_D \mu(t\tau E)\, dm(t) = \int_{D\tau} \mu(tE)\, dm(t) = \nu(D\tau \times E)$$

because m is right invariant. Thus, again by (5) and by the invariance of m,

$$\mu_D(\tau E) = \int_E d\mu(x) \int_{D\tau} J(t, x)\, dm(t) = \int_E I_D(\tau, x)\, d\mu(x). \qquad \ldots\ldots(8)$$

From (8), for $\tau = e$, $\mu_D(E) = \int_E I_D(e, x)\, d\mu(x)$. This can be written in the other form

$$\int \chi_E(x)\, d\mu_D(x) = \int \chi_E(x) I_D(e, x)\, d\mu(x).$$

Since this equality holds for each Baire set E, we deduce easily that it remains valid if we put any function $f \in B_+(G)$ in place of χ_E. Let, in particular, $f(x) = J_D(\tau, x)\chi_E(x)$, where τ, E are fixed. Then, since $I_D(e, x) > 0$ (cf. Lemma 1),

$$\int_E J_D(\tau, x)\, d\mu_D(x) = \int_E I_D(\tau, x)\, d\mu(x).$$

This, together with (8), implies that $\mu_D(\tau E) = \int_E J_D(\tau, x)\, d\mu_D(x)$. Hence J_D is a μ_D-Jacobian.

To prove the second part of our lemma, observe that, for all compact sets Q, C, $\mu_Q(C) < \infty$ by Lemma 1 and hence, by (8) with $\tau = e$, $D = Q$, $E = C$, we have that $I_Q(e, x)$ is finite for almost all $x \in C$. Since G is a union of a countable increasing sequence $\mathbf{Q}$ of compact sets, almost every $x \in C$ has the property that $I_Q(e, x)$ is finite for each $Q \in \mathbf{Q}$, and hence for every compact set. Since C is arbitary, almost every $x \in G$ has this property.

Suppose that $I_Q(e, x_0) < \infty$ for each compact Q. Let C be compact and let Q be a compact set containing DC. Then, for $\tau \in C$,

$$I_D(\tau, x_0) = \int_{D\tau} J(t, x_0)\, dm(t) \leqslant \int_Q J(t, x_0)\, dm(t) = I_Q(e, x_0) < \infty.$$

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Hence $I_D(\tau, x_0)$ is bounded above on C. To prove that I_D is bounded below by a positive constant, consider a set $V \in \mathbf{N}$ such that $VV^{-1}V \subset D$. Let $\{Vt_1, \ldots, Vt_n\}$ be a maximal disjoint family of sets of the form Vt_r which are contained in DC (this family is finite since DC is bounded). For every $\tau \in C$, $D\tau$ contains at least one Vt_i, $i \leqslant n$. To see this we note that $V\tau$ cannot be disjoint to all Vt_i, by the above maximality condition. So, for some i, $V\tau \cap Vt_i \neq \varnothing$, $t_i \in V^{-1}V\tau$ and $Vt_i \subset D\tau$. Since $J(t, x_0) > 0$, we have, for each $\tau \in C$,

$$I_D(\tau, x_0) = \int_{D\tau} J(t, x_0)\, dm(t) \geqslant \int_{Vt_i} J(t, x_0)\, dm(t) = I_i > 0$$

for a certain $i \leqslant n$. Hence $I_D(\tau, x_0) \geqslant \min\{I_1, \ldots, I_n\} > 0$. So we see that $I_D(\tau, x_0)$ is bounded away from zero and infinity on C and thus the same is true for $J_D(\tau, x_0)$.

This completes the proof of Lemma 2.

We note that from Lemma 2 and Theorem C we have, for every s, t,

$$J_D(st, x) = J_D(s, tx) J_D(t, x) \qquad \text{.................................(9)}$$

for almost all x.

Lemma 3. *There is an $x_0 \in G$ such that $J_D(tx_0^{-1}, x_0)$, considered as a function of t, is bounded away from zero and infinity on every compact set and moreover, for almost all s,*

$$J_D(stx_0^{-1}, x_0) = J_D(s, t) J_D(tx_0^{-1}, x_0) \qquad \text{..............................(10)}$$

holds for almost all t.

Proof. A subset E of a measure space X will be called almost equal to X if $X - E$ has measure zero. Let X, Y be measure spaces and let $X \times Y$ be the product space with the product measure. For $E \subset X \times Y$, $E(x)$ denotes the section of E determined by $x \in X$, i.e. the set of all $y \in Y$ such that $\langle x, y \rangle \in E$. It follows from Fubini's theorem (cf. [1], § 36, Theorem A, p. 147) that E is almost equal to $X \times Y$ if and only if, for almost all $x \in X$, $E(x)$ is almost equal to Y.

Let $M \subset G \times G \times G$ be the set of all triples $\langle s, t, x \rangle$ satisfying (9). Since, for every s, t, (9) holds almost everywhere, $M(s, t)$ is almost equal to G and thus M is almost equal to $G^3 = G \times G \times G$. Consequently, for almost every x, $M(x)$ is almost equal to $G^2 = G \times G$. Applying Lemma 2, we see that there is an x_0 such that $J_D(t, x_0)$ is bounded away from zero and infinity on every compact set and moreover $M(x_0)$ is almost equal to G^2. Using again Fubini's theorem, we deduce that almost all s have the property that $M(s, x_0)$ is almost equal to G. But then, by Theorem D, $M(s, x_0)x_0$ also is almost equal to G and we have, for almost all t, $t \in M(s, x_0)x_0$, $tx^{-1} \in M(s, x_0)$. This means that (9) holds, with $x = x_0$ and tx_0^{-1} in place of t. Hence (10) follows and Lemma 3 is proved.

We are now in position to complete our proof. We define on G a measure η by

$$\eta(E) = \int_E J_D^{-1}(tx_0^{-1}, x_0)\, d\mu_D(t).$$

We now prove that η is left invariant. We denote by Q the set of all s such that (10) holds for almost all t. Thus $\mu(G - Q) = 0$ by Lemma 3. Let τ, E be arbitrary. For all $s \in Q$, since J_D is a Jacobian,

$$\eta(sE) = \int \chi_E(s^{-1}t) J_D^{-1}(tx_0^{-1}, x_0)\, d\mu_D(t) = \int_E J_D(s, t) J_D^{-1}(stx_0^{-1}, x_0)\, d\mu_D(t) = \eta(E).$$

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In particular, if E is replaced by τE, $\eta(s\tau E) = \eta(\tau E)$ holds for all $s \in Q$. From $\mu(G-Q) = 0$ and Theorem D, $\mu(G - Q\tau^{-1}) = 0$. Hence $Q \cap Q\tau^{-1} \neq \emptyset$ and if $s \in Q \cap Q\tau^{-1}$, then we have $\eta(s\tau E) = \eta(\tau E)$ and also, since $s\tau \in Q$, $\eta(s\tau E) = \eta(E)$. Thus $\eta(E) = \eta(\tau E)$.

Since $J_D^{-1}(tx_0^{-1}, x_0)$ is bounded away from zero and infinity on compact sets, η is equivalent to μ_D and therefore to μ. We have also that η is finite on compact sets and thus it is a Haar measure.

This completes the proof of the main theorem.

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