Can. J. Math., Vol. XXV, No. 5, 1973, pp. 1090–1102

A TECHNIQUE FOR STUDYING THE BOUNDEDNESS AND EXTENDABILITY OF CERTAIN TYPES OF OPERATORS

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1. Introduction. For $1 \leqq p < \infty$, μ real, let $L_{\mu,p}$ denote the collection of functions f, Lebesgue measurable on $(0, \infty)$, and such that $||f||_{\mu,p} < \infty$, where

$$(1.1) \qquad ||f||_{\mu,p} = \left\{ \int_0^\infty t^{\mu-1}|f(t)|^p dt \right\}^{1/p}.$$

Also, if X and Y are Banach spaces, denote by $[X, Y]$ the collection of bounded linear operators from X to Y; $[X, X]$ denote by $[X]$. Let $\mathscr{S}_\mu$ denote the collection of operators $S \in [L_{\mu,2}]$, which are defined in terms of a kernel k, associated with S, by an equation of the form

$$(1.2) \qquad (Sf)(x) = x^{-(\mu-1)/2} \frac{d}{dx} x^{-(\mu-1)/2} \int_0^\infty k(xt)f(t) \frac{dt}{t},$$

and let $\mathscr{T}_\mu$ denote the collection of operators $T \in [L_{\mu,2}]$, which are defined in terms of a kernel l, associated with T, by an equation of the form

$$(1.3) \qquad (Tf)(x) = x^{-(\mu-1)/2} \frac{d}{dx} x^{(\mu-1)/2} \int_0^\infty l(x/t)f(t)dt.$$

In a recent paper [**6**], we considered particular operators of the form $R_2^{-1}R_1$, where either both R_1 and R_2 belonged to $\mathscr{S}_\mu$ for some μ, or both R_1 and R_2 belonged to $\mathscr{T}_\mu$ for some μ. By associating with a certain function, analytic in a strip, an operator in $[L_{\mu,p}]$ for a range of values of μ and p, we were able to extend $R_2^{-1}R_1$ to other $L_{\mu,p}$ spaces as an element of $[L_{\mu,p}]$. The technique used there seems of some general interest, and our first objective in this paper is to prove a general result as to when an operator in $[L_{\mu,p}]$ can be defined by the method used in [**6**]. This is accomplished in Theorem 1.

Our second objective is to show when $R_2^{-1}R_1$ can be extended, and we achieve this in Theorem 2.

In [**6**] we applied our results to relate the ranges of R_1 and R_2, and our final objective is to place the technique used there in a general setting. This is done in Theorem 3.

In section 2 below we prove a number of preliminary lemmas. In section 3 we show how to associate an operator of $[L_{\mu,p}]$ with a function analytic in a strip, the results being summed up in Theorem 1. Section 4 is devoted to determining necessary and sufficient conditions that transformations be in $\mathscr{S}_\mu$ or $\mathscr{T}_\mu$, while in section 5 we give conditions that $R_2^{-1}R_1$ exist and be extendable.

Received June 29, 1972. This research was supported by N.R.C. Grant No. A4048.

https://doi.org/10.4153/CJM-1973-116-9 Published online by Cambridge University Press

In section 6 we show how the extendability of R_2 can be used to extend R_1, and relate their ranges, while in section 7 we give two examples of the use of this process.

2. Preliminaries. In this section we shall prove two lemmas giving some properties of the spaces $L_{\mu,p}$, define the Mellin transformation, and state a lemma giving its principal properties. First we need a definition.

Definition 2.1. *If* $1 \leqq p < \infty$, $f \in L_{\mu,p}$, *we define* $C_{\mu,p}$ *by*

$$(C_{\mu,p}f)(t) = e^{\mu t/p}f(e^t).$$

LEMMA 2.1. $C_{\mu,p}$ *is an isometric isomorphism of* $L_{\mu,p}$ *onto* $L_p(-\infty, \infty)$.

Proof. See [**6**, Lemma 2.1].

Definition 2.2. *Denote by* C_0 *the collection of functions, continuous on* $(0, \infty)$ *and vanishing outside some interval* (a, b), *where* $0 < a < b < \infty$.

LEMMA 2.2. C_0 *is dense in* $L_{\mu,p}$. *Indeed if* $f \in L_{\mu_1,p_1} \cap L_{\mu_2,p_2}$ *and* $\epsilon > 0$, *then* g *exists in* C_0 *so that* $\|f - g\|_{\mu_i,p_i} < \epsilon$, $i = 1, 2$.

Proof. See [**6**, Lemmas 2.2 and 2.3].

Definition 2.3. *For* $f \in L_{\mu,p}$, $1 \leqq p \leqq 2$, *let*

$$(\mathscr{M}f)((\mu/p) + it) = (C_{\mu,p}f)\hat{}(t),$$

where $\hat{F}$ *is the Fourier transform of* F, *defined by*

$$\hat{F}(t) = \int_{-\infty}^{\infty} e^{itu}F(u)du$$

when $F \in L_1(-\infty, \infty) \cap L_p(-\infty, \infty)$, *and by continuity on* $L_p(-\infty, \infty)$ *when* $1 < p \leqq 2$. $\mathscr{M}$ *will be called the Mellin transformation.*

LEMMA 2.3. *If* $1 \leqq p \leqq 2$, $\mathscr{M} \in [L_{\mu,p}, L_{p'}(-\infty, \infty)]$. *If* $p = 2$, $\mathscr{M}$ *is unitary if* $L_2(-\infty, \infty)$ *has measure* $dt/2\pi$.

Proof. See [**6**, Lemma 4.1].

3. A class of operators. We first define a class of analytic functions, and then show that with each member of this class we can associate an operator in $[L_{\mu,p}]$ for a range of values of μ and p.

Definition 3.1. *We say* $m \in \mathscr{A}$ *if there are extended real numbers* $\alpha(m)$ *and* $\beta(m)$, *with* $\alpha(m) < \beta(m)$, *so that*

(a) $m(s)$ *is analytic in the strip* $\alpha(m) < \operatorname{Re} s < \beta(m)$,

(b) *in every closed sub-strip*, $\sigma_1 \leqq \operatorname{Re} s \leqq \sigma_2$, *where* $\alpha(m) < \sigma_1 \leqq \sigma_2 < \beta(m)$, $m(s)$ *is bounded,*

(c) *for* $\alpha(m) < \sigma < \beta(m)$, $|m'(\sigma + it)| = O(|t|^{-1})$, *as* $|t| \to \infty$.

https://doi.org/10.4153/CJM-1973-116-9 Published online by Cambridge University Press

LEMMA 3.1. *If* $m \in \mathscr{A}$, *then for each* σ, $\alpha(m) < \sigma < \beta(m)$, *and for each* p, $1 < p < \infty$, $m(\sigma + it)$ *is an* $L_p(-\infty, \infty)$ *multiplier. If the operator, in* $[L_p(-\infty, \infty)]$ *for* $1 < p < \infty$, *generated by* $m(\sigma + it)$ *is denoted by* $T_{m,\sigma}$, *then for* $1 < p \leqq 2$, $F \in L_p(-\infty, \infty)$,

$$(3.1) \qquad (T_{m,\sigma}F)^\wedge(t) = m(\sigma + it)\hat{F}(t).$$

If $1 < p \leqq 2$, $\alpha(m) < \sigma < \beta(m)$, $T_{m,\sigma}$ *is one-to-one on* $L_p(-\infty, \infty)$ *unless* $m \equiv 0$. *If* $m^{-1} \in \mathscr{A}$, *then for* $\max(\alpha(m), \alpha(m^{-1})) < \sigma < \min(\beta(m), \beta(m^{-1}))$, $1 < p < \infty$, $T_{m,\sigma}$ *is a one-to-one mapping of* $L_p(-\infty, \infty)$ *onto itself, and*

$$(3.2) \qquad (T_{m,\sigma})^{-1} = T_{m^{-1},\sigma}.$$

Proof. The first statement follows from [**7**, Chapter 4, Theorem 3] as does (3.1) when $p = 2$, and thus for $F \in L_p(-\infty, \infty) \cap L_2(-\infty, \infty)$. But this last space is dense in $L_p(-\infty, \infty)$, and from [**8**, Theorem 74] both sides of (3.1) represent bounded operators from $L_p(-\infty, \infty)$ to $L_{p'}(-\infty, \infty)$ since $1 < p \leqq 2$, and $m(\sigma + it)$ is bounded. Thus by continuity, (3.1) is true for $1 < p \leqq 2$.

The next statement follows from (3.1), for since $m(s)$ is analytic, $m(\sigma + it) \neq 0$ a.e., and thus if $T_{m,\sigma}F = 0$ a.e., $\hat{F} = 0$ a.e. and $F = 0$ a.e.

From (3.1), if $\max(\alpha(m), \alpha(m^{-1})) < \sigma < \min(\beta(m), \beta(m^{-1}))$, then for $F \in L_2(-\infty, \infty)$, $(T_{m^{-1},\sigma}T_{m,\sigma}F)^\wedge(t) = (T_{m,\sigma}T_{m^{-1},\sigma}F)^\wedge(t) = \hat{F}(t)$ a.e., and hence $T_{m^{-1},\sigma}T_{m,\sigma} = T_{m,\sigma}T_{m^{-1},\sigma} = I$ on $L_2(-\infty, \infty)$. But then by the denseness of $L_2(-\infty, \infty) \cap L_p(-\infty, \infty)$ in $L_p(-\infty, \infty)$ and the continuity of all three operators appearing in this last equation, it must hold for $1 < p < \infty$, and the remainder of the lemma follows.

LEMMA 3.2. *Suppose* $m \in \mathscr{A}$, $1 < p < \infty$, $\alpha(m) < \mu/p < \beta(m)$, *and let*

$$H_{m,\mu,p} = C_{\mu,p}^{-1}T_{m,\mu/p}C_{\mu,p}.$$

Then $H_{m,\mu,p} \in [L_{\mu,p}]$. *If* $f \in L_{\mu_1,p_1} \cap L_{\mu_2,p_2}$, *where* $1 < p_i < \infty$, $\alpha(m) < \mu_i/p_i < \beta(m)$, *then* $H_{m,\mu_1,p_1}f = H_{m,\mu_2,p_2}f$ *a.e.*

Proof. That $H_{m,\mu,p} \in [L_{\mu,p}]$ follows from Lemmas 2.1 and 3.1. For the remainder, suppose first that $f \in C_0$, and let

$$F(s) = \int_0^\infty x^{s-1}f(x)dx.$$

Clearly F is entire. Now

$$C_{\mu_1,p_1}H_{m,\mu_1,p_1}f = T_{\mu_1/p_1}C_{\mu_1,p_1}f;$$

but clearly $C_{\mu_1,p_1}f \in L_2(-\infty, \infty)$, and hence by Lemma 3.1, so is $C_{\mu_1,p_1}H_{m,\mu_1,p_1}f$, and from (3.1),

$$(C_{\mu_1,p_1}H_{m,\mu_1,p_1}f)^\wedge(t) = m((\mu_1/p_1) + it)(C_{\mu_1,p_1}f)^\wedge(t).$$

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But $C_{\mu_1,p_1}f$ is clearly also in $L_1(-\infty, \infty)$, and hence

$$(C_{\mu_1,p_1}f)\hat{}(t) = \int_{-\infty}^{\infty} e^{itu}(C_{\mu_1,p_1}f)(u)du = \int_{-\infty}^{\infty} e^{(\mu_1 u/p_1)+iut}f(e^u)du$$
$$= \int_0^{\infty} x^{(\mu_1/p_1+it-1)}f(x)dx = F((\mu_1/p_1) + it).$$

Hence, from [**8**, Theorem 48],

$$(C_{\mu_1,p_1}H_{m,\mu_1,p_1}f)(u) = \lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R} e^{-iut}m((\mu_1/p_1) + it)F((\mu_1/p_1) + it)dt,$$

the limit being in the topology of $L_2(-\infty, \infty)$. But then there is a sequence $\{R_j\}$, with $R_j \to \infty$ as $j \to \infty$, so that

$$(C_{\mu_1,p_1}H_{m,\mu_1,p_1}f)(u) = \lim_{j\to\infty}\frac{1}{2\pi}\int_{-R_j}^{R_j} e^{-iut}m((\mu_1/p_1) + it)F((\mu_1/p_1) + it)dt$$

a.e. on $(-\infty, \infty)$, or

$$(H_{\mu_1,p_1}f)(x) = \lim_{j\to\infty}\frac{1}{2\pi}\int_{-R_j}^{R_j} x^{-(\mu_1/p_1)-it}m((\mu_1/p_1) + it)F((\mu_1/p_1) + it)dt$$
$$= \lim_{j\to\infty}\frac{1}{2\pi i}\int_{(\mu_1/p_1)-iR_j}^{(\mu_1/p_1)+iR_j} x^{-s}m(s)F(s)ds,$$

a.e. on $(0, \infty)$.

Similarly

$$(C_{\mu_2,p_2}H_{\mu_2,p_2}f)(u) = \lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R} e^{-iut}m((\mu_2/p_2) + it)F((\mu_2/p_2) + it)dt$$
$$= \lim_{j\to\infty}\frac{1}{2\pi}\int_{-R_j}^{R_j} e^{-iut}m((\mu_2/p_2) + it)F((\mu_2/p_2) + it)dt,$$

the limits being in the topology of $L_2(-\infty, \infty)$. But then there is a subsequence $\{S_j\}$ of $\{R_j\}$ so that

$$(C_{\mu_2,p_2}H_{\mu_2,p_2}f)(u) = \lim_{j\to\infty}\frac{1}{2\pi}\int_{-S_j}^{S_j} e^{-iut}m((\mu_2/p_2) + it)F((\mu_2/p_2) + it)dt$$

almost everywhere on $(-\infty, \infty)$, or

$$(H_{\mu_2,p_2}f)(x) = \lim_{j\to\infty}\frac{1}{2\pi}\int_{-S_j}^{S_j} x^{-(\mu_2/p_2)-it}m((\mu_2/p_2) + it)F((\mu_2/p_2) + it)dt$$
$$= \lim_{j\to\infty}\frac{1}{2\pi i}\int_{(\mu_2/p_2)-iS_j}^{(\mu_2/p_2)+iS_j} x^{-s}m(s)F(s)ds,$$

a.e. on $(0, \infty)$.

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Hence, since $\{S_j\}$ is a subsequence of $\{R_j\}$, we have for almost all $x \in (0, \infty)$,

(3.3) $$(H_{m,\mu_1,p_1}f)(x) - (H_{m,\mu_2,p_2}f)(x) = \lim_{j\to\infty} \frac{1}{2\pi i}\left\{\int_{(\mu_1/p_1)-iS_j}^{(\mu_1/p_1)+iS_j} - \int_{(\mu_2/p_2)-iS_j}^{(\mu_2/p_2)+iS_j}\right\} x^{-s}m(s)F(s)ds.$$

If $\mu_1/p_1 = \mu_2/p_2$, the right hand side of this equation is zero, and

$$(H_{\mu_1,p_1}f) = (H_{\mu_2,p_2}f) \quad \text{a.e.}$$

If $\mu_1/p_1 \neq \mu_2/p_2$, let γ be the rectangle with vertices $(\mu_1/p_1) \pm iS_j$ and $(\mu_2/p_2) \pm iS_j$. Then since γ is contained in the strip $\alpha(m) < \operatorname{Re} s < \beta(m)$, m is analytic in this strip, and F is entire, we have for $x > 0$,

$$\int_\gamma x^{-s}m(s)F(s)ds = 0,$$

from which (3.3) can be written

(3.4) $$(H_{m,\mu_1,p_1}f)(x) - (H_{m,\mu_2,p_2}f)(x) = \lim_{j\to\infty}\frac{1}{2\pi i}\left\{\int_{\mu_1/p_1}^{\mu_2/p_2} x^{-\sigma-iS_j}m(\sigma+iS_j)F(\sigma+iS_j)d\sigma - \int_{\mu_1/p_1}^{\mu_2/p_2} x^{-\sigma+iS_j}m(\sigma-iS_j)F(\sigma-iS_j)d\sigma\right\},$$

almost everywhere.

But by the Riemann-Lebesgue lemma, $F(\sigma \pm iS_j) \to 0$ as $j \to \infty$; also

$$|F(\sigma \pm iS_j)| \leqq \int_0^\infty x^{\sigma-1}|f(x)|dx,$$

which is clearly bounded on the interval of integration since $f \in C_0$; further by Definition 3.1 (b), $|m(\sigma \pm iS_j)| \leqq K$, where K is a constant, for σ in the interval of integration; and $|x^{-\sigma \pm iS_j}| = x^{-\sigma}$ is clearly bounded on the interval of integration. Hence by the theorem of dominated convergence, the two integrals in (3.4) tend to zero as $j \to \infty$, and hence

$$H_{m,\mu_1,p_1}f = H_{m,\mu_2,p_2}f \quad \text{a.e.}$$

Now if $f \in L_{\mu_1,p_1} \cap L_{\mu_2,p_2}$, then by Lemma 2.1, there is a sequence $\{g_n\}$ of functions of C_0 so that $||f - g_n||_{\mu_i,p_i} \to 0$ as $n \to \infty$, $i = 1, 2$. But then, as $n \to \infty$,

$$||H_{m,\mu_1,p_1}f - H_{m,\mu_1,p_1}g_n||_{\mu_1,p_1} \to 0$$

and hence there is a subsequence $\{n_i\}$ such that

$$H_{m,\mu_1,p_1}f = \lim_{i\to\infty} H_{m,\mu_1,p_1}g_{n_i} \quad \text{a.e.}$$

However, as $i \to \infty$

$$||H_{m,\mu_2,p_2}f - H_{m,\mu_2,p_2}g_{n_i}||_{\mu_2,p_2} \to 0,$$

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so that there is a subsequence $\{n_i'\}$ of $\{n_i\}$ so that

$$H_{m,\mu_2,p_2}f = \lim_{i\to\infty} H_{m,\mu_2,p_2}g_{n_i'} \quad \text{a.e.}$$

Hence, for almost all x, since $g_{n_i'} \in C_0$,

$$\begin{aligned} H_{m,\mu_1,p_1}f &= \lim_{i\to\infty} H_{m,\mu_1,p_1}g_{n_i} = \lim_{i\to\infty} H_{m,\mu_1,p_1}g_{n_i'} \\ &= \lim_{i\to\infty} H_{m,\mu_2,p_2}g_{n_i'} = H_{m,\mu_2,p_2}f, \end{aligned}$$

as was to be proved.

In view of the last part of Lemma 3.2, it appears that $H_{m,\mu,p}$ is independent of μ and p, and so we will rename it.

Definition 3.2. *If* $m \in \mathscr{A}$, $1 < p < \infty$, $\alpha(m) < \mu/p < \beta(m)$, *we define* H_m *by*

$$H_m = C_{\mu,p}^{-1} T_{\mu/p} C_{\mu,p}.$$

The chief properties of H_m are summed up in the following theorem.

THEOREM 1. *If* $m \in \mathscr{A}$, *then for each* μ *and* p *such that* $1 < p < \infty$ *and* $\alpha(m) < \mu/p < \beta(m)$, $H_m \in [L_{\mu,p}]$. *If* $1 < p \leqq 2$, *and* $f \in L_{\mu,p}$,

$$(3.5) \qquad (\mathscr{M} H_m f)((\mu/p) + it) = m((\mu/p) + it)((\mathscr{M} f)((\mu/p) + it).$$

If $1 < p \leqq 2$, $\alpha(m) < \mu/p < \beta(m)$, H_m *is one-to-one on* $L_{\mu,p}$, *unless* $m \equiv 0$. *If* $m^{-1} \in \mathscr{A}$, *then for* $\max\ (\alpha(m), \alpha(m^{-1})) < \mu/p < \min\ (\beta(m), \beta(m^{-1}))$, $1 < p < \infty$, H_m *is a one-to-one mapping of* $L_{\mu,p}$ *onto itself, and*

$$(3.6) \qquad (H_m)^{-1} = H_{m^{-1}}.$$

Proof. This follows immediately from Lemma 3.1.

4. Transformations of $\mathscr{S}_\mu$ and $\mathscr{T}_\mu$. In this section we find necessary and sufficient conditions that transformations S and T be in $\mathscr{S}_\mu$ and $\mathscr{T}_\mu$ respectively, and equivalent forms of (1.2) and (1.3), that are easier to work with. The results are summed up in the following lemma.

LEMMA 4.1. (a) *A transformation* $S \in [L_{\mu,2}]$ *is in* $\mathscr{S}_\mu$ *if and only if there is a function* ω, *bounded a.e. on* $(-\infty, \infty)$, *so that for all* $f \in C_0$

$$(4.1) \qquad (\mathscr{M} Sf)(\tfrac{1}{2}\mu + it) = \omega(t)(\mathscr{M} f)(\tfrac{1}{2}\mu - it) \quad \text{a.e.}$$

When $S \in \mathscr{S}_\mu$, (4.1) *holds for all* $f \in L_{\mu,2}$. *Conversely, given* ω, *bounded a.e. on* $(-\infty, \infty)$, (4.1) *defines a transformation* $S \in \mathscr{S}_\mu$, *with kernel* $k \in L_{-\mu,2}$ *given by* $(\mathscr{M} k)(-\frac{1}{2}\mu + it) = \omega(t)/(\frac{1}{2} - it)$ *a.e.*

(b) *A transformation* $T \in [L_{\mu,2}]$ *is in* $\mathscr{T}_\mu$ *if and only if there is a function* ω, *bounded a.e. on* $(-\infty, \infty)$, *so that for all* $f \in C_0$

$$(4.2) \qquad (\mathscr{M} Tf)(\tfrac{1}{2}\mu + it) = \omega(t)(\mathscr{M} f)(\tfrac{1}{2}\mu + it) \quad \text{a.e.}$$

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When $T \in \mathscr{T}_\mu$, (4.2) *holds for all* $f \in L_{\mu,2}$. *Conversely, given* ω, *bounded a.e. on* $(-\infty, \infty)$, (4.2) *defines a transformation* $T \in \mathscr{T}_\mu$, *with kernel* $l \in L_{\mu-2,2}$ *given by* $(\mathscr{M} l)(\frac{1}{2}\mu - 1 + it) = \omega(t)/(\frac{1}{2} - it)$ *a.e.*

Proof. (a) is known when $\mu = 1$ (see Kober [**4**]), and by minor changes of variables, the $\mathscr{S}_\mu$ case can be changed to the $\mathscr{S}_1$ case. (b) follows from (a) once it is noticed that $T \in \mathscr{T}_1$ if and only if $TU \in \mathscr{S}_1$, where $(Uf)(x) = x^{-1}f(x^{-1})$.

5. Existence and extendability of $R_2^{-1}R_1$**.** The theorem below gives conditions under which $R_2^{-1}R_1$ exists and can be extended. Throughout the remainder of the paper we will suppose ω_1 and ω_2 are bounded a.e. on $(-\infty, \infty)$, and λ is a real number, and we let S_1 and S_2 be the transformations of $\mathscr{S}_\lambda$ associated with ω_1 and ω_2 respectively by (4.1), and let T_1 and T_2 be the transformations of $\mathscr{T}_\lambda$ associated with ω_1 and ω_2 respectively by (4.2).

THEOREM 2. *Suppose* ω_1 *and* ω_2 *are bounded a.e. on* $(-\infty, \infty)$ *and that there is an* $m \in \mathscr{A}$, *with* $\alpha(m) < \frac{1}{2}\lambda < \beta(m)$, *so that* $m(\frac{1}{2}\lambda + it) = \omega_1(t)/\omega_2(t)$ *a.e. Then* $S_2^{-1}S_1$ *and* $T_2^{-1}T_1$ *exist and belong to* $[L_{\lambda,2}]$, *and* $S_2^{-1}S_1$ *can be extended to* $L_{\mu,p}$, *uniquely as an element of* $[L_{\mu,p}]$, *for all* μ *and* p *satisfying* $1 < p < \infty$, $(\lambda - \beta(m)) < \mu/p < (\lambda - \alpha(m))$, *while* $T_2^{-1}T_1$ *can be extended to* $L_{\mu,p}$, *uniquely as an element of* $[L_{\mu,p}]$, *for all* μ *and* p *satisfying* $1 < p < \infty, \alpha(m) < \mu/p < \beta(m)$. *If, in addition,* $1 < p \leqq 2$, *the extended operators are one-to-one.*

If also $m^{-1} \in \mathscr{A}$, *then* $S_2^{-1}S_1$ *is a one-to-one mapping of* $L_{\mu,p}$ *onto itself if* $1 < p < \infty$,

$$\max((\lambda - \beta(m)), (\lambda - \beta(m^{-1}))) < \mu/p < \min((\lambda - \alpha(m)), (\lambda - \alpha(m^{-1}))),$$

while $T_2^{-1}T_1$ *is a one-to-one mapping of* $L_{\mu,p}$ *onto itself if* $1 < p < \infty$,

$$\max(\alpha(m), \alpha(m^{-1})) < \mu/p < \min(\beta(m), \beta(m^{-1})).$$

Proof. Since $\omega_1(t)/\omega_2(t)$ is defined a.e., $\omega_2(t) \neq 0$ a.e., and hence if $S_2 f = 0$ a.e., then from (4.1) $(\mathscr{M} f)(\frac{1}{2}\lambda - it) = 0$ a.e., and $f = 0$ a.e., and thus S_2^{-1} exists. Similarly T_2^{-1} exists.

To show $S_2^{-1}S_1$ exists, we must show that the range of S_1 is a subset of the range of S_2; this is equivalent to showing that if $f \in L_{\lambda,2}$, then there is a $g \in L_{\lambda,2}$ so that $S_2 g = S_1 f$. But since

$$m(\tfrac{1}{2}\lambda + it) = \omega_1(t)/\omega_2(t) \quad \text{a.e.}$$

and, from Definition 3.1 $m(\frac{1}{2}\lambda + it)$ is bounded, it follows that

$$\omega_1/\omega_2 \in L_\infty(-\infty, \infty).$$

Hence, since the Mellin transformation is a unitary mapping of $L_{\lambda,2}$ onto $L_2(-\infty, \infty)$, there is a $g \in L_{\lambda,2}$ so that

$$(\mathscr{M} g)(\tfrac{1}{2}\lambda + it) = (\omega_1(-t)/\omega_2(-t))(\mathscr{M} f)(\tfrac{1}{2}\lambda + it) \quad \text{a.e.}$$

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But then, from (4.1), for almost all t

$$\begin{aligned}(\mathcal{M} S_2 g)(\tfrac{1}{2}\lambda + it) &= \omega_2(t)(\mathcal{M} g)(\tfrac{1}{2}\lambda - it) = \omega_2(t)(\omega_1(t)/\omega_2(t))(\mathcal{M} f)(\tfrac{1}{2}\lambda - it) \\ &= \omega_1(t)(\mathcal{M} f)(\tfrac{1}{2}\lambda - it) = (\mathcal{M} S_1 f)(\tfrac{1}{2}\lambda + it),\end{aligned}$$

and $S_2 g = S_1 f$ a.e., so that $S_2^{-1}S_1$ exists. Also

$$||S_2^{-1}S_1 f||_{\lambda,2} = ||g||_{\lambda,2} = ||\mathcal{M} g||_2 \leqq K||\mathcal{M} f||_2 = K||f||_{\lambda,2},$$

where K is an essential upper bound for ω_1/ω_2, and $S_2^{-1}S_1 \in [L_{\lambda,2}]$.

Similarly, if we define h by

$$(\mathcal{M} h)(\tfrac{1}{2}\lambda + it) = (\omega_1(t)/\omega_2(t))(\mathcal{M} f)(\tfrac{1}{2}\lambda + it) \quad \text{a.e.},$$

then from (4.2), for almost all t

$$\begin{aligned}(\mathcal{M} T_2 h)(\tfrac{1}{2}\lambda + it) &= \omega_2(t)(\mathcal{M} h)(\tfrac{1}{2}\lambda + it) \\ &= \omega_2(t)(\omega_1(t)/\omega_2(t))(\mathcal{M} f)(\tfrac{1}{2}\lambda + it) \\ &= \omega_1(t)(\mathcal{M} f)(\tfrac{1}{2}\lambda + it) = (\mathcal{M} T_1 f)(\tfrac{1}{2}\lambda + it),\end{aligned}$$

and $T_2 h = T_1 f$ a.e., so that $T_2^{-1}T_1$ exists. Also

$$||T_2^{-1}T_1 f||_{\lambda,2} = ||h||_{\lambda,2} = ||\mathcal{M} h||_2 \leqq K||\mathcal{M} f||_2 = K||f||_{\lambda,2},$$

and $T_2^{-1}T_1 \in [L_{\lambda,2}]$.

Let $\tilde{m}(s) = m(\lambda - s)$; clearly $\tilde{m} \in \mathcal{A}$, $\alpha(\tilde{m}) = \lambda - \beta(m)$, and $\beta(\tilde{m}) = \lambda - \alpha(m)$. Hence from Theorem 1, $H_{\tilde{m}} \in [L_{\mu,p}]$ if $1 < p < \infty$, $(\lambda - \beta(m)) < \mu/p < (\lambda - \alpha(m))$. Note that $(\lambda - \beta(m)) < \frac{1}{2}\lambda < (\lambda - \alpha(m))$, and hence if $f \in L_{\lambda,2}$, then from (4.1) and (3.5), for almost all t

$$\begin{aligned}(\mathcal{M} S_2 H_{\tilde{m}} f)(\tfrac{1}{2}\lambda + it) &= \omega_2(t)(\mathcal{M} H_{\tilde{m}} f)(\tfrac{1}{2}\lambda - it) \\ &= \omega_2(t)\tilde{m}(\tfrac{1}{2}\lambda - it)(\mathcal{M} f)(\tfrac{1}{2}\lambda - it) \\ &= \omega_2(t)m(\tfrac{1}{2}\lambda + it)(\mathcal{M} f)(\tfrac{1}{2}\lambda - it) \\ &= \omega_2(t)(\omega_1(t)/\omega_2(t))(\mathcal{M} f)(\tfrac{1}{2}\lambda - it) \\ &= \omega_1(t)(\mathcal{M} f)(\tfrac{1}{2}\lambda - it) \\ &= (\mathcal{M} S_1 f)(\tfrac{1}{2}\lambda + it),\end{aligned}$$

so that $S_2 H_{\tilde{m}} f = S_1 f$ a.e., $S_2 H_{\tilde{m}} = S_1$ on $L_{\lambda,2}$, and $H_{\tilde{m}} = S_2^{-1}S_1$ on $L_{\lambda,2}$.

Hence we can extend $S_2^{-1}S_1$ to $L_{\mu,p}$, if $1 < p < \infty$, $(\lambda - \beta(m)) < \mu/p < (\lambda - \alpha(m))$, by defining it to be $H_{\tilde{m}}$, and then $S_2^{-1}S_1 \in [L_{\mu,p}]$. This extension will be unique as an element of $[L_{\mu,p}]$, for it coincides with $S_2^{-1}S_1$ on $L_{\mu,p} \cap L_{\lambda,2}$, and this set is dense in $L_{\mu,p}$, since it contains C_0. The remaining statements about $S_2^{-1}S_1$ in the statement of Theorem 2 are just paraphrases of statements about $H_{\tilde{m}}$ in Theorem 1.

In a similar way $H_m = T_2^{-1}T_1$ on $L_{\lambda,2}$, and thus we can extend $T_2^{-1}T_1$ to $L_{\mu,p}$, if $1 < p < \infty$, $\alpha(m) < \mu/p < \beta(m)$, by defining it to be H_m, and then $T_2^{-1}T_1 \in [L_{\mu,p}]$; this is the unique extension as an element of $[L_{\mu,p}]$; and the remaining statements about $T_2^{-1}T_1$ are paraphrases of those about H_m in Theorem 1.

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6. Extension and range of R_1. In many cases R_2 can be extended from $L_{\lambda,2}$ to other spaces $L_{\mu,p}$ for a collection P of pairs (p, μ), as a bounded operator from $L_{\mu,p}$ to $L_{\nu,q}$ for a range of values of (q, ν), depending on (p, μ). In our next theorem, we show that when this is so, and the hypotheses of Theorem 2 are satisfied, it may be possible to extend R_1, and that then there is a relation between the range of R_1 and that of R_2.

THEOREM 3. *Suppose that ω_1 and ω_2 are bounded a.e., and that there is an $m \in \mathscr{A}$ with $\alpha(m) < \frac{1}{2}\lambda < \beta(m)$, so that $m(\frac{1}{2}\lambda + it) = \omega_1(t)/\omega_2(t)$ a.e. Then*

(a) *if S_2 can be extended to $L_{\mu,p}$ for a collection P of pairs (p, μ), as an element of $[L_{\mu,p}, L_{\nu,q}]$, for a range of values of (q, ν) depending on (p, μ), then for all μ and p so that $(p, \mu) \in P$, $1 < p < \infty$, $(\lambda - \beta(m)) < \mu/p < (\lambda - \alpha(m))$, S_1 can be extended to $L_{\mu,p}$, uniquely as an element of $[L_{\mu,p}, L_{\nu,q}]$, and for such μ and p, $S_1(L_{\mu,p}) \subseteq S_2(L_{\mu,p})$;*

(b) *if T_2 can be extended to $L_{\mu,p}$ for a collection P of pairs (p, μ), as an element of $[L_{\mu,p}, L_{\nu,q}]$, for a range of values of (q, ν) depending on (p, μ), then for all μ and p so that $(p, \mu) \in P$, $1 < p < \infty$, $\alpha(m) < \mu/p < \beta(m)$, T_1 can be extended to $L_{\mu,p}$, uniquely as an element of $[L_{\mu,p}, L_{\nu,q}]$, and for such μ and p, $T_1(L_{\mu,p}) \subseteq T_2(L_{\mu,p})$.*

Further (c) *if $m^{-1} \in \mathscr{A}$, then for $(p, \mu) \in P$, $1 < p < \infty$,*

$$\max((\lambda - \beta(m)), (\lambda - \beta(m^{-1}))) < \mu/p < \min((\lambda - \alpha(m)), (\lambda - \alpha(m^{-1}))),$$

$S_1(L_{\mu,p}) = S_2(L_{\mu,p})$, and for $(p, \mu) \in P$, $1 < p < \infty$,

$$\max(\alpha(m), \alpha(m^{-1})) < \mu/p < \min(\beta(m), \beta(m^{-1})),\ T_1(L_{\mu,p}) = T_2(L_{\mu,p}).$$

Proof. We shall only prove (a) and that part of (c) referring to S_1 and S_2, the proof of (b) and the rest of (c) being similar.

We extend S_1 by defining it to be $S_2(S_2^{-1}S_1)$. Since by Theorem 2, for the indicated values of p and μ, $S_2^{-1}S_1 \in [L_{\mu,p}]$ and by hypothesis $S_2 \in [L_{\mu,p}, L_{\nu,q}]$, then $S_1 \in [L_{\mu,p}, L_{\nu,q}]$, and it is the unique such extension, since it coincides with S_1 on $L_{\mu,p} \cap L_{\lambda,2}$, and this set is dense in $L_{\mu,p}$ since it contains C_0.

To show $S_1(L_{\mu,p}) \subseteq S_2(L_{\mu,p})$, we must show that if $f \in L_{\mu,p}$, there is a $g \in L_{\mu,p}$, so that $S_2 g = S_1 f$. But if we let $g = S_2^{-1}S_1 f$, then $S_2 g = S_1 f$.

To show (c) for S_1 and S_2, it is enough to notice that under the hypotheses of (c), the general hypothesis of the theorem is true with ω_1 and ω_2 interchanged, if m is replaced by m^{-1}, and the hypotheses of (a) are true with S_1 and S_2 interchanged, if P is replaced by $Q = \{(p, \mu) | (p, \mu) \in P, 1 < p < \infty, (\lambda - \beta(m)) < \mu/p < (\lambda - \alpha(m))\}$, and the conclusion of (c) follows.

7. Applications. We shall give two applications of our results, the first when R_1 and R_2 are in $\mathscr{S}_\mu$ for a particular μ, and the second when they are in $\mathscr{T}_\mu$.

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For our first application let $\eta > -1$, and let H_η be the Hankel transformation; that is if $f \in C_0$

$$(H_\eta f)(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\eta(xt) f(t) dt,$$

and for $0 \leqq \zeta < \eta + 1$, let $(H_{\eta,\zeta} f)(x) = x^{-\zeta}(H_\eta F)(x)$, where $F(t) = t^{-\zeta} f(t)$; that is

$$(H_{\eta,\zeta} f)(x) = \int_0^\infty (xt)^{\frac{1}{2}-\zeta} J_\eta(xt) f(t) dt.$$

If we integrate both sides of this equation from zero to x, it has the form (1.2) for $\mu = 1$, with kernel

$$k_{\eta,\zeta}(x) = \int_0^x t^{\frac{1}{2}-\zeta} J_\eta(t) dt.$$

H_η is studied in [**8**, Chapter 8, §§4 and 5], and by minor changes of variables in the results of those sections, it is easy to show that $k_{\eta,\zeta} \in L_{-1,2}$ and

$$(\mathscr{M} k_{\eta,\zeta})(-\tfrac{1}{2} + it) = \omega_{\eta,\zeta}(t)/(\tfrac{1}{2} - it)$$

where

$$\omega_{\eta,\zeta}(t) = 2^{it-\zeta}\Gamma(\tfrac{1}{2}(\eta - \zeta + 1 + it))/\Gamma(\tfrac{1}{2}(\eta + \zeta + 1 - it)).$$

But from [**2**, 1.18(6)],

$$|\Gamma(x + iy)| \sim (2\pi)^{\frac{1}{2}} |y|^{x-\frac{1}{2}} e^{-\pi|y|/2}, \tag{7.1}$$

uniformly in x for x in any finite interval, and thus

$$|\omega_{\eta,\zeta}(t)| \sim |2t|^{-\zeta} \quad \text{as} \quad |t| \to \infty.$$

Hence since $0 \leqq \zeta < \eta + 1$, $\omega_{\eta,\zeta}$ is bounded a.e., and $H_{\eta,\zeta} \in \mathscr{S}_1$.

We shall take $H_{\eta,\zeta}$ as S_1 in Theorems 2 and 3, and for S_2 we shall take the Fourier cosine transformation $\mathscr{F}_c = H_{-\frac{1}{2}}$. Since both transformations are in $\mathscr{S}_1$, we must find a function $m \in \mathscr{A}$, with $\alpha(m) < \frac{1}{2} < \beta(m)$, so that

$$m(\tfrac{1}{2} + it) = \omega_{\eta,\zeta}(t)/\omega_{-\frac{1}{2},0}(t) = 2^{-\zeta}(\Gamma(\tfrac{1}{2}(\tfrac{1}{2} - it)) \cdot \Gamma(\tfrac{1}{2}(\eta - \zeta + 1 + it)))/(\Gamma(\tfrac{1}{2}(\tfrac{1}{2} + it))\Gamma(\tfrac{1}{2}(\eta + \zeta + 1 - it))).$$

An analytic function with the right value at $\frac{1}{2} + it$ is

$$m_{\eta,\zeta}(s) = 2^{-\zeta}(\Gamma(\tfrac{1}{2}(1 - s))\Gamma(\tfrac{1}{2}(\eta - \zeta + \tfrac{1}{2} + s)))/(\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}(\eta + \zeta + \tfrac{3}{2} - s))).$$

$m_{\eta,\zeta}(s)$ is analytic in the strip $\zeta - \eta - \frac{1}{2} < \operatorname{Re} s < 1$, and if $\zeta - \eta - \frac{1}{2} < \sigma_1 \leqq \sigma_2 < 1$, then from (5.1), uniformly in σ for $\sigma_1 \leqq \sigma \leqq \sigma_2$, $|m_{\eta,\zeta}(\sigma + it)| \sim |2t|^{-\zeta}$ as $|t| \to \infty$. Hence since $\zeta \geqq 0$, $m_{\eta,\zeta}(s)$ is bounded in the strip $\sigma_1 \leqq \operatorname{Re} s \leqq s_2$. Also

$$m'_{\eta,\zeta}(s) = \tfrac{1}{2} m_{\eta,\zeta}(s)\{\psi(\tfrac{1}{2}(\eta - \zeta + \tfrac{1}{2} + s)) - \psi(\tfrac{1}{2}(1 - s)) + \psi(\tfrac{1}{2}(\eta + \zeta + \tfrac{3}{2} - s)) - \psi(\tfrac{1}{2}s)\},$$

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where $\psi(s) = \Gamma'(s)/\Gamma(s)$. But from [**2**, 1.18(7)],

$$\psi(z) = \log z + (2z)^{-1} + O(|z|^{-2}) \quad \text{as} \quad z \to \infty \text{ in } |\arg z| \leqq \pi - \delta.$$

Hence as $|y| \to \infty$,

$$\begin{aligned}(7.2) \quad \psi(x+iy) &= \log (x+iy) + (2(x+iy))^{-1} + O(|x+iy|^{-2}) \\ &= \log iy - i((x-\tfrac{1}{2})/y) + O(y^{-2}),\end{aligned}$$

and thus as $|t| \to \infty$

$$m'_{\eta,\zeta}(\sigma+it) = m(\sigma+it)\{(-i\zeta/t) + O(t^{-2})\},$$

so that $m(\sigma+it) = O(|t|^{-1})$, as $|t| \to \infty$ for $\zeta - \eta - \frac{1}{2} < \sigma < 1$. Thus $m_{\eta,\zeta} \in \mathscr{A}$, with $\alpha(m_{\eta,\zeta}) = \zeta - \eta - \frac{1}{2}$, $\beta(m_{\eta,\zeta}) = 1$, so that $\alpha(m_{\eta,\zeta}) < \frac{1}{2} < \beta(m_{\eta,\zeta})$.

Hence by Theorem 2, since $\mathscr{F}_c^{-1} = \mathscr{F}_c, \mathscr{F}_c H_{\eta,\zeta}$ can be extended to $L_{\mu,p}$ as an element of $[L_{\mu,p}]$ for all μ and p such that $1 < p < \infty, 0 < \mu/p < \eta - \zeta + \frac{3}{2}$, $0 \leqq \zeta < \eta + 1$, and is one-to-one if $1 < p \leqq 2$. Clearly $(m_{\eta,0})^{-1} \in \mathscr{A}$, $\alpha((m_{\eta,0})^{-1}) = 0$, $\beta((m_{\eta,0})^{-1}) = \eta + \frac{3}{2}$, and hence if $1 < p < \infty$,

$$\max (0, -\eta - \tfrac{1}{2}) < \mu/p < \min (1, \eta + \tfrac{3}{2}),$$

$\mathscr{F}_c H_\eta$ is a bounded one-to-one mapping of $L_{\mu,p}$ onto itself.

Now by [**5**, Theorem 1], if $1 < p < \infty$, $\max (p^{-1}, p'^{-1}) \leqq \mu < 1$, $p \leqq q \leqq 1/(1-\mu)$, then $\mathscr{F}_c \in [L_{p\mu,p}, L_{q(1-\mu),q}]$. Hence from Theorem 3, if $1 < p < \infty$, $\max (p^{-1}, p'^{-1}) \leqq \mu < 1$, $p \leqq q \leqq 1/(1-\mu)$, $\mu < \eta - \zeta + \frac{3}{2}$ and $0 \leqq \zeta < \eta + 1$, $H_{\eta,\zeta} \in [L_{p\mu,p}, L_{q(1-\mu),q}]$, or writing this in terms of H_η, $H_\eta \in [L_{p(\mu+\zeta),p}, L_{q(1-\mu-\zeta),q}]$.

But, if $1 < p \leqq q < \infty$, $\max (p^{-1}, q'^{-1}) \leqq \nu < \eta + \frac{3}{2}$, there are numbers μ and ζ with $\max (p^{-1}, p'^{-1}) \leqq \mu < 1$, $q \leqq 1/(1-\mu)$, $\mu < \eta - \zeta + \frac{3}{2}$ and $0 \leqq \zeta < \eta + 1$, so that $\nu = \mu + \zeta$. To see this, note first that if we define $\zeta(\mu) = \nu - \mu$, for $\max (p^{-1}, q'^{-1}) \leqq \mu < 1$ then the range of ζ is $(\nu - 1, \nu - \max (p^{-1}, q'^{-1})]$, and this intersects $[0, \eta + \frac{1}{2})$, since $\nu - 1 < \eta + \frac{1}{2}$, and $\nu - \max (p^{-1}, q'^{-1}) \geqq 0$. Hence letting ζ be any point of this intersection, and $\mu = \nu - \zeta$, we have $\max (p^{-1}, p'^{-1}) \leqq \max (p^{-1}, q'^{-1}) \leqq \mu < 1$, $q \leqq 1/(1-\mu)$ since $\mu \geqq q'^{-1}$, $\mu = \nu - \zeta < \eta - \zeta + \frac{3}{2}$, $0 \leqq \zeta < \eta + \frac{1}{2} < \eta + 1$, and $\nu = \mu + \zeta$.

Hence we have shown that if $1 < p \leqq q < \infty$, $\max (p^{-1}, q'^{-1}) \leqq \nu < \eta + \frac{3}{2}$, $H_\eta \in [L_{p\nu,p}, L_{q(1-\nu),q}]$.

If we take $1 < p \leqq 2$, $\mu = 1/p$, $q = p'$, this result becomes $H_\eta \in [L_p, L_{p'}]$ if $p > (\eta + \frac{3}{2})^{-1}$, which is well-known if $\eta \geqq -\frac{1}{2}$, see [**1**], since then $(\eta + \frac{3}{2})^{-1} < 1$, but is less well-known if $-1 < \eta < -\frac{1}{2}$.

Also, from Theorem 3, $H_{\eta,\zeta}(L_{p\mu,p}) \subseteq \mathscr{F}_c(L_{p\mu,p})$ if $1 < p < \infty$, $\max (p^{-1}, p'^{-1}) \leqq \mu < 1$, and $\mu < \eta - \zeta + \frac{3}{2}$, and since $(m_{\eta,0})^{-1} \in \mathscr{A}$, $H_\eta(L_{p\mu,p}) = \mathscr{F}_c(L_{p\mu,p})$ if $1 < p < \infty$, $\max (p^{-1}, p'^{-1}) \leqq \mu < 1$ and $\mu < \eta + \frac{3}{2}$.

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For our second application let

$$(7.3) \qquad (I_{\nu,\alpha,\xi}f)(x) = \frac{\nu x^{-\nu(\xi+\alpha-1)}}{\Gamma(\alpha)} \int_0^x (x^\nu - t^\nu)^{\alpha-1} t^{\nu\xi-1} f(t)dt,$$

where $\mathrm{Re}\,\alpha > 0$, $\nu > 0$, and ξ is a complex number. It is well-known that $I_{\nu,\alpha,\xi} \in [L_{\mu,p}]$ if $1 \leqq p < \infty$, $\mu/p < \nu \mathrm{Re}\,\xi$; see [**6**, Corollary 3.1]. From [**6**, Corollary 4.1], if $\frac{1}{2}\lambda < \nu\,\mathrm{Re}\,\xi$, $f \in L_{\lambda,2}$,

$$(\mathscr{M} I_{\nu,\alpha,\xi} f)(\tfrac{1}{2}\lambda + it) = \omega_{\nu,\alpha,\xi}(t)(\mathscr{M} f)(\tfrac{1}{2}\lambda + it)$$

where

$$\omega_{\nu,\alpha,\xi}(t) = \Gamma(\xi - ((\tfrac{1}{2}\lambda + it)/\nu))/\Gamma(\xi + \alpha - ((\tfrac{1}{2}\lambda + it)/\nu)).$$

But from (5.1)

$$|\omega_{\nu,\alpha,\xi}(t)| \sim |t|^{-\mathrm{Re}\alpha} \quad \text{as} \quad |t| \to \infty,$$

and hence since $\frac{1}{2}\lambda < \nu\,\mathrm{Re}\,\xi$, ω is bounded a.e., and $I_{\nu,\alpha,\xi} \in \mathscr{T}_\lambda$.

We shall take $T_1 = I_{\nu_1,\alpha_1,\xi_1}$, $T_2 = I_{\nu_2,\alpha_2,\xi_2}$ in Theorems 2 and 3. Transformations of the form $T_2^{-1}T_1$ have been considered by Erdélyi [**3**]. Since both transformations are in $\mathscr{T}_\lambda$, if $\frac{1}{2}\lambda < \min(\nu_1\,\mathrm{Re}\,\xi_1, \nu_2\,\mathrm{Re}\,\xi_2)$, we must find $m \in \mathscr{A}$ so that

$$(7.4) \qquad m(\tfrac{1}{2}\lambda + it) = \omega_{\nu_1,\alpha_1,\xi_1}(t)/\omega_{\nu_2,\alpha_2,\xi_2}(t) \quad \text{a.e.}$$

Clearly an analytic function satisfying (7.4) is

$$m(s) = (\Gamma(\xi_1 - (s/\nu_1))\Gamma(\xi_2 + \alpha_2 - (s/\nu_2)))/ \\ (\Gamma(\xi_1 + \alpha_1 - (s/\nu_1))\Gamma(\xi_2 - (s/\nu_2))),$$

which is analytic for $-\infty < \mathrm{Re}\,s < \min(\nu_1\,\mathrm{Re}\,\xi_1, \nu_2\,\mathrm{Re}\,(\xi_2 + \alpha_2))$. Since from (7.1) and (7.2),

$$|m(\sigma + it)| \sim |t|^{\mathrm{Re}(\alpha_2 - \alpha_1)}, \text{ and}$$

$$|m'(\sigma + it)| \sim |m(\sigma + it)|\{(\mathrm{Re}(\alpha_2 - \alpha_1)/t) + O(t^{-2})\},$$

if $\mathrm{Re}\,\alpha_2 \leqq \mathrm{Re}\,\alpha_1$, $m \in \mathscr{A}$ with $\alpha(m) = -\infty$, $\beta(m) = \min(\nu_1\,\mathrm{Re}\,\xi_1, \nu_2\,\mathrm{Re}\,(\xi_2 + \alpha_2))$. Also $m^{-1} \in \mathscr{A}$, if $\mathrm{Re}\,\alpha_1 = \mathrm{Re}\,\alpha_2$, with $\alpha(m^{-1}) = -\infty$, $\beta(m^{-1}) = \min(\nu_1\,\mathrm{Re}\,(\xi_1 + \alpha_1), \nu_2\,\mathrm{Re}\,\xi_2)$. Thus from Theorems 2 and 3, and using [**6**, Lemma 3.4] it follows that if $\mathrm{Re}\,\alpha_2 \leqq \mathrm{Re}\,\alpha_1$, $(I_{\nu_2,\alpha_2,\xi_2})^{-1} I_{\nu_1,\alpha_1,\xi_1}$ exists and belongs to $[L_{\mu,p}]$ if $1 < p < \infty$, $\mu/p < \min(\nu_1\,\mathrm{Re}\,\xi_1, \nu_2\,\mathrm{Re}\,\xi_2)$, and can be extended to $L_{\mu,p}$ as an element of $[L_{\mu,p}]$ if $\mu/p < \min(\nu_1\,\mathrm{Re}\,\xi_1, \nu_2\,\mathrm{Re}\,(\xi_2 + \alpha_2))$. It is one-to-one if $1 < p \leqq 2$ or $\mu/p < \min(\nu_1\,\mathrm{Re}\,\xi_1, \nu_2\,\mathrm{Re}\,\xi_2)$ and onto if $\mathrm{Re}\,\alpha_1 = \mathrm{Re}\,\alpha_2$ and $\mu/p < \min(\nu_1\,\mathrm{Re}\,\xi_1, \nu_2\,\mathrm{Re}\,\xi_2)$.

Further, if $\mu/p < \min(\nu_1\,\mathrm{Re}\,\xi_1, \nu_2\,\mathrm{Re}\,\xi_2)$, $\mathrm{Re}\,\alpha_1 \leqq \mathrm{Re}\,\alpha_2$,

$$I_{\nu_2,\alpha_2,\xi_2}(L_{\mu,p}) \subseteq I_{\nu_1,\alpha_1,\xi_1}(L_{\mu,p}),$$

with equality if $\mathrm{Re}\,\alpha_1 = \mathrm{Re}\,\alpha_2$.

https://doi.org/10.4153/CJM-1973-116-9 Published online by Cambridge University Press

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https://doi.org/10.4153/CJM-1973-116-9 Published online by Cambridge University Press