A SPECTRAL SEQUENCE FOR COHOMOTOPY

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1. Introduction. For a prime number p let $\mathfrak{C}(\hat{p})$ be the class of finite abelian groups whose orders are prime to p. For a finitely generated abelian group G, let G_p be the sum of the free and p-primary components of G. Our aim in this paper is to prove the following theorem.

THEOREM. *Suppose that*
(i) $H^i(X;Z) = 0$ *for* $i > k$,
(ii) $H^i(X;Z) \in \mathfrak{C}(\hat{p})$ *for* $i > k - d$.
Then there exists a spectral sequence with

$$E_1^{r,0} = H^{k+r}(X;Z)_p,$$
$$E_1^{r,q} = H^{k+r+2q(p-1)-1}(X;Z_p) \quad \text{for } q > 0,$$
$$E_1^{r,q} = 0 \quad \text{for } q < 0;$$

and the differential $d_1^{r,q}: E_1^{r,q} \to E_1^{r+1,q+1}$ *is given by*

$$d_1^{r,0} = P^1: H^{k+r}(X;Z)_p \to H^{k+r+2p-2}(X;Z_p),$$
$$d_1^{r,q} = (q+1)P^1\delta - q\delta P^1: H^{k+r+2q(p-1)-1}(X;Z_p) \to H^{k+r+2(q+1)(p-1)}(X;Z_p).$$

When $0 \leqq -r \leqq d + 2p(p-1) - 2$, *the spectral sequence converges to* $\Sigma^{k+r}(X)_p = (\{X, S^{k+r}\}_p)$. *That is, there exists a filtration*

$$\Sigma^{k+r}(X)_p = A^{r,0} \supset A^{r,1} \supset \ldots \supset A^{r,p-1} \supset A^{r,p} = 0,$$

where

$$E_\infty^{r,q} = A^{r,q}/A^{r,q+1}.$$

When $-r > d + 2p(p-1) - 2$, then $A^{r,0}$ does not give much information about $\Sigma^{k+r}(X)_p$.

Turning this result around, suppose given a space X of dimension k, then we can compute (up to extension) $\Sigma^n(X)_p$ if $k - n \leqq d + 2p(p-1) - 2$. (Here d is a function of p.) Putting these results together for different p, we can compute $\Sigma^n(X)$ up to an "indeterminacy" corresponding to primes less than $\frac{1}{2}(3 + (5 + 2(k-n))^{\frac{1}{2}})$. This is roughly the square root of the indeterminacy given in (**5**, p. 702, Theorem).

The Theorem also gives some information about stable homotopy groups. If $H_i(X;Z) = 0$ for $i < n$ and $H_i(X;Z) \in \mathfrak{C}(\hat{p})$ for $i < n + d$, the dual

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sequence (involving homology groups and the dual operations) will converge to $\Sigma_{n+r}(X)_p$ when $r \leqq d + 2p(p-1) - 2$. The idea of a homotopy (mod $\mathfrak{C}$) reduction of a space is introduced and the (mod $\mathfrak{C}(\hat{p})$) reduction of an appropriate space in the Postnikov system of S^n is studied. Aside from Theorem 1, the results in this paper rely heavily on the work of Toda (**16**) and of Adem (**2**). The construction of the spectral sequence is based on (**15**).

The theorems in this paper have no content when $p = 2$. Therefore, we will assume throughout that $p \neq 2$. We will also assume that all the spaces discussed are simply connected.

2. The homotopy (mod $\mathfrak{C}$) reduction.

Definition. Let $\mathfrak{C}$ be a class of finite abelian groups. For a finitely generated abelian group G, $\bar{G}$ is the quotient group of G determined by adding the relation $a = 0$ for every $a \in G$ such that the cyclic group generated by a is in $\mathfrak{C}$. When the class $\mathfrak{C}$ is $\mathfrak{C}(\hat{p})$, we will write G_p for $\bar{G}$. Furthermore, write $G_{\mathfrak{C}}$ for the largest subgroup of G which is in $\mathfrak{C}$.

Definition. $\bar{X}$ is a homotopy (mod $\mathfrak{C}$) reduction of X if $\pi_i(\bar{X}) = \overline{\pi_i(X)}$ and there exists a weak homotopy equivalence (whe) (mod $\mathfrak{C}$) $f: X \to \bar{X}$ such that $f_*: \pi_n(X) \to \pi_n(\bar{X})$ is the canonical projection when $\pi_n(X)$ is the first non-trivial homotopy group of X.

Note that $f_*: \pi_i(X) \to \pi_i(\bar{X})$ does not have to be the canonical projection when $i > n$.

THEOREM 1. *If X has only a finite number of non-trivial homotopy groups, then it has a homotopy* (mod $\mathfrak{C}$) *reduction, and any two such reductions are of the same weak homotopy type* (mod $\mathfrak{C}$).

That any two reductions have the same weak homotopy type mod $\mathfrak{C}$ follows directly from their definition; cf. (**5**). The existence part requires two lemmas.

LEMMA 1. *If the diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{k_1} & Y_1 \\ f\downarrow & & \downarrow g \\ X_2 & \xrightarrow{k_2} & Y_2 \end{array}$$

is homotopy commutative and if f and g are weak homotopy equivalences (mod $\mathfrak{C}$), *then there exists a map*

$$h: E_{k_1} \to E_{k_2}$$

which is a whe (mod $\mathfrak{C}$).

Proof. Let $a_1 = gk_1: X_1 \to Y_2$ and $a_2 = k_2 f: X_1 \to Y_2$. Then a_1 is homotopic to a_2 and there exists a homotopy equivalence

$$b: E_{a_1} \to E_{a_2}.$$

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Now the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{k_1} & Y_1 \\ & \searrow^{a_1} & \downarrow g \\ & & Y_2 \end{array}$$

is commutative, hence there exists a map

$$h_1 \colon E_{k_1} \to E_{a_1}$$

which commutes with the projection onto X_1. This gives rise to the following commutative diagram

$$\begin{array}{ccccccccc} \pi_{i+1}(X_1) & \xrightarrow{k_{1*}} & \pi_{i+1}(Y_1) & \to & \pi_i(E_{k_1}) & \to & \pi_i(X_1) & \xrightarrow{k_{1*}} & \pi_i(Y_1) \\ \| & & \downarrow g_* & & \downarrow h_{1*} & & \| & & \downarrow g_* \\ \pi_{i+i}(X_1) & \xrightarrow{a_{1*}} & \pi_{i+i}(Y_2) & \to & \pi_i(E_{a_1}) & \to & \pi_i(X_1) & \xrightarrow{a_{1*}} & \pi_i(Y_2) \end{array}$$

where the rows are exact and the four outside vertical maps are isomorphisms (mod $\mathfrak{C}$). Therefore,

$$h_{1*} \colon \pi_i(E_{k_1}) \to \pi_i(E_{a_1})$$

is a (mod $\mathfrak{C}$) isomorphism for all i and h_1 is a whe (mod $\mathfrak{C}$).

Similarly, it can be shown that there exists a whe (mod $\mathfrak{C}$) $h_2 \colon E_{a_2} \to E_{k_2}$, and hence the composition

$$h = h_2 b h_1 \colon E_{k_1} \to E_{k_2}$$

is a whe (mod $\mathfrak{C}$).

LEMMA 2. *Suppose that* $\pi_n(X) = \pi_{n-1}(X) = 0$ *and* $f\colon X \to \bar{X}$ *is a homotopy* (mod $\mathfrak{C}$) *reduction of* X. *Let* $k\colon X \to K(G, n)$ *be any map. Then there exists a map* $\bar{k}\colon \bar{X} \to K(\bar{G}, n)$ *such that* $E_{\bar{k}}$ *is a homotopy* (mod $\mathfrak{C}$) *reduction of* E_k.

Proof. Let $p\colon K(G, n) \to K(\bar{G}, n)$ be the map induced by the natural projection $G \to \bar{G}$. Let $\bar{\lambda}$ be the fundamental class in $H^n(K(\bar{G}, n), \bar{G})$. The cohomology homomorphism:

$$f^* \colon H^n(\bar{X}, \bar{G}) \to H^n(X, \bar{G})$$

is a (mod $\mathfrak{C}$) isomorphism. Therefore, for some q such that $Z_q \in \mathfrak{C}$, the class $qk^*p^*(\bar{\lambda})$ lies in $\operatorname{Im} f^*$. Now

$$qk^*p^*(\bar{\lambda}) = k^*(qp^*(\bar{\lambda})) = k^*(qp)^*(\bar{\lambda}).$$

Thus, there exists a map $\bar{k}$ which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{k} & K(G, n) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle qp} \\ \bar{X} & \xrightarrow{\bar{k}} & K(\bar{G}, n) \end{array}$$

homotopy commutative.

Since $Z_q \in \mathfrak{C}$, the map qp is a weak homotopy equivalence (mod $\mathfrak{C}$), and, of course, so is f. Thus, by Lemma 1, there exists a whe (mod $\mathfrak{C}$) h: $E_k \to E_{\bar{k}}$.

Now we have only to check that $\pi_i(E_{\bar{k}}) = \overline{\pi_i(E_k)}$. Since $\pi_n(X) = \pi_{n-1}(X) = 0$, we know that $\pi_n(\bar{X}) = \pi_{n-1}(\bar{X}) = 0$ and $\pi_i(E_{\bar{k}}) = \pi_i(X)$ for all $i \neq n-1$ and $\pi_{n-1}(E_{\bar{k}}) = \bar{G}$. Furthermore, $\pi_i(E_k) = \pi_i(X)$ for all $i \neq n-1$ and $\pi_{n-1}(E_k) = G$. This completes the proof.

Proof of Theorem 1. The proof is by induction on the number of non-trivial homotopy groups of X. When $X = K(G, r)$, take $\bar{X} = K(\bar{G}, r)$ and let p: $X \to \bar{X}$ be the map induced by the canonical projection $G \to \bar{G}$.

Suppose that X has n non-trivial homotopy groups and that the theorem is true for all spaces having fewer than n non-trivial homotopy groups. Let X^{n-1} be the space in the Postnikov system of X made up of the first $n-1$ (non-trivial) homotopy groups of X. Then X^{n-1} has a (mod $\mathfrak{C}$) reduction $\overline{X^{n-1}}$ and there exists a whe (mod $\mathfrak{C}$) f: $X^{n-1} \to \overline{X^{n-1}}$. Suppose that the highest non-trivial homotopy group of X is in dimension $r-1$. Then there is a map k: $X^{n-1} \to K(\pi_{r-1}(X), r)$ such that $X = E_k$. Furthermore, $\pi_{r-1}(X^{n-1}) = \pi_r(X^{n-1}) = 0$. Hence, by Lemma 2, there exists a map $\bar{k}$: $\overline{X^{n-1}} \to K\overline{(\pi_{r-1}(X), r)}$ such that $E_{\bar{k}}$ is a (mod $\mathfrak{C}$) reduction of E_k. Thus, we have constructed a homotopy (mod $\mathfrak{C}$) reduction of X.

COROLLARY 1.1. *If* $\overline{\pi_*(X)}$ *is finitely generated, then* X *has a homotopy* (mod $\mathfrak{C}$) reduction.

Proof. Since $\overline{\pi_*(X)}$ is finitely generated, there exists an m such that $\pi_i(X) \in \mathfrak{C}$ for all $i > m$. Then construct the (mod $\mathfrak{C}$) reduction of the space X^m in the Postnikov system of X made up of the first m homotopy groups.

3. A periodicity theorem for cohomology. The next theorem is stronger than what we need at this point. However, it may be of some interest in itself.

Notation. Let $n(X)$ and $m(X)$ be the smallest and greatest integers for which $\pi_i(X)_p \neq 0$.

THEOREM 2. *Suppose that* $\pi_*(X)_p$ *is finitely generated and* $n(X) > 2p(p-1)$. *Then*

$$P^1: H^k(X; Z_p) \to H^{k+2p-2}(X; Z_p)$$

is an isomorphism when $m(X) + 3 < k < n(X) + 2(p-1)^2 - 1$.

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Assertion. It is sufficient to prove this theorem when $\pi_*(X)$ has only free and p-primary summands.

Proof. Since $\pi_*(X)_p$ is finitely generated, X has a homotopy reduction modulo the class $\mathfrak{C}(\hat{p})$. That is, there exists a space $\bar{X}$, a whe (mod $\mathfrak{C}(\hat{p})$) $f\colon X \to \bar{X}$ and $\pi_*(\bar{X})$ has only free and p-primary summands. Consider the commutative diagram

$$\begin{array}{ccc} H^k(X; Z_p) & \xrightarrow{P^1} & H^{k+2p-2}(X; Z_p) \\ \uparrow f^* & & f^* \uparrow \\ H^k(\bar{X}; Z_p) & \xrightarrow{P^1} & H^{k+2p-2}(\bar{X}; Z_p) \end{array}$$

Since $f^*\colon H^n(X; Z_p) \to H^n(X; Z_p)$ is an isomorphism (mod $\mathfrak{C}(\hat{p})$), ker f^* and coker f^* are finite groups whose orders are prime to p. However, they are a subgroup and a quotient group of groups whose orders are powers of p. Therefore, ker f^* = coker $f^* = 0$. Hence, f^* is an isomorphism and the assertion is proved.

We now need some information about the action of P^1 on the spaces $K(Z, n)$, $K(Z_p, n)$, and $K(Z_{p^r}, n)$. Let $\mathbf{S}$ be the Steenrod algebra of reduced p-powers and $\mathbf{S}^i$ the subgroup that raises the degree precisely by i. Let

$$\mathbf{S} < t = \sum_{i<t} \mathbf{S}^i.$$

The following table gives an additive basis for $\mathbf{S}^i$ when $2 < i < 2p^2$:

$$\begin{aligned} &\mathbf{S}^{2t(p-1)}\colon P^t, \\ &\mathbf{S}^{2t(p-1)+1}\colon \delta P^t, P^t\delta, \\ &\mathbf{S}^{2t(p-1)+2}\colon \delta P^t\delta. \end{aligned}$$

The multiplication table is given by the Adem relations.

LEMMA 3. *Define a homomorphism* $h\colon \mathbf{S}^i \to \mathbf{S}^{i+2p-2}$ *by* $h(\theta) = P^i \cdot \theta$. *Then h is an isomorphism when* $2 < i < 2(p-1)^2$.

Proof. When $i = 2t(p-1)$ and $1 \leqq t < p-1$, then $P^1P^t = tP^{t+1}$ is not zero and generates $\mathbf{S}^{2(t+1)(p-1)}$.

When $i = 2t(p-1) + 2$ and $1 \leqq t < p-1$, then $P^1\delta P^t\delta = t\delta P^{t+1}\delta$ is not zero and generates $\mathbf{S}^{2(t+1)(p-1)+2}$.

When $i = 2t(p-1) + 1$ and $1 \leqq t < p-1$, we compute P^1 on the basis δP^t, $P^t\delta$:

$$P^1\delta P^t = t\delta P^{t+1} + P^{t+1}\delta \quad \text{and} \quad P^1P^t\delta = tP^{t+1}\delta.$$

Clearly, no non-trivial linear combination of these two can be zero. Thus, $h\colon \mathbf{S}^{2t(p-1)+1} \to \mathbf{S}^{2(t+1)(p-1)+1}$ is a monomorphism. However, these are both vector spaces of dimension two over Z_p. Therefore, h is an isomorphism.

(A) Let λ be a generator of $H^n(K(Z_p, n); Z_p)$. If $i < n$, then the homomorphism $e: \mathbf{S}^i \to H^{n+i}(K(Z_p, n); Z_p)$ defined by $\theta \to \theta(\lambda)$ is an isomorphism, and furthermore the diagram

$$\begin{array}{ccc} \mathbf{S}^i & \xrightarrow{e} & H^{n+i}(K(Z_p, n); Z_p) \\ h\downarrow & & P^1 \downarrow \\ \mathbf{S}^{i+2p-2} & \xrightarrow{e} & H^{n+i+2p-2}(K(Z_p, n); Z_p) \end{array}$$

is commutative (**6**). Using Lemma 3 we have: If $n > 2p(p-1)$ and $2 < i < 2(p-1)^2$, then

$$P^1: H^{n+i}(K(Z_p, n); Z_p) \to H^{n+2p-2+i}(K(Z_p, n); Z_p)$$

is an isomorphism.

(B) Let λ be a generator of $H^n(K(Z, n); Z_p)$. If $i < n$, then the homomorphism $e: \mathbf{S}^i \to H^{n+i}(K(Z, n); Z_p)$ defined by $\theta \to \theta(\lambda)$ is onto and has the kernel $\mathbf{S}^i\delta$. If $2 < i < 2(p-1)^2$, then $P^1\mathbf{S}^i\delta = \mathbf{S}^{i+2p-2}\delta$. (This follows from the proof of Lemma 3.) Thus, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{S}^i\delta & \to & \mathbf{S}^i & \to & H^{n+i}(K(Z, n); Z_p) & \to & 0 \\ & & h\downarrow & & \searrow h & & P^i\downarrow & & \\ 0 & \to & \mathbf{S}^{i+2p-2}\delta & \to & \mathbf{S}^{i+2p-2} & \to & H^{n+i+2p-2}(K(Z, n); Z_p) & \to & 0 \end{array}$$

where the rows are exact. When $2 < i < 2(p-1)^2$, then the left and middle vertical homomorphisms are isomorphisms, and therefore P^1 is an isomorphism in this range.

(C) If $n > 2p(p-1)$ and $2 < i < 2(p-1)^2$, then

$$P^1: H^{n+i}(K(Z_{p^r}, n); Z_p) \to H^{n+2p-2+i}(K(Z_{p^r}, n); Z_p)$$

is an isomorphism. This is proved by induction on r. The case $r = 1$ is the content of statement (A). Assume that statement (C) is true for $r - 1$ and also that

$$H^{n+2(p-1)^2-1}(K(Z_{p^{r-1}}, n); Z_p) = H^{n+2p(p-1)}(K(Z_{p^{r-1}}, n); Z_p) = 0.$$

Let $f: K(Z_{p^r}, n) \to K(Z_{p^{r-1}}, n)$ be the fibring induced by the projection $Z_{p^r} \to Z_{p^{r-1}}$. The fibre has the homotopy type of a $K(Z_p, n)$. The sequence

$$\begin{array}{ccccccc} H^{n+i-1}(K_1; Z_p) \to H^{n+i}(K_{r-1}; Z_p) \to H^{n+i}(K_r; Z_p) \to H^{n+i}(K_1; Z_p) \\ \downarrow \\ H^{n+i+1}(K_{r-1}; Z_p) \end{array}$$

is exact when $i < n - 2$. (Here, $K_d = K(Z_{p^d}, n)$.) When $i = 2(p-1)^2 - 1$,

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then $H^{n+i}(K_{r-1}; Z_p) = 0$ by hypothesis and $H^{n+i}(K_1; Z_p) = 0$ by Lemma 3 and the table preceding it. Therefore

$$(*) \qquad H^{n+i}(K_r; Z_p) = 0.$$

The same holds for $i = n + 2p(p-1) - 1$. And hence, trivially

$$(**) \qquad P^1\colon H^{n+2(p-1)^2-1}(K_r; Z_p) \to H^{n+2p(p-1)-1}(K_r; Z_p)$$

is an isomorphism.

When $i + 1 < 2(p-1)^2$, then, by statement (A) and the induction hypothesis, P^1 is an isomorphism on the four outside groups. Therefore, by the Five Lemma:

$$P^1\colon H^{n+i}(K_r; Z_p) \to H^{n+2p-2+i}(K_r; Z_p)$$

is an isomorphism for $i < 2(p-1)^2 - 1$. Combining this with $(*)$ and $(**)$, we recapture the induction hypothesis and the proof is complete.

We will now write $H^n(X)$ for the nth cohomology group of X with coefficients Z_p. The only class we consider for the rest of this paper is the class $\mathfrak{C}(\hat{p})$, and objects such as $\bar{G}$, $\bar{X}$ will be understood to be relative to this class.

Statements (A), (B), and (C) are summed up in the following lemma.

LEMMA 4. *If G is a finitely generated abelian group and*
(i) $G = G_p$,
(ii) $n > 2p(p-1)$,
then $P^1\colon H^{n+i}(K(G, n)) \to H^{n+i+2p-2}(K(G, n))$ *is an isomorphism for*

$$2 < i < 2(p-1)^2.$$

Proof. G is a direct sum of infinite cyclic groups and of cyclic groups whose orders are powers of p. $K(G, n)$ is the product of the corresponding Eilenberg-MacLane spaces, say $K(G, n) = \Pi_j K_j$, where $K_j = K(Z, n)$ or $K_j = K(Z_{p^r}, n)$ for some r. The natural inclusion $h\colon \vee_j K_j \to \Pi_j K_j$ induces cohomology isomorphisms up to dimension $2n - 1$ which is at least $n + 2p(p-1) - 1$. We have

$$H^k\left(\bigvee_j K_j\right) = \sum_j H^k(K_j)$$

and by statements (A), (B), and (C), P^1 is an isomorphism on each direct summand in the range indicated. Furthermore, P^1 commutes with the isomorphisms induced by the inclusion, and this completes the proof.

Proof of Theorem 2. Recall that we may assume that $\pi_i(X) = \pi_i(X)_p$ for all i. Then the proof is by induction on $N(X)$, the number of non-trivial homotopy groups of X. When $N(X) = 1$ a stronger statement has been proved (Lemma 4). Now assume that the theorem is true for all spaces Y with $N(Y) \leqq r$. Suppose that $N(X) = r + 1$. Let

$$f\colon X \to K(\pi_{n(X)}(X), n(X))$$

be a fibring representing the fundamental class of X. Call the fibre F and write $K_0 = K(\pi_{n(X)}(X), n(X))$. Then $n(F) > n(X)$, $m(F) = m(X)$, and $N(F) = r$. Furthermore, $\pi_i(F) = \pi_i(F)_p$ for all i.

Now consider the Serre exact sequence:

$$H^{k-1}(F) \to H^k(K_0) \to H^k(X) \to H^k(F) \to H^{k+1}(K_0).$$

By our induction hypothesis, P^1 is an isomorphism on both $H^{k-1}(F)$ and $H^k(F)$ when

$$m(F) + 4 < k < n(F) + 2(p-1)^2 - 1.$$

By Lemma 4, P^1 is an isomorphism on both $H^k(K_0)$ and $H^{k+1}(K_0)$ when

$$n(X) + 2 < k < n(X) + 2(p-1)^2 - 1.$$

Using the equalities and inequalities listed above, we see that P^1 is an isomorphism on the four outside groups if

$$(***) \qquad m(X) + 4 < k < n(X) + 2(p-1)^2 - 1.$$

Then by the Five Lemma, it is also an isomorphism from $H^k(X)$ to $H^{k+2p-2}(X)$ when k is in this range. However, this does not recapture the induction hypothesis and another (similar) argument is necessary.

Now let X^r be the space in the Postnikov system of X made up of the first r homotopy groups of X. Let $g: X \to X^r$ be a fibring with fibre

$$K_1 \quad (= K(\pi_{m(X)}(X), m(X))).$$

Then $N(X^r) = r$, $m(X^r) < m(X)$, and $n(X^r) = n(X)$. Again we have an exact sequence

$$H^{k-1}(K_1) \to H^k(X^r) \to H^k(X) \to H^k(K_1) \to H^{k+1}(X^r).$$

By the induction hypothesis, P^1 is an isomorphism on both $H^k(X^r)$ and $H^{k+1}(X^r)$ when

$$m(X^r) + 3 < k < n(X^r) + 2(p-1)^2 - 2.$$

By Lemma 4, P^1 is an isomorphism on both $H^{k-1}(K_1)$ and $H^k(K_1)$ when

$$m(X) + 3 < k < m(X) + 2(p-1)^2.$$

Altogether, P^1 is an isomorphism on the four outside groups if

$$(****) \qquad m(X) + 3 < k < n(X) + 2(p-1)^2 - 2.$$

By the Five Lemma, P^1: $H^k(X) \to H^{k+2p-2}(X)$ is an isomorphism when k is in the range defined by (****). However, P^1 is also an isomorphism when k is in the range defined by (***). Hence, P^1 is an isomorphism when k is in the union of these ranges, i.e. when

$$m(X) + 3 < k < n(X) + 2(p-1)^2 - 1.$$

This completes the induction and the proof of Theorem 2.

4. A (mod $\mathfrak{C}(\hat{p})$) **study of** S^n**.** Toda (**16**) has shown that when

$$n > 2p(p-1) > k+1,$$

we have $\pi_{n+k}(S^n) \in \mathfrak{C}(\hat{p})$ unless $k = 0$ or k is of the form $2t(p-1)-1$, and

$$\pi_{n+2t(p-1)-1}(S^n) \approx Z_p \ (\text{mod } \mathfrak{C}(\hat{p}))$$

for $1 \leqq t < p$.

Take n large and let Y be the space in the Postnikov system for S^n made up out of the homotopy groups of S^n up to (and including) dimension

$$n + 2p(p-1) - 2.$$

Let X^p be a (mod $\mathfrak{C}(\hat{p})$) reduction of Y; we shall study the Postnikov system of X^p. (We know that X^p is well-determined up to $\mathfrak{C}(\hat{p})$. In fact, as will be shown later, it is unique up to homotopy type.) We have

$$\pi_n(X^p) = Z, \ \pi_{n+2t(p-1)-1}(X^p) = Z_p \quad \text{when } 1 \leqq t < p,$$

and all other homotopy groups of X^p are trivial. Let X^r be the space in the Postnikov system of X^p made up of r non-trivial groups. Let

$$K^r = K(Z_p, n + 2r(p-1)).$$

Then the Postnikov system of X^p has the diagram

$$\begin{array}{ccccccccccc}
 & & & & \Omega K^r & & \Omega K^{r-1} & & & & K(Z,n) \\
 & & & & \downarrow i_{r+1} & & \downarrow i_r & & & & \| \\
X^p \to X^{p-1} & \to & \ldots & & X^{r+1} & \to & X^r & \xrightarrow{f_{r-1}} & X^{r-1} & \to \ldots \to & X^1 \\
 & & & & \downarrow \theta^{r+1} & & \downarrow \theta^r & & & & \downarrow \theta^1 \\
 & & & & K^{r+1} & & K^r & & & & K^1
\end{array}$$

where ΩK^{r-1} is the fibre of $f_{r-1}: X^r \to X^{r-1}$ and X^{r+1} has the same homotopy type as $E\theta r$. Let $W_r = \theta^r \cdot i_r$ for $r > 1$ and let $W_1 = \theta^1$. Then we may think of W^r ($r > 1$) as a class in $H^{n+2r(p-1)}(K(Z_p, n+2(r-1)(p-1)-1))$ and by means of the isomorphism e identify it with an element $\mathbf{S}^{2p-1}$. Similarly, $W_1 \in H^{n+2p-2}(K(Z,n); Z_p)$ may be identified with an element in $\mathbf{S}^{2p-2}$ since

$$e: \mathbf{S}^{2p-2} \to H^{n+2p-2}(K(Z,n); Z_p)$$

is an isomorphism. The problem is to determine these elements. This diagram and the notation introduced will be used repeatedly in what follows.

In the construction of the Postnikov system for X^p, θ^r was chosen with an indeterminacy which corresponds to coefficient automorphisms in

$$H^{n+2r(p-1)}(X^r; Z_p);$$

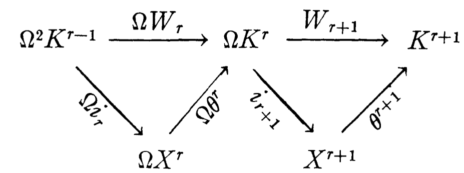
that is, it is determined up to non-zero scalar multiplication. When we say that W_r "equals" some named cohomology operation, this will mean that they correspond under a non-zero scalar multiplication. Bearing this in mind, we have the following result.

THEOREM 3.

$$W_1 = P^1, \quad W_r = rP^1\delta - (r-1)\delta P^1, \qquad r = 2, 3, \ldots, p-1.$$

Proof. We first assume that $W_r \neq 0$. Since P^1 generates $\mathbf{S}^{2p-2}$, we may take $W_1 = P^1$, or $W_1 = P^1(\lambda) \in H^{n+2p-2}(K(Z,n);Z_p)$, where λ is a generator of $H^n(K(Z,n);Z_p)$.

The diagram



is commutative, and $i_{r+1} \cdot \Omega\theta^r = 0$. Therefore, $W_{r+1} \cdot \Omega W_r = 0$, or as stable operations $W_{r+1}W_r = 0$; in particular, $W_2(P^1(\lambda)) = 0$. Let $W_2 = aP^1\delta + b\delta P^1$. Then

$$(aP^1\delta + b\delta P^1)P^1\lambda = (a+2b)P^2(\lambda) = 0$$

(since $\delta\lambda = 0$). Since, by assumption, $W_2 \neq 0$, we must have $W_2 = 2P^1\delta - \delta P^1$.

Let $aP^1\delta + b\delta P^1$ and $cP^1\delta + d\delta P^1$ be any two elements of $\mathbf{S}^{2p-1}$. Using the Adem relations, we obtain

$$(\dagger) \qquad (aP^1\delta + b\delta P^1)(cP^1\delta + d\delta P^1) = (ac + 2bc + bd)\delta P^2\delta.$$

Take $\gamma \in \mathbf{S}^{2p-1}$. Define $\bar{\gamma}: \mathbf{S}^{2p-1} \to \mathbf{S}^{4p-2}$ by $\bar{\gamma}(\alpha) = \alpha\gamma$, ($\dagger$) implies that $\bar{\gamma} = 0$ if and only if $\gamma = 0$. Since $\mathbf{S}^{4p-2}$ is one-dimensional (as a Z_p vector space) and $\mathbf{S}^{2p-1}$ is two-dimensional, for $\gamma \neq 0$, ker $\bar{\gamma}$ is one-dimensional. Hence, if W_r and W_{r+1} are both not equal to zero, we may take W_{r+1} as any generator of ker $\overline{W}_r$. Thus, to prove the theorem by induction, it is only necessary to check that

$$[(r+1)P^1\delta - r\delta P^1][rP^1\delta - (r-1)P^1\delta] = 0.$$

This readily follows from ($\dagger$).

It is still necessary to prove that $W_r \neq 0$. This requires some preliminary information about a test space.

Let $n - 1 > k = 2t(p-1) - 1$, where $1 \leqq t < p$. Choose a class $\alpha \in \pi_{n+k}(S^n)$ which is of order p. (Such a class exists since the group has a Z_p summand.) Represent α by a map $\bar{\alpha}: S^{n+k} \to S^n$ and call the cone C_α.

LEMMA 5. *The co-Hurewicz homomorphism* $h^n: \pi^n(C_\alpha) \to H^n(C_\alpha; Z)$ *is a* (mod $\mathfrak{C}(\hat{p})$) *monomorphism and* $\operatorname{Im} h^n = pH^n(C_\alpha; Z)$.

Proof. Let $i: S^n \to C_\alpha$ be the inclusion. There is an exact cohomotopy sequence

$$\pi^n(S^{n+k+1}) \to \pi^n(C_\alpha) \xrightarrow{i^*} \pi^n(S^n) \xrightarrow{\bar{\alpha}^*} \pi^n(S^{n+k}).$$

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Here, $\pi^n(S^{n+k+1}) = \pi_{n+k+1}(S^n) \in \mathfrak{C}(\hat{p})$ since $k + 1$ is not of the required form $(k + 1 = 2t(p - 1))$. Therefore, i^* is a (mod $\mathfrak{C}(\hat{p})$) monomorphism. $\overline{\alpha^*}$ takes the identity map in $\pi^n(S^n)$ to α in $\pi^n(S^{n+k})$. α has order p, and hence $\operatorname{Im} i^* = \ker \overline{\alpha^*} = p\pi^n(S^n)$. Consider the commutative diagram

$$\begin{array}{ccc} \pi^n(C_\alpha) & \xrightarrow{i^*} & \pi^n(S^n) \\ h^n \downarrow & & \downarrow \overline{h^n} \\ H^n(C_\alpha; Z) & \xrightarrow{i_\#} & H^n(S^n; Z) \end{array}$$

Since $i_\#$ and $\overline{h^n}$ are isomorphisms, $\ker h^n = \ker i^* \in \mathfrak{C}(\hat{p})$ and

$$\begin{aligned} \operatorname{Im} h^n &= (i_\#)^{-1} \overline{h^n} i^*(\pi^n(C_\alpha)) \\ &= (i_\#)^{-1} \overline{h^n}(p\pi^n(S^n)) \\ &= (i_\#)^{-1} pH^n(S^n; Z) \\ &= pH^n(C_\alpha; Z). \end{aligned}$$

Y is the space in the Postnikov system for S^n made up out of the homotopy groups of S^n up to dimension $n + 2p(p - 1) - 2$ and X^P is a (mod $\mathfrak{C}(\hat{p})$) reduction of Y. Thus, we have maps $g: S^n \to Y$ and $f: Y \to X^p$ such that

(i) $g_*: \pi_i(S^n) \to \pi_i(Y)$ is an isomorphism for $i \leqq n + 2p(p - 1) - 2$, and

(ii) $f_*: \pi_i(Y) \to \pi_i(X^P)$ is an isomorphism (mod $\mathfrak{C}(\hat{p})$) for all i and is onto (i.e., an isomorphism) when $i = n$.

Let λ_1 and λ_2 be maps representing the fundamental classes of S^n and X^P, respectively. Then the following diagram is commutative up to sign

$$\begin{array}{ccccc} S^n & \xrightarrow{g} & Y & \xrightarrow{f} & X^P \\ & \searrow^{\lambda_1} & & & \downarrow \lambda_2 \\ & & & & K(Z, n) \end{array}$$

If the signs do not fit, change λ_1 to its negative (without changing notation). Furthermore, $\lambda_2 = f_0 f_1 \ldots f_{p-1}$ (the composition of the projections in the Postnikov system for X^P). Let S^2L be a space of dimension less than

$$n + 2p(p - 1) - 2.$$

LEMMA 6. *If* $\alpha: S^2L \to K(Z, n)$ *lifts to* $\bar{\alpha}: S^2L \to X^p$ $(\alpha = \lambda_2\bar{\alpha})$, *then for some* q *prime to* p, $q\alpha$ *lifts* $\beta: S^2L \to S^n$ *(i.e.,* $q\alpha = \lambda_1\beta$*).*

Proof. Since $\dim S^2L < n + 2p(p - 1) - 2$, $g_*: [S^2L, S^n] \to [S^2L, Y]$ is an isomorphism. Furthermore, $f_*: [S^2L, Y] \to [S^2L, X^P]$ is an isomorphism (mod $\mathfrak{C}(\hat{p})$). Thus, for some q prime to p, $q\bar{\alpha} \in \operatorname{Im} f_*$, say $q\bar{\alpha} = f_*(\gamma)$; take $\beta = (g)^{-1}\gamma$. Then $\lambda_{1*}(\beta) = (\lambda_2 fg)_*\beta = q\alpha$. Furthermore, note that

$$\lambda_{1*}: [S^2L, S^n] \to [S^2L, K(Z, n)]$$

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is, up to sign, the co-Hurewicz homomorphism in that dimension.

The following lemma will complete the proof of Theorem 3.

LEMMA 7. *$W_r \neq 0$ for $r = 1, 2, \ldots, p-1$.*

Proof. Take $r > 1$. We have the following diagram

$$\begin{array}{ccccc} \Omega K^{r-1} & \xrightarrow{i_r} & X^r & \xrightarrow{\theta^r} & K^r = K(Z_p, n + 2r(p-1)) \\ & & \downarrow{\scriptstyle f_{r-1}} & & \\ & & X^{r-1} & & \end{array}$$

If $W_r = \theta^r \cdot i_r = 0$, then by the Serre exact sequence, there exists a γ: $X^{r-1} \to K^r$ such that $\gamma f_{r-1} = \theta^r$. γ is a class in $H^{n+2r(p-1)}(X^{r-1}; Z_p)$ and $m(X^{r-1}) = n + 2(r-2)(p-1) - 1$. Therefore, by Theorem 2,

$$P^1 \colon H^{n+2(r-1)(p-1)}(X^{r-1}) \to H^{n+2r(p-1)}(X^{r-1})$$

is an isomorphism and $\gamma = P^1 \bar{\gamma}$. (The coefficient group has been dropped; it is Z_p.) Thus, $\theta^r = P^1(\bar{\gamma} f_{r-1})$, where

$$\bar{\gamma} f_{r-1} \in H^{n+2(r-1)(p-1)}(X^r).$$

Let $\alpha_r \in \pi_{n+2r(p-1)-1}(S^n)$ be an element of order p. Represent α_r by a map $S^{n+2r(p-1)-1} \to S^n$ and let C_{α_r} be the cone. Notice that C_{α_r} is a double suspension since α_r is well within the stable range. Let g: $C_{\alpha_r} \to K(Z, n)$ represent a generator of $H^n(C_{\alpha_r}; Z)$. Since $H^{n+2t(p-1)}(C_{\alpha_r}; Z_p)$ is trivial when $1 \leqq t \leqq p-1$ except for $t = r$, the only obstruction to lifting g to X^p is in dimension $n + 2r(p-1)$. Let g_r be the (unique) lift of g to X^r. Now we have the following commutative diagram:

$$\begin{array}{ccccc} C_{\alpha_r} & \xrightarrow{g_r} & X^r & \xrightarrow{\theta^r} & K(Z_p, n + 2r(p-1)) \\ & & \downarrow{\scriptstyle f_{r-1}} & & \uparrow{\scriptstyle P^1} \\ & & X^{r-1} & \xrightarrow{\bar{\gamma}} & K(Z_p, n + 2(r-1)(p-1)). \end{array}$$

Here $\theta^r g_r = P^1(\bar{\gamma} f_{r-1} g_r) = 0$ since $\bar{\gamma} f_{r-1} g_r \in H^{n+2(r-1)(p-1)}(C_{\alpha_r}) = 0$. Therefore, g_r lifts to $\bar{g}$: $C_{\alpha_r} \to X^p$ and for some q prime to p, qg lifts to β: $C_{\alpha_r} \to S^n$ (Lemma 6). But then the class of $qg \in H^n(C_\alpha; Z)$ is in the image of the co-Hurewicz homomorphism. This contradicts Lemma 5. Hence, $W_r \neq 0$ for $r > 1$.

When $r = 1$, $W_1 = \theta^1 = 0$ implies the existence of the lift to X^p and the remainder of the argument is the same. This completes the proof of Theorem 3.

5. The (mod p) **cohomology of the spaces X^r.** Before going on to construct the spectral sequence of the Theorem, we will compute some of the cohomology of the spaces X^r and show that X^p is completely determined up to homotopy type. The inequality $n > 2p(p-1)$ will be assumed throughout this section, and cohomology groups will have coefficients in Z_p. Furthermore, the notation of the preceding section will be carried over.

THEOREM 4. *When $n < i < n + 2(p-1)$, we have*

$$H^i(K(Z_p, n + 2r(p-1))) \xrightarrow{(\theta^r)^*} H^i(X^r) \to 0$$

is exact, and the kernel of $(\theta^r)^*$ is generated by $(P^{j-1}W_{r+1})\lambda_r$ and $(\delta P^j\delta)\lambda_r$, where λ_r is the fundamental class in

$$H^{n+2r(p-1)}(K(Z_p, n + 2r(p-1)))$$

and $1 \leqq j < p - r$.

COROLLARY 4.1. *If V^p is another* (mod $\mathfrak{C}(\hat{p})$) *reduction of Y (the space in the Postnikov system for S^n consisting of the homotopy groups of S^n to dimension $n + 2p(p-1) - 2$), then there exists a homotopy equivalence $h\colon X^p \to V^p$.*

Proof of Corollary 4.1. Let V^r be the space in a Postnikov system for V^p made up of the first r non-trivial homotopy groups of V^p, and let

$$\phi^r \in H^{n+2r(p-1)}(V^r)$$

be the Postnikov invariants. Then Theorem 4 holds for $\{V^r, \phi^r\}$ and $H^{n+2r(p-1)}(V^r)$ and $H^{n+2r(p-1)}(X^r)$ are generated by ϕ^r and θ^r, respectively.

Suppose that $h_r\colon X^r \to V^r$ is a homotopy equivalence. Then

$$(h_r)^*\colon H^{n+2r(p-1)}(V^r) \to H^{n+2r(p-1)}(X^r)$$

is an isomorphism and $(h_r)^*\phi^r = c\theta^r$, where $0 \neq c \in Z_p$. Therefore, X^{r+1} is of the same homotopy type as V^{r+1}. We have only to check that X^1 and V^1 have the same homotopy type. This is true since each is a $K(Z, n)$.

We now begin to prove Theorem 4.

(A) Let $n < i < n + 2p(p-1)$. Then

$$(\theta^1)^*\colon H^i(K(Z_p, n + 2p - 2)) \to H^i(X^1)$$

is onto.

Proof. We have $X^1 = K(Z, n)$ and

$$H^i(X^1) = \begin{cases} Z_p & \text{if } i = n + 2r(p-1), \\ Z_p & \text{if } i = n + 2r(p-1) + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $1 \leqq r < p$ and $n < i < n + 2p(p-1)$. Let λ be a generator of $H^n(X^1)$. Then $H^{n+2r(p-1)}(X^1)$ and $H^{n+2r(p-1)+1}(X^1)$ are generated by $P^r\lambda$ and $\delta P^r\lambda$, respectively. If $\bar{\lambda}$ is a generator of $H^{n+2p-2}(K(Z_p, n + 2p - 2))$, then

$P^1\lambda = (\theta^1)^*\bar{\lambda}$ (up to a non-zero scalar multiple) since θ^1, as a cohomology operation, is P^1. Therefore,

$$(\theta^1)^* P^{r-1}\bar{\lambda} = P^{r-1}(\theta^1)^*\bar{\lambda} = P^{r-1}P^1\lambda$$

and

$$(\theta^1)^*\delta P^{r-1}\bar{\lambda} = \delta P^{r-1}(\theta^1)^*\bar{\lambda} = \delta P^{r-1}P^1\lambda.$$

These two classes are non-zero mutiples of $P^r\lambda$ and $\delta P^r\lambda$, respectively, and hence $(\theta^1)^*$ is onto. We shall now write K^r for $K(Z_p, n + 2r(p-1))$.

(B) Let λ be the fundamental class of K^r. Then $(\theta^r)^*\delta\lambda \neq 0$.

Proof. Let $\bar{\lambda}$ be the fundamental class of ΩK^{r-1}. The maps

$$\Omega K^{r-1} \xrightarrow{i_r} X^r \xrightarrow{\theta^r} K^r$$

induce cohomology homomorphisms

$$H^i(K^r) \xrightarrow{(\theta^r)^*} H^i(X^r) \xrightarrow{(i_r)^*} H^i(\Omega K^{r-1}) \quad \text{and} \quad (i_r)^*(\theta^r)^*\lambda = W_r\bar{\lambda}.$$

Furthermore,

$$(i_r)^*(\theta^r)^*\delta\lambda = \delta(i_r)^*(\theta^r)^*\lambda = \delta W_r\bar{\lambda} = r\delta P^1\delta\bar{\lambda} \neq 0.$$

Therefore, $(\theta^r)^*\delta\lambda \neq 0$.

(C) Let λ be the fundamental class of K^r. Take $0 \neq \alpha \in \mathbf{S}^{2d(p-1)+j}$, where $1 \leqq d < p - r$ and $j = 0, 1, 2$. Then if $(\theta^r)^*\alpha\lambda = 0$, α must be a multiple of $P^{d-1}W_{r+1}$ if $j = 1$ or $\delta P^d\delta$ if $j = 2$; and when $j = 0$, no such α exists.

Proof. It is sufficient to check this statement when $d = 1$; the other cases will follow since $(P^1)^{d-1}$ is an isomorphism on both spaces in the ranges under discussion, $(P^1)^{d-1}W_{r+1}$ is a non-zero multiple of $P^{d-1}W_{r+1}$ and $(P^1)^{d-1}\delta P^1\delta$ is a non-zero multiple of $\delta P^d\delta$.

As in part (B), $(\theta^r)^*\alpha\lambda = 0$ implies $\alpha W_r\bar{\lambda} = 0$, where $\bar{\lambda}$ is the fundamental class of ΩK^{r-1}. Thus, the operation αW_r must be zero. When $d = 1$ and $j = 0$, α is a multiple of P^1 and $P^1W_r \neq 0$. When $d = 1$ and $j = 1$, $\alpha W_r = 0$ if and only if α is a multiple of W_{r+1}; see Theorem 3. When $d = 1$ and $j = 2$, α is a multiple of $\delta P^1\delta$, and $\delta P^1\delta W_r = 0$ since it raises dimension by $4p - 1$.

(D) $(\theta^r)^*: H^i(K^r) \to H^i(X^r)$ is onto when $n < i < n + 2p(p-1)$.

Proof. This has already been proved when $r = 1$ (statement (A)). The proof is by induction on r. Suppose it is true for r. Since $(\theta^r)^*$ is onto, the Serre exact sequence for the fibring

$$X^{r+1} \xrightarrow{f_r} X^r \xrightarrow{\theta^r} K^r$$

decomposes into short exact sequences

$$0 \to H^{i-1}(X^{r+1}) \xrightarrow{(f_r)^*} H^i(K^r) \xrightarrow{(\theta^r)^*} H^i(X^r) \to 0.$$

Thus, $H^{i-1}(X^{r+1}) \neq 0$ only when $(\theta^r)^*: H^i(K^r) \to H^i(X^r)$ has a non-trivial kernel. By part (C), this can happen only when $i = n + 2(r+d)(p-1) + j$, where $1 \leqq d < p - r$ and $j = 1$ or 2. Now, by part (B),

$$0 \neq (\theta^{r+1})^*\lambda \in H^{n+2(r+1)(p-1)}(X^{r+1})$$

and

$$0 \neq (\theta^{r+1})^*\delta\lambda \in H^{n+2(r+1)(p-1)+1}(X^{r+1}),$$

where λ is the fundamental class of K^{r+1}.

By Theorem 2,

$$(P^1)^{d-1}: H^{n+2(r+1)(p-1)+j-1}(X^{r+1}) \to H^{n+2(r+d)(p-1)+j-1}(X^{r+1})$$

is an isomorphism for $1 \leqq d < p - r$ and $j = 1, 2$. Hence, these groups are all non-trivial. Therefore part (C) describes the kernel of $(\theta^r)^*$ exactly and since $(P^1)^{d-1}$ commutes with $(\theta^{r+1})^*$, $(\theta^{r+1})^*$ is onto. This completes the induction and the proof of Theorem 4.

We now begin to prove the main Theorem. Let $Y^0 = X^p$. Recall that

$$\pi_n(Y^0) = Z, \quad \pi_{n+2q(p-1)-1}(Y^0) = Z_p \quad \text{when } 1 \leqq q \leqq p-1,$$

and all other homotopy groups of Y^0 are trivial. Consider the Moore-Postnikov system of the map $pt \to Y^0$. Let Y^q be the space in this system obtained after killing the first q non-trivial homotopy groups of Y^0. Let

$$K^q = K(Z_p, n + 2q(p-1) - 1)$$

when $1 \leqq q \leqq p - 1$ and let $K^0 = K(Z, n)$. Then the Moore-Postnikov system has the diagram

$$\begin{array}{ccccccccccccc}
 & & & & & & K^q & & & & & & Y^0 \\
 & & & & & & \uparrow \phi^q & & & & & & \| \\
Y^p & \to & Y^{p-1} & \to \cdots \to & Y^{q+1} & \to & Y^q & \xrightarrow{g_q} & Y^{q-1} & \to \cdots \to & Y^1 & \to & Y^0 \\
\| & & & & & & \uparrow j_{q-1} & & & & & & \\
\text{pt.} & & & & & & \Omega K^{q-1} & & & & & &
\end{array}$$

where ϕ^q is the fundamental class of Y^q,

$$\Omega K^{q-1} \xrightarrow{j_{q-1}} Y^q \xrightarrow{g_q} Y^{q-1}$$

is a fibring, and $Y^{q+1} = E_{\phi^q}$.

The map $c_{q-1} = \phi^q j_{q-1}: \Omega K^{q-1} \to K^q$ also appears in the Postnikov system for ΩY^0 $(= \Omega X^p)$. Therefore,

$$c_{q-1} = \Omega(\phi^q j_q): \Omega K(Z_p, n + 2(q-1)(p-1) - 1) \to \Omega K(Z_p, n + 2q(p-1)).$$

Since we are in the stable range, c_{q-1}, as a cohomology operation, is equal to $W_q = qP^1\delta - (q-1)\delta P^1$ (or is equal to P^1 when $q = 0$).

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THEOREM 5. *Suppose that $X = S^2L$ is a double suspension and $H^i(X, Z) = 0$ for $i > n$. Then there exists a spectral sequence with*

$$E_1^{r,0} = H^{n+r}(X; Z),$$
$$E_1^{r,q} = H^{n+r+2q(p-1)-1}(X; Z_p) \quad \text{when } q > 0,$$
$$E_1^{r,q} = 0 \quad \text{when } q < 0,$$

with the differential $d_1^{r,q}\colon E_1^{r,q} \to E_1^{r+1,q+1}$ given by

$$d_1^{r,0} = P^1\colon H^{n+r}(X; Z) \to H^{n+r+2p-2}(X; Z_p)$$

and

$$d_1^{r,q} = (q+1)P^1\delta - q\delta P^1\colon H^{n+r+2q(p-1)-1}(X; Z_p) \to H^{n+r+2(q+1)(p-1)}(X; Z_p)$$

when $q > 0$. This spectral sequence converges to $[S^{-r}X, Y^0]$ (when $r > 0$, everything is trivial). Thus, there exists a filtration

$$[S^rX, Y^0] = A^{r,0} \supset A^{r,1} \supset \ldots \supset A^{r,p} = 0,$$

where $E_\infty^{r,q} = A^{r,q}/A^{r,q+1}$.

The proof follows (**15**). Take the Moore-Postnikov system for the map pt. $\to Y^0$. For $q < 0$ and $q > p$ let $Y^q =$ pt. and $K^q =$ pt. Let

$$D^{r,q} = [S^{-r}X, Y^q], \qquad r \leqq 0, -\infty < q < \infty,$$
$$E^{r,q} = [S^{-r}Z, K^q], \qquad r \leqq 0, -\infty < q < \infty,$$

and

$$E^{r,q} = D^{r,q} = 0 \quad \text{when } r > 0.$$

Since X is a double suspension, these are all abelian groups. Let

$$g^{r,q} = (g_q)_*\colon [S^{-r}X, Y^q] \to [S^{-r}X, Y^{q-1}],$$
$$\phi^{r,q} = (\phi^q)_*\colon [S^{-r}X, Y^q] \to [S^{-r}X, K^q],$$

and let $j^{r,q}$ be the composition

$$[S^{-r}X, K^q] \xrightarrow{e} [S^{-(r+1)}X, \Omega K_q] \xrightarrow{(j_q)_*} [S^{-(r+1)}X, Y^{q+1}],$$

where e is the natural isomorphism.

Since $Y^{q+1} \to Y^q \to K^q$ is a fibring, we have exact sequences

(††) $$\ldots \to E^{r-1,q} \to D^{r,q+1} \to D^{r,q} \to E^{r,q} \to \ldots$$

provided that $r \leqq 0$. However, $E^{0,q} = D^{0,q} = 0$ unless $q = 0$, and the homomorphism $(\phi^0)_*\colon [X, Y^0] \to [X, K(Z, n)]$ is an isomorphism (since $\dim X \leqq n$ and ϕ^0 is the fundamental class of Y^0). Therefore, the sequences (††) are exact for all r and q.

Now, following (**15**) $d_1^{r,q}\colon E_1^{r,q} \to E_1^{r+1,q+1}$ is given by $\phi^{r+1,q+1}\cdot j^{r,q}$. Thus, $d_1^{r,q}$ is the composition

$$[S^{-r}X, K^q] \xrightarrow{e} [S^{-(r+1)}X, \Omega K^q] \xrightarrow{(\phi^{q+1}\cdot j_q)_*} [S^{-(r+1)}X, K^{q+1}].$$

Now, e is the suspension isomorphism in cohomology and $\phi^{q+1}j_q = C_q$ is the cohomology operation $(q+1)P^1\delta - q\delta P^1$ (if $q > 0$) or the operation P^1 (when $q = 0$). Since stable operations commute with suspension we have

$$d^{r,0} = P^1\colon H^{n+r}(X; Z) \to H^{n+2p-2+r}(X; Z_p)$$

and

$$d^{r,q} = (q+1)P^1\delta - q\delta P^1\colon H^{n+r+2q(p-1)-1}(X; Z_p) \to H^{n+r+2(q+1)(p-1)}(X; Z_p)$$

when $q > 0$. (When $q < 0$ all the groups are trivial.) Let

$$A^{r,q} = \mathrm{Im}([S^{-r}X, Y^q] \to [S^{-r}X, Y^0]).$$

Then

$$[S^{-r}X, Y^0] = A^{r,0} \supset A^{r,1} \supset \ldots \supset A^{r,p} = 0$$

and $A^{r,q}/A^{r,q+1} = E_\infty^{r,q}$.

Note that the only groups in E_1 which may contain non-trivial subgroups that are in $\mathfrak{C}(\hat{p})$ are the groups $E_1^{r,0} = H^{n+r}(X, Z)$. Clearly, all differentials vanish on $(E_1^{r,0})_{\mathfrak{C}(\hat{p})}$ and no element of $(E_1^{r,0})_{\mathfrak{C}(\hat{p})}$ is in the image of a differential. Thus, $E_\infty^{r,0} = (E_\infty^{r,0})_p \oplus (E_1^{r,0})_{\mathfrak{C}(\hat{p})}$ and $A^{r,0} = (A^{r,0})_p \oplus (E_1^{r,0})_{\mathfrak{C}(\hat{p})}$. If we modify the spectral sequence by having

$$\tilde{E}_1^{r,0} = (E_1^{r,0})_p = H^{n+r}(X; Z)_p,$$

then the convergence would be to $[S^{-r}X, Y^0]_p$. Recall that we have a map $f\colon S^n \to Y^0$ $(=X^p)$ such that $f_*\colon \pi_i(S^n) \to \pi_i(Y^0)$ is an isomorphism (mod $\mathfrak{C}(\hat{p})$) for $i < n + 2p(p-1) - 1$. Therefore, by (**4**, Lemma 14), if X is a double suspension and $H^i(X; Z) \in \mathfrak{C}(\hat{p})$ for $i > n + 2p(p-1) - 2$, then

$$f_*[X, S^n] \to [X, Y^0]$$

is an isomorphism (mod $\mathfrak{C}(\hat{p})$). Hence, $[X, S^n]_p \approx [X, Y^0]_p$.

Proof of the Theorem. We are given

(i) $H^i(X; Z) = 0$ for $i > k$, and

(ii) $H^i(X; Z) \in \mathfrak{C}(\hat{p})$ for $i > k - d$.

Choose n large. Let $X' = S^{n-k}X$. Then $H^i(X'; Z) = 0$ for $i > k$ and by Theorem 5 and the remark above, the spectral sequence with

$$\begin{aligned} E_1^{r,0} &= H^{n+r}(X'; Z)_p = H^{k+r}(X; Z)_p, \\ E_1^{r,q} &= H^{n+r+2q(p-1)-1}(X'; Z_p) \\ &= H^{k+r+2q(p-1)-1}(X; Z_p) \end{aligned}$$

if $q > 0$ and $E_1^{r,q} = 0$ if $q < 0$ and with the same differentials (i.e., $d_1^{r,q} = C_q$) converges to $[S^{-r}X', Y^0]_p$. Now suppose that $0 \leqq -r \leqq d + 2p(p-1) - 2$. Then if $i > n + 2p(p-1) - 2$, we have

$$i - n + k + r > n + 2p(p-1) - 2 - n + k - (d + 2p(p-1) - 2) = k - d.$$

Therefore,

$$H^i(S^{-r}X'; Z) = H^i(S^{n-k-r}X; Z) = H^{i-n+k+r}(X; Z) \in \mathfrak{C}(\hat{p})$$

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when $i > n + 2p(p-1) - 2$. Using the remark above, this implies that $[S^{-r}X', Y^0]_p = [S^{-r}X', S^n]_p$. Since n is large, we are in the stable range and

$$[S^{-r}X', S^n]_p = \{S^{n-k-r}X, S^n\}_p = \{X, S^{k+r}\}_p = \Sigma^{k+r}(X)_p.$$

Thus, the spectral sequence converges to this last group provided that $0 \leqq -r \leqq d + 2p(p-1) - 2$. (When $r > 0$, the statement is also true since everything is trivial.)

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