Non-abelian gauge symmetries and interactions

The previous chapter showed how the fact that the phase of the electron wave-function is not an observable quantity leads to the concept of gauge symmetry, which in turn introduces the gauge potentials, and which then provides the basic framework for describing gauge interactions. The chapter before that showed that the classification of mesons and hadrons in the quark model uncovers that quarks have an additional degree of freedom – dubbed *color*, and the corresponding symmetry with the structure of the $SU(3)_c$ group. Since the physical states that may be detected must be "colorless," i.e., $SU(3)_c$-invariant, it follows that the color of any individual quark cannot be detected either, and so can be changed arbitrarily. This arbitrariness – as a function of space and time! – of the color change in quarks while maintaining the hadron composed of those quarks "colorless" is the essence of the so-called gauge principle. When applied to the local changes in the (matrix-valued) phases of wave-functions, the resulting theories are called "Yang–Mills theories"; Chapter 9 will show that the application of the same idea to local changes of parametrization of the spacetime itself leads to Einstein's theory of gravity.

6.1 The gauge symmetry of color

The first non-abelian (non-commutative) gauge theory was proposed by Oskar Klein in 1938, but that proposal was too early and remained undeveloped, unapplied, and forgotten. Non-abelian gauge theories were taken seriously only after 1954–5, after the publication of the works of C.-N. Yang and R. L. Mills, and independently, R. Shaw's dissertation mentored by A. Salam. However, both of the (currently) well-known applications, the $SU(2)_w$ theory of weak nuclear interactions and the $SU(3)_c$ theory of strong nuclear interactions, required several decades of development and novel ideas for a general acceptance and contemporary formulation of these theories, as well as their embedding in the "Standard Model."

The next few sections consider the $SU(3)_c$ gauge theory as a theoretical system for describing strong nuclear interactions. Chapter 7 will focus on the $SU(2)_w$ gauge theory of weak nuclear interactions, the $SU(2)_w \times U(1)_Q$ theory of electroweak interactions and finally the "Standard Model," based on the $SU(3)_c \times SU(2)_w \times U(1)_Q$ gauge theory. Generally, all gauge theories based on a

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group of symmetries that act by local changes of some generalized phases [☞ e.g., relation (6.2)] are called "Yang–Mills" gauge theories.

6.1.1 The $SU(3)_c$ gauge symmetry and gluons

Section 2.3.13 [☞ discussion on p. 61] showed that quarks have an additional 3-dimensional degree of freedom called "color." That is, the wave-function of any quark is a superposition

$$\Psi_n(\mathrm{x}) = \hat{\mathrm{e}}_\alpha \Psi_n^\alpha(\mathrm{x}) = \hat{\mathrm{e}}_\mathrm{r}\Psi_n^\mathrm{r}(\mathrm{x}) + \hat{\mathrm{e}}_\mathrm{y}\Psi_n^\mathrm{y}(\mathrm{x}) + \hat{\mathrm{e}}_\mathrm{b}\Psi_n^\mathrm{b}(\mathrm{x}) = \begin{bmatrix} \Psi_n^\mathrm{r}(\mathrm{x}) \\ \Psi_n^\mathrm{y}(\mathrm{x}) \\ \Psi_n^\mathrm{b}(\mathrm{x}) \end{bmatrix}, \quad n = u, d, s, c, b, t. \tag{6.1}$$

This matrix representation of the quark wave-functions makes it evident that a local change of the phase (5.14b) of quark wave-functions, in general, becomes

$$\Psi_n(\mathrm{x}) \to e^{ig_c \boldsymbol{\varphi}(\mathrm{x})/\hbar}\, \Psi_n(\mathrm{x}), \qquad \boldsymbol{\varphi}(\mathrm{x}) := \varphi^a(\mathrm{x})\, Q_a, \tag{6.2}$$

where Q_a, $j = 1,\ldots,8$ are eight 3×3 matrices that generate the $SU(3)_c$ *gauge group* [☞ Appendix A.4], and equation (6.2) is the *gauge transformation*. This $SU(3)_c$ symmetry is exact, and must not be confused with the approximate $SU(3)_f$ symmetry discussed in Section 4.4; the group structure of $SU(3)_c$ is identical with that of $SU(3)_f$ but the application is quite different. One usually uses the Gell-Mann matrices (A.71) although, of course, any other basis of Hermitian 3×3 traceless matrices serves just as well.

Digression 6.1 The non-abelian analogue of the simple formal argument in Digression 5.1 on p. 166 shows that the relation (6.2) changes the state operator for a quark by a similarity transformation, rather than leaving it invariant as in the abelian case of Digression 5.1 on p. 166. It is thus not as obvious that all generalized, non-abelian phase transformations (6.2) should be symmetries. Nevertheless, the physical motivation for requiring the transformation (6.2) to be a symmetry remains – and not just because quarks and their particular color states are not directly observable; see Conclusion 11.8 on p. 444.

By the way, the ninth linearly independent matrix generator is proportional to the unit 3×3 matrix and simply produces an overall, diagonal, phase-change

$$\Psi_n(\mathrm{x}) \to e^{ig_c\varphi^0(\mathrm{x})\mathbb{1}Q/\hbar}\, \Psi_i(\mathrm{x}) = e^{iq_n(g_c/g_e)\varphi^0(\mathrm{x})/\hbar}\, \Psi_n(\mathrm{x}), \tag{6.3}$$

which *looks* like the transformation (5.14b). Here, $\Psi_n(\mathrm{x})$ is the eigenfunction of the operator Q, of which the eigenvalue equals the electric charge of the quark q_n represented by $\Psi_n(\mathrm{x})$.[1] This provides essentially the same representation of the gauge transformation of the electrodynamics and chromodynamics interaction. It is clear that $\mathbb{1}$ commutes with all Q_a, whereby the matrices $\{\mathbb{1}; Q_1, \ldots, Q_8\}$ generate the $U(1) \times SU(3)_c$ group – except that the first factor cannot be identified with the gauge symmetry of electrodynamics straightforwardly because of the difference in the magnitudes of the respective charges and the corresponding factor (g_c/g_e) in the exponent (6.3). In addition, one *implicitly* considers the phase-change (6.2) to be limited to quarks, whereby the

[1] Unfortunately, the letter q is standardly used for charge, for the transfer 4-momentum, and for the general symbol-synonym for "quark." Herein, the 4-vector is denoted by q, and the electric charge of the quark q_n is denoted by q_n – the eigenvalue of the operator Q of the eigenfunctions that are identified with the quark q_n.

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transformation (6.3) would correspond to an interaction that is limited to the hadrons and excludes leptons. Although that is a perfectly consistent possibility, such an interaction does not exist in Nature.

Akin to the analysis (5.5)–(5.7), replace $\partial_\mu \to D_\mu$ in the Dirac equation (5.34), and require that with respect to the gauge transformation (6.2) this new Dirac equation should remain unchanged:

$$[i\hbar \not{\partial} - mc]\mathbf{\Psi}_n(\mathrm{x}) = 0 \;\to\; [i\hbar \not{\partial}' - mc]\mathbf{\Psi}'_n(\mathrm{x}) = 0, \tag{6.4a}$$

$$D_\mu \;\to\; D'_\mu := U_{\boldsymbol{\varphi}}\, D_\mu\, U_{\boldsymbol{\varphi}}^{-1}, \tag{6.4b}$$

$$U_{\boldsymbol{\varphi}} := e^{ig_c \boldsymbol{\varphi}/\hbar}, \tag{6.4c}$$

where the matrix representation (6.1) is understood, so that $\not{\partial} = \boldsymbol{\gamma}^\mu D_\mu$ acts as a *double* matrix derivative operator: both as a 4×4-matrix upon the spinor components (because of the $\boldsymbol{\gamma}$-matrices) and as a 3×3-matrix upon colors:

$$\not{\partial}\mathbf{\Psi}_n \equiv \boldsymbol{\gamma}^\mu D_\mu \mathbf{\Psi}_n, \quad \text{i.e.,} \quad (\not{\partial}\mathbf{\Psi}_n)^\alpha \equiv \boldsymbol{\gamma}^\mu D_\mu{}^\alpha_\beta\, \Psi_n^\beta, \quad \text{i.e.,} \quad (\not{\partial}\mathbf{\Psi}_n)^{\alpha A} \equiv (\gamma^\mu)^A{}_B D_\mu{}^\alpha_\beta\, \Psi_n^{\beta B}, \tag{6.5}$$

where repeated α, β indices are summed over colors, red–yellow–blue. The indices A, B, which indicate the Dirac spinor components (5.51), have been written out explicitly only in the third version (6.5).

As in the procedure (5.5)–(5.14a), one finds that

$$D_\mu := \mathbb{1}\,\partial_\mu + \tfrac{ig_c}{\hbar c} A^a_\mu\, Q_a \qquad \text{so that} \qquad \not{\partial}'(e^{ig_c\boldsymbol{\varphi}(\mathrm{x})/\hbar}\,\mathbf{\Psi}_n) = e^{ig_c\boldsymbol{\varphi}(\mathrm{x})/\hbar}(\not{\partial}_\mu \mathbf{\Psi}_n), \tag{6.6a}$$

$$A'^a_\mu Q_a = A^a_\mu\, U_{\boldsymbol{\varphi}}\, Q_a U_{\boldsymbol{\varphi}}^{-1} + \tfrac{\hbar c}{ig_c} U_{\boldsymbol{\varphi}}(\partial_\mu U_{\boldsymbol{\varphi}}^{-1}) = A^a_\mu\, U_{\boldsymbol{\varphi}}\, Q_a U_{\boldsymbol{\varphi}}^{-1} - c(\partial_\mu \varphi^a) Q_a, \tag{6.6b}$$

$$\text{i.e.,} \qquad \mathbb{A}'_\mu = U_{\boldsymbol{\varphi}}\, \mathbb{A}_\mu U_{\boldsymbol{\varphi}}^{-1} - c(\partial_\mu \boldsymbol{\varphi}), \qquad \mathbb{A}_\mu := A^a_\mu Q_a \;\text{ and }\; \boldsymbol{\varphi} := \varphi^a Q_a, \tag{6.6c}$$

where Q_a are 3×3 Hermitian matrices that close the $SU(3)_c$ algebra:

$$[Q_a, Q_b] = i\, f_{ab}{}^c\, Q_c. \tag{6.6d}$$

Since the single electric charge operator Q in electrodynamics is here replaced by eight operators Q_a, it follows that the photon 4-vector gauge potential $A_\mu(\mathrm{x})$ must be replaced by eight *gluon* 4-vector gauge potentials, $A^a_\mu(\mathrm{x})$, $a = 1, \ldots, 8$. Of course, the parameter of electromagnetic interaction, g_e, is also replaced by the parameter of chromodynamic interactions, g_c.

Linearization of equation (6.6b) – the first-order expansion in $\varphi^a(\mathrm{x})$ – produces the gauge transformation of the gluon 4-vector gauge potentials:

$$\delta A^a_\mu = -(D_\mu\, \boldsymbol{\varphi})^a := -c(\partial_\mu \varphi^a) + \tfrac{ig_c}{\hbar c} A^b_\mu\, (\widetilde{Q}_b)_c{}^a\, \varphi^c = -(\partial_\mu \varphi^a) - \tfrac{g_c}{\hbar c} A^b_\mu\, f_{bc}{}^a\, \varphi^c, \tag{6.6e}$$

$$(\widetilde{Q}_b)_c{}^a = i\, f_{bc}{}^a, \qquad \text{where} \qquad [\widetilde{Q}_b, \widetilde{Q}_c] = i\, f_{bc}{}^a\, \widetilde{Q}_a. \tag{6.6f}$$

Here, $\widetilde{Q}_a$ are Hermitian 8×8 traceless matrices that close the same $SU(3)_c$ algebra as the 3×3 matrices Q_a, i.e., the $\frac{1}{2}\boldsymbol{\lambda}_j$'s in relations (A.71). Thus, the operators Q_a – and, in particular, the matrices $\frac{1}{2}\boldsymbol{\lambda}_a$ – provide a 3-dimensional representation of the $SU(3)_c$ group, i.e., a matrix action of the $SU(3)_c$ group upon the 3-dimensional vector space (6.1). In turn, $\widetilde{Q}_a$ provide an 8-dimensional representation of the $SU(3)_c$ group – a matrix action of the $SU(3)_c$ group upon the 8-dimensional vector space $\{\boldsymbol{\varphi}(\mathrm{x}) := \varphi^a(\mathrm{x})\, Q_a\}$.[2]

[2] To be precise, this matrix-valued function $\boldsymbol{\varphi}(\mathrm{x})$ represents a vector space in every spacetime point x; their union forms a so-called *vector bundle* over spacetime. Gauge theories are therefore properly described by the geometry of vector bundles and their connections, here represented by the matrix 4-vector gauge potentials $\mathbb{A}_\mu$.

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Comment 6.1 *In the general case, when the G-covariant derivative D_μ acts upon functions $f_{\mathcal{A}}(\mathrm{x})$ that span the d-dimensional representation of the group G, so $\mathcal{A} = 1, \ldots, d$, we have that*

$$\big(D_\mu f(\mathrm{x})\big)_{\mathcal{A}} := \big(\partial_\mu f_{\mathcal{A}}(\mathrm{x})\big) + \tfrac{ig_G}{\hbar c} A^a_\mu(\mathrm{x})\,[Q_a]_{\mathcal{A}}{}^{\mathcal{B}}\, f_{\mathcal{B}}(\mathrm{x}), \tag{6.7}$$

where the $d \times d$ matrices Q_a generate the group G, the 4-vectors $A^a_\mu(\mathrm{x})$ are the gauge potentials, and g_G is the magnitude of the charge of the corresponding gauge interaction.

Comparing with electrodynamics, recall that the transformation operators U_φ commute and imply the result (5.89)

$$D'_\mu = U_\varphi\, D_\mu\, U_\varphi^{-1} \quad \Rightarrow \qquad A'_\mu = A_\mu - c(\partial_\mu \lambda). \tag{6.8}$$

For the non-abelian group of chromodynamics, $SU(3)_c$, we have (expanding equation (6.6c) only to first order in φ^a)

$$D'_\mu = U_{\boldsymbol{\varphi}}\, D_\mu\, U_{\boldsymbol{\varphi}}^{-1} \quad \Rightarrow \quad (A')^a_\mu = A^a_\mu - c(D_\mu \varphi^a) = A^a_\mu - c(\partial_\mu \varphi^a) + \tfrac{g_c}{\hbar} A^b_\mu\, f_{bc}{}^a\, \varphi^c. \tag{6.9}$$

Also, in electrodynamics we have

$$F_{\mu\nu}(A') = F_{\mu\nu}(A), \tag{6.10}$$

because the fields $\vec{E}, \vec{B}$ are *invariant* with respect to the action of the electromagnetic $U(1)$ gauge transformation (5.14a) [☞ discussion of the definitions (5.15)]. In the non-abelian case, however, direct computation shows that

$$\big(\partial_\mu (A')^a_\nu - \partial_\nu (A')^a_\mu\big) \neq (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu), \tag{6.11}$$

and even

$$\big(\partial_\mu (A')^a_\nu - \partial_\nu (A')^a_\mu\big) \neq U_{\boldsymbol{\varphi}} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) U_{\boldsymbol{\varphi}}^{-1}. \tag{6.12}$$

Note, however, that both in electrodynamics and in chromodynamics the derivatives D_μ are by definition covariant:

$$U(1): \quad D'_\mu = U_\varphi\, D_\mu\, U_\varphi^{-1}, \qquad SU(3): \quad D'_\mu = U_{\boldsymbol{\varphi}}\, D_\mu\, U_{\boldsymbol{\varphi}}^{-1}, \tag{6.13}$$

that is, the change by means of a similarity transformation. It then follows that arbitrary (operatorial) polynomials in the D_μ's are also covariant. Finally, in electrodynamics we have that

$$\big[\, D_\mu\,,\, D_\nu \,\big] = \big[\, \partial_\mu + \tfrac{iq}{\hbar c} A_\mu\,,\, \partial_\nu + \tfrac{iq}{\hbar c} A_\nu \,\big] = +\tfrac{iq}{\hbar c}(\partial_\mu A_\nu - \partial_\nu A_\mu) = \tfrac{iq}{\hbar c} F_{\mu\nu}. \tag{6.14}$$

This result provides an interpretation of the fields $\vec{E}, \vec{B}$ (components of the $F_{\mu\nu}$ tensor) as curvatures in the geometry followed by electrically charged particles.[3] Indeed, in the presence of an electromagnetic field, electrically charged particles move in trajectories of which the curvature is determined by the fields $\vec{E}, \vec{B}$, i.e., the components of the $F_{\mu\nu}$ tensor. Nudged by this result, and the fact that all formal operatorial functions of $SU(3)_c$-covariant derivatives D_μ will also be $SU(3)_c$-covariant, we define

$$\begin{aligned} \mathbb{F}_{\mu\nu} &:= \tfrac{\hbar c}{ig_c}\big[\, D_\mu\,,\, D_\nu \,\big] = \tfrac{\hbar c}{ig_c}\big[\, \partial_\mu + \tfrac{ig_c}{\hbar c} A^b_\mu\, Q_b\,,\, \partial_\nu + \tfrac{ig_c}{\hbar c} A^c_\nu\, Q_c \,\big] \\ &= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) Q_a + \tfrac{\hbar c}{ig_c} (\tfrac{ig_c}{\hbar c})^2\, A^b_\mu\, A^c_\nu\, [Q_b, Q_c] \;=\; F^a_{\mu\nu} Q_a, \end{aligned} \tag{6.15a}$$

$$F^a_{\mu\nu} := (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - \tfrac{g_c}{\hbar c}\, f^a{}_{bc}\, A^b_\mu\, A^c_\nu, \tag{6.15b}$$

where we used the defining relation (A.70) of the $SU(3)$ group generators.

[3] Many a Student may find this interpretation unusual. However, Chapter 9 about gravity will, hopefully, clarify: The commutator of G-covariant derivatives provides the curvature stemming from G-gauge symmetry.

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Comment 6.2 *The interaction parameter, g_c, is, in the literature, often absorbed by redefining the gluon 4-vector gauge potential, $g_c A^a_\mu \mapsto A^a_\mu$, for visibility and ease of computing. In final expressions, however, factors of g_c must be returned for comparison with experiments.*

It follows that this *matrix* $\mathbb{F}_{\mu\nu}$ transforms covariantly, as expected:

$$\begin{aligned}\mathbb{F}_{\mu\nu} \;\to\; \mathbb{F}'_{\mu\nu} &:= \tfrac{i\hbar c}{g_c}\big[\, D'_\mu \,,\, D'_\nu \,\big] = \tfrac{i\hbar c}{g_c}\big[\, U_{\boldsymbol{\varphi}} D_\mu U_{\boldsymbol{\varphi}}^{-1} \,,\, U_{\boldsymbol{\varphi}} D_\nu U_{\boldsymbol{\varphi}}^{-1} \,\big] = \tfrac{i\hbar c}{g_c} U_{\boldsymbol{\varphi}}\big[\, D_\mu \,,\, D_\nu \,\big] U_{\boldsymbol{\varphi}}^{-1} \\ &= U_{\boldsymbol{\varphi}}\, \mathbb{F}_{\mu\nu} U_{\boldsymbol{\varphi}}^{-1}. \end{aligned} \tag{6.16}$$

Thus, the $\mathbb{F}_{\mu\nu}$ tensor is in general covariant – but not invariant – with respect to the action of non-abelian (non-commutative) symmetries such as $SU(3)_c$. Only the field tensor of an abelian symmetry is invariant with respect to the action of this symmetry, as is the case with the electromagnetic field tensor $F_{\mu\nu}$, which is invariant with respect to the $U(1)_Q$ symmetry.

The relation (6.15) also implies that these matrix-represented covariant operators D_μ act upon other matrix-represented quantities by means of commutation. This implies that the covariant derivative of the gauge field, $\mathbb{F}_{\mu\nu}$, itself equals

$$D_\mu(\mathbb{F}_{\nu\rho}) = [D_\mu, \mathbb{F}_{\mu\nu}] = \tfrac{\hbar c}{i g_c}\big[\, D_\mu \,, [\, D_\nu \,,\, D_\rho \,]\big]. \tag{6.17}$$

Using the so-called Jacobi identity,

$$\big[\, A \,, [\, B \,,\, C \,]\big] + \big[\, B \,, [\, C \,,\, A \,]\big] + \big[\, C \,, [\, A \,,\, B \,]\big] \equiv 0, \tag{6.18}$$

the relation (6.17) implies that

$$\varepsilon^{\mu\nu\rho\sigma} D_\mu(\mathbb{F}_{\nu\rho}) = \tfrac{\hbar c}{i g_c}\varepsilon^{\mu\nu\rho\sigma}\big[\, D_\mu \,, [\, D_\nu \,,\, D_\rho \,]\big] = 0, \tag{6.19}$$

which generalizes the Bianchi identity (5.87) for electrodynamics.

6.1.2 The Lagrangian density for chromodynamics

Since the $SU(3)$ generators are Hermitian traceless matrices, $\mathrm{Tr}[Q_a] = 0$, it is also true that

$$\mathrm{Tr}[\mathbb{F}_{\mu\nu}] = F^a_{\mu\nu}\, \mathrm{Tr}[Q_a] = 0. \tag{6.20}$$

However, as the trace of a product of two (Hermitian or not) traceless matrices need not be zero, there is no group-theoretical reason for $\mathrm{Tr}[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}]$ to vanish. In turn, the "trace" function is invariant with respect to similarity transformations of its argument:

$$\mathrm{Tr}[\mathbb{X}] \;\to\; \mathrm{Tr}[\mathrm{S}\,\mathbb{X}\,\mathrm{S}^{-1}] = \mathrm{Tr}[\mathbb{X}\,\mathrm{S}^{-1}\,\mathrm{S}] = \mathrm{Tr}[\mathbb{X}]. \tag{6.21}$$

It follows that

$$\begin{aligned}\mathrm{Tr}[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}] \;\to\; \mathrm{Tr}[\mathbb{F}'_{\mu\nu}\mathbb{F}'^{\,\mu\nu}] &= \mathrm{Tr}[U_{\boldsymbol{\varphi}}\mathbb{F}_{\mu\nu}U_{\boldsymbol{\varphi}}^{-1}U_{\boldsymbol{\varphi}}\mathbb{F}^{\mu\nu}U_{\boldsymbol{\varphi}}^{-1}] = \mathrm{Tr}[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}U_{\boldsymbol{\varphi}}^{-1}\,U_{\boldsymbol{\varphi}}] \\ &= \mathrm{Tr}[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}] \end{aligned} \tag{6.22}$$

is invariant with respect to $SU(3)_c$ transformations. Up to a suitably chosen sign and coefficient [☞ Exercises 5.1.3 and 5.1.4 on p. 171], this then provides a Lagrangian density for $SU(3)_c$ gluons, analogous to the Lagrangian density for photons (5.76). Lorentz-invariance is evident since $\mathbb{F}_{\mu\nu}$ is a rank-2 tensor, and $\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}$ is a scalar contraction, just as in electrodynamics [☞ Digression 5.7 on p. 183].

A Lagrangian density that, via Hamilton's principle of minimal action, produces the Dirac equation for quarks that interact with gluons is obtained by direct generalization of the Lagrangian

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density (5.118). That is, the $\Psi(x)$ representing an electron is replaced by $\Psi^{\alpha}_{n}(x)$, which represents the nth species (flavor) of quark and of the color α. Using the notation (6.5), we then have

$$\begin{aligned}\mathscr{L}_{QCD} &= \sum_n \mathrm{Tr}\left[\overline{\mathbf{\Psi}}_n(x)\left[i\hbar c\not{\partial} - m_n c^2\right]\mathbf{\Psi}_n(x)\right] - \tfrac{1}{4}\,\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right] \\ &= \sum_n \overline{\Psi}_{\alpha\, n}(x)\left[i\boldsymbol{\gamma}^{\mu}\left(\hbar c\,\delta^{\alpha}_{\beta}\partial_{\mu} + i g_c\, A^{a}_{\mu}\left(\tfrac{1}{2}\lambda_a\right)^{\alpha}{}_{\beta}\right) - m_n c^2\delta^{\alpha}_{\beta}\right]\Psi^{\beta}_{n}(x) - \tfrac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a}. \end{aligned} \tag{6.23}$$

As in QED, variation by A^{a}_{μ} yields the equations of motion akin to Gauss's law:

$$D_{\mu}F^{a\,\mu\nu} = g_c\sum_n \overline{\Psi}_{n\alpha A}(\gamma^{\nu})^{A}{}_{B}\left(\tfrac{1}{2}\lambda^{a}\right)^{\alpha}{}_{\beta}\Psi^{\beta B}_{n}, \tag{6.24}$$

where the right-hand side expression may be identified as the quark contribution to the color current:

$$j^{a\,\mu}_{(q)} := g_c\sum_n \overline{\Psi}_{n\alpha A}(\gamma^{\mu})^{A}{}_{B}\left(\tfrac{1}{2}\lambda^{a}\right)^{\alpha}{}_{\beta}\Psi^{\beta B}_{n}. \tag{6.25}$$

However, the left-hand side of equation (6.24) contains terms nonlinear in A^{a}_{μ}, by which this differs fundamentally from equation (5.78). For example, in electrodynamics it is true that

$$\partial_{\nu}j^{\nu}_{e} = \frac{4\pi\epsilon_0 c}{4\pi}\partial_{\nu}\partial_{\mu}F^{\mu\nu} \equiv 0, \qquad \text{since} \qquad F_{\mu\nu} = -F_{\nu\mu}, \tag{6.26}$$

which then immediately produces the continuity equation, i.e., charge conservation. For the equations of motion (6.24) this argument is not true:

$$\partial_{\nu}j^{a\,\nu}_{(q)} = \partial_{\nu}D_{\mu}F^{a\,\mu\nu} \neq 0. \tag{6.27}$$

Digression 6.2 The quark current $j^{a\,\nu}_{(q)}$ does satisfy a gauge-covariant version of the continuity equation:

$$D_{\nu}j^{a\,\nu}_{(q)} \overset{(6.24)}{=} D_{\nu}D_{\mu}F^{a\,\mu\nu} = -\tfrac{1}{2}[D_{\mu},D_{\nu}]F^{a\,\mu\nu} \overset{(6.15)}{=} -\tfrac{1}{2}f^{a}{}_{bc}F^{b}_{\mu\nu}F^{c\,\mu\nu} = 0, \tag{6.28a}$$

where we used the definition of the quark current (6.25), that $D_{\nu}D_{\mu}F^{a\,\mu\nu} = -D_{\mu}D_{\nu}F^{a\,\mu\nu}$ because of the antisymmetry $F^{\alpha\,\mu\nu} = -F^{a\,\nu\mu}$, as well as that $f^{a}{}_{bc} = -f^{a}{}_{cb}$ is antisymmetric with respect to the exchange $b \leftrightarrow c$, whereas $F^{a}_{\mu\nu}F^{b\,\mu\nu} = F^{a\,\mu\nu}F^{b}_{\mu\nu} = F^{b}_{\mu\nu}F^{a\,\mu\nu}$ is symmetric. However, the equation (6.28a) does not imply (purely quark) color conservation; following the computation (2.67) now produces

$$\begin{aligned} 0 = D_{\mu}j^{a\,\mu}_{(q)} &= \partial_{\mu}j^{a\,\mu}_{(q)} - \tfrac{g_c}{\hbar c}f^{a}{}_{bc}\,A^{b}_{\mu}\,j^{c\,\mu}_{(q)} \\ \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_V \mathrm{d}^3\vec{r}\,j^{a\,0}_{(q)}\Big) &= -\oint_{\partial V}\mathrm{d}^2\vec{r}\cdot\vec{j}^{\,a}_{(q)} + \tfrac{g_c}{\hbar c}f^{a}{}_{bc}\Big(\int_V \mathrm{d}^3\vec{r}\,A^{b}_{\mu}\,j^{c\,\mu}_{(q)}\Big), \end{aligned} \tag{6.28b}$$

where the additional right-hand side term does not simplify and certainly does not vanish in general.

However, by the example of (6.6e)–(6.6f),

$$D_{\mu}F^{a\,\mu\nu} = \partial_{\mu}F^{a\,\mu\nu} - \tfrac{g_c}{\hbar c}f_{bc}{}^{a}\,A^{b}_{\mu}F^{c\,\mu\nu}, \tag{6.29}$$

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and moving the second term, $-\frac{g_c}{\hbar c} f_{bc}{}^a A^b_\mu F^{c\,\mu\nu}$, from the so-written left-hand side of the relation (6.24) to its right-hand side, we obtain

$$D_\mu F^{a\,\mu\nu} = j^{a\,\nu}_{(q)} \quad \Rightarrow \quad \partial_\mu F^{a\,\mu\nu} = J^{a\,\nu}_{(c)} \quad \Rightarrow \quad \partial_\nu J^{a\,\nu}_{(c)} = 0, \tag{6.30}$$

since $F^{a\,\mu\nu} = -F^{a\,\nu\mu}$ but $\partial_\nu \partial_\mu = +\partial_\mu \partial_\nu$. Here,

$$J^{a\,\nu}_{(c)} := j^{a\,\mu}_{(q)} + \tfrac{g_c}{\hbar c} f_{bc}{}^a A^b_\mu F^{c\,\mu\nu}, \tag{6.31}$$

$$Q^a_{(c)} := \int \mathrm{d}^3\vec{r}\; J^{a\,0}_{(c)} = g_c \int \mathrm{d}^3\vec{r}\; \Big(\sum_n [\overline{\Psi}_n \boldsymbol{\gamma}^\mu \tfrac{1}{2}\boldsymbol{\lambda}^a \Psi_n] \;+\; \tfrac{1}{\hbar c} f_{bc}{}^a A^b_\mu F^{c\,\mu\nu} \Big) \tag{6.32}$$

are, respectively, the chromodynamical (gauge) current density for which the continuity equation, i.e., color charge conservation, holds, and the corresponding chromodynamics (gauge) charge $Q^a_{(c)}$ that is conserved in time according to Noether's theorem.

Conclusion 6.1 *The continuity equation for the (chromodynamics) current (6.31), i.e., the conservation law for the (chromodynamical "color") charge (6.32) is guaranteed by the antisymmetry of the tensor (of chromodynamics) fields (6.15). This conclusion holds for all gauge theories.*

In contrast to this qualitative and conceptual similarity in all gauge theories, the specific results (6.15)–(6.32) also indicate two fundamental differences in comparison with electromagnetism:

Conclusion 6.2 *The chromodynamics (non-abelian gauge) field tensor $F^a_{\mu\nu}$ is nonlinear in the gluon 4-vector potentials A^a_μ. By contrast, the electromagnetic field tensor $F_{\mu\nu}$ is a linear function of the photon 4-vector potential.*

Conclusion 6.3 *The chromodynamics (non-abelian gauge) current (6.31) and corresponding charge (6.32) have contributions both from quarks and from gluons! By contrast, photons have no electromagnetic charge and do not contribute to the electric current.*

Example 6.1 For illustration, consider the $SU(3)_c$-covariant equations of motion (6.24) where we fix $\nu = 0$, and where we use equation (6.29):

$$\partial_\mu F^{a\,\mu 0} - \tfrac{g_c}{\hbar c} f^a{}_{bc} A^b_\mu F^{c\,\mu 0} = j^{a\,0}_{(q)}, \qquad a, b, c = 1, \ldots, 8. \tag{6.33}$$

Just as in electrodynamics [☞ Section 5.2.2], we define

$$\vec{E}^a := \hat{\mathrm{e}}_i F^{a\,i0}, \qquad \rho^a_{(q)} := j^{a\,0}_{(q)}, \qquad \vec{A}^a := -\hat{\mathrm{e}}^i A^a_i, \tag{6.34}$$

where we are free to absorb all numerical factors in these definitions, and where we may fix $A^a_0 = 0$, $a = 1, \ldots, 8$. The equations (6.33) then reduce to

$$\vec{\nabla}\cdot\vec{E}^a = \rho^a_{(q)} - \tfrac{g_c}{\hbar c} f^a{}_{bc} \vec{A}^b \cdot \vec{E}^c, \tag{6.35}$$

where $\rho^a_{(q)}$ is evidently a source of the chromo-electric field $\vec{E}^a$, but where the chromodynamics vector potentials and fields – *of other colors* – themselves contribute to the

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source! Equation (6.35) is the generalization of Gauss's law for non-commutative (non-abelian) charges – here, of the chromodynamics "color." For example, using the concrete values (A.71) of $f^a{}_{bc}$, we see that

$$\vec{\nabla}\cdot\vec{E}^1 = \rho^1_{(q)} - \tfrac{g_c}{\hbar c}\big(2\vec{A}^{[2}\cdot\vec{E}^{3]} + \vec{A}^{[4}\cdot\vec{E}^{7]} + \vec{A}^{[5}\cdot\vec{E}^{6]}\big), \tag{6.36}$$

where $X^{[a}Y^{b]} := \frac{1}{2}(X^aY^b - X^bY^a)$. That is, the indicated (nonlinear!) coupling of the chromodynamics potentials and fields serves as an *additional* source (or sink, depending on the overall sign) for chromodynamics fields, besides the quark source $\rho^a_{(q)}$.

It follows that the equations of motion for non-abelian gauge theory – obtained by varying the Lagrangian density $\mathscr{L}_{QCD}$ (6.23) with respect to A^a_μ and Ψ^α_n:

1. cannot be expressed without explicit use of the 4-vector gauge potentials, A^k_μ,
2. form a nonlinearly coupled system of differential equations.

Because of Conclusion 6.2 above, the chromodynamics generalizations of the electric and the magnetic fields are rarely used in quantum chromodynamics. Chromodynamics is expressed using the 4-vector potentials A^a_μ, the quanta of which are interpreted as gluons. Conclusion 6.3 indicates a fundamentally larger complexity and technical demand in chromodynamics – and then also the relative simplicity of electrodynamics in comparison with chromodynamics.[4]

With this in mind, the fact that the exploration of chromodynamics is still a very active research field should not come as a surprise. In about four decades, many different approaches in this exploration have been developed from the need to "extract" from this conceptually successful theoretical model concrete quantitative predictions for precise comparison with experiments, but also for better theoretical understanding of the model itself. Among these approaches, may it suffice here to mention three:

Lattice QCD In this approach, the otherwise continuous spacetime is replaced by a lattice of a small spacing. The equations of motion are then solved numerically, and one estimates the forms of those solutions in the limit when the spacetime lattice spacing tends to zero.

Large-*N* QCD Since no experiment can identify any one of the colors in any one real physical process, the contributions to the physical processes must be summed and averaged over all colors. If N is to denote the number of "colors," summing over colors tends to produce factors of N while averaging tends to incur factors of $\frac{1}{N}$. The contributions to the various processes may thus be classified according to the exponent in the overall factor N^ν. Such re-organizing of the computations sometimes permits summing contributions that are all $O(N^\nu)$, albeit from different orders of perturbation as counted by powers of g_s or $\hbar$, and this produces results not derivable otherwise.

QCD strings The original motivation for introducing strings into the physics of elementary particles was the fact that hadrons (mesons and baryons) in collisions at sufficiently high energies show a structure that appears filamentary in a first approximation. The results from the quark model and quantum chromodynamics soon surpassed the precision of this filamentary approximation. However, recent results in the mathematical analysis of superstrings – and

[4] This insight will hopefully not discourage the Students who are already acquainted with this "relative simplicity" of electrodynamics, and so also the "relative simplicity" of the exam problems in that course.

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especially the so-called AdS/CFT, i.e., gravitation/gauge duality – led to new methods, the application of which to the original problem of hadronic physics produces new results and new avenues for exploration.

To sum up: chromodynamics exhibits the generalization of the Maxwell equations:

$$D_\mu \mathbb{F}^{\mu\nu} = \mathbb{J}^\nu_{(q)} \qquad \text{and} \qquad \varepsilon^{\mu\nu\rho\sigma} D_\mu(\mathbb{F}_{\nu\rho}) = 0, \tag{6.37}$$

where

$$\mathbb{J}^\nu_{(q)} := g_c \Big(\sum_n \overline{\Psi}_{n\alpha A} (\gamma^\mu)^A{}_B \big(\tfrac{1}{2}\lambda^a\big)^\alpha{}_\beta \Psi^{\alpha A}_n \Big) Q_a \tag{6.38}$$

is the quark contribution to the chromodynamics current. This current density, however, does not satisfy the continuity equation and $\int \mathrm{d}^3\vec{r}\, \mathbb{J}^0_{(q)}$ is not a conserved color charge. Instead, there exists a redefinition of the (Gaussian) first half of equations (6.37):

$$\partial_\mu \mathbb{F}^{\mu\nu} = \mathbb{J}^\nu_{(c)}, \qquad \text{where} \qquad \mathbb{J}^\nu_{(c)} := \mathbb{J}^\nu_{(q)} + \tfrac{ig_c}{\hbar c}[\mathbb{A}_\mu, \mathbb{F}^{\mu\nu}], \tag{6.39}$$

so that

$$\partial_\nu \mathbb{J}^\nu_{(c)} = 0, \qquad \text{and so also} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathrm{d}^3\vec{r}\, \mathbb{J}^0_{(c)} = -c \oint_{\partial V} \mathrm{d}^2\vec{\sigma} \cdot \vec{\mathbb{J}}_{(c)}. \tag{6.40}$$

It should be clear that in the application to an abelian (*commutative*) group where $f_{ab}{}^c = 0$, the relations (6.37)–(6.30) reduce to the Maxwell equations, the definition of the electric current density, the continuity equation and the electric charge conservation in electrodynamics, respectively.

6.1.3 Exercises for Section 6.1

✎ ***6.1.1*** *Expanding the $\sigma = 0$ component of the system of equations (6.19), obtain the chromodynamic equivalent of Gauss's law for the chromomagnetic field, and show that the nonlinear coupling of gluons also provides an effective chromomagnetic source term in this equation.*

✎ ***6.1.2*** *Determine the gauge covariance of equation (6.24); prove that the left-hand side and the right-hand side of the equality both transform the same.*

✎ ***6.1.3*** *As in the previous exercise, determine separately the gauge covariance of the left-hand side and the right-hand side of the equality $\partial_\mu F^{a\,\mu\nu} = J^{a\,\nu}_{(c)}$ in the result (6.30), where $J^{a\,\nu}_{(c)}$ is defined by the equation (6.31).*

✎ ***6.1.4*** *As in the previous exercise and using the definition (6.31), determine the gauge covariance of the chromodynamics continuity equation $\partial_\nu J^{a\,\nu}_{(c)} = 0$ in the result (6.30).*

6.2 Concrete calculations

Conceptually, quantum-chromodynamics processes are analyzed in the same way as the quantum-electrodynamics ones, via computations that begin with the rules for Feynman diagrams. Adapting Procedure 5.2 on p. 193, we then have the analogous algorithm. However, QCD computations for diagrams with closed loops require exceptional care, additional rules and even additional, so-called

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ghost fields with so-called BRST symmetry[5] – which is beyond our present scope. The presently given algorithm therefore suffices only for chromodynamics diagrams with no gluon loops.

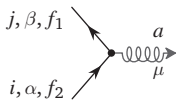
Procedure 6.1 *The contribution to the amplitude $\mathfrak{M}$ corresponding to a given Feynman diagram (with no gluon loops) for the chromodynamics processes with quarks, antiquarks and gluons is computed following the algorithm [☞ textbooks [445, 425, 586] for a derivation]:*

1. ***Notation***
   (a) *Energy–momentum: Denote incoming and outgoing 4-momenta by $p_1, p_2, \ldots$, and the spins $s_1, s_2, \ldots$ Denote the "internal" 4-momenta (assigned to lines that connect two vertices inside the diagram) by $q_1, q_2, \ldots$*
   (b) *Orientation: For a spin-$\frac{1}{2}$ particle, orient the line in the 4-momentum direction, oppositely for antiparticles. Orient external gluon lines in the direction of time (herein, upward). Orient the internal gluon lines arbitrarily, but use the so-chosen orientation consistently.*
   (c) *Polarization: Assign every external line the polarization factor:*

Spin-$\frac{1}{2}$ quark	*incoming*	$u^s_f \chi^\alpha$	s = *spin projection* $= \uparrow, \downarrow$ α = *quark color* $= r, y, b$
	outgoing	$\overline{u}_{f,s}\, \chi^\dagger_\alpha$	f = *quark flavor:* $u, d, s, \ldots$
Spin-$\frac{1}{2}$ antiquark	*incoming*	$\overline{v}_{f,s}\, \chi^\dagger_\alpha$	(*$\simeq$ spin-$\frac{1}{2}$ quark, travels backwards in time*)
	outgoing	$v^s_f \chi^\alpha$	
Gluon	*incoming*	$\epsilon^\mu \chi^a$	$\epsilon^\mu p_\mu = 0$ *and* $\epsilon^0 = 0$
	outgoing	$\epsilon^{\mu *} \chi^{a *}$	

(6.41)

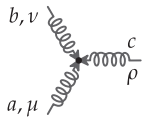
2. ***Vertices*** *To each vertex assign the factor according to type:*
   (a) *Quark–gluon vertex:*[6]



$$\rightarrow \quad -ig_c \boldsymbol{\gamma}^\mu \, \delta^{f_1}_{f_2} \left(\tfrac{1}{2}\lambda_a\right)^\beta{}_\alpha . \tag{6.42}$$

This factor clearly corresponds to the term $-g_c \overline{\Psi}_\alpha \boldsymbol{\gamma}^\mu A^a_\mu (\lambda_a)^\alpha{}_\beta \Psi^\beta$ in the Lagrangian (6.23), and represents the elementary gluon–quark interaction.

   (b) *3-gluon vertex:*



$$\rightarrow \quad \begin{aligned} -g_c f^{abc} [&\eta_{\mu\nu}(k_1 - k_2)_\rho \\ &+ \eta_{\nu\rho}(k_2 - k_3)_\mu \\ &+ \eta_{\rho\mu}(k_3 - k_1)_\nu]. \end{aligned} \tag{6.43}$$

[5] The name of this symmetry is an acronym from the names of the discoverers: C. M. Becchi, A. Rouet and R. Stora [44], and I. V. Tyutin [526]. The method of using ghost fields itself is usually called after L. Faddeev and V. Popov, the physicists who were among the first to use the method; B. DeWitt, who published very similar ideas at the same time but in a technically much more demanding fashion, is unfortunately almost never cited in the invention of this method. Unlike electrodynamics, the non-abelian nature of the gluon interactions in quantum chromodynamics unavoidably couples all degrees of freedom in the gluon 4-vectors A^a_μ, so that the unphysical degrees of freedom (the longitudinal polarization and the temporal component) cannot be consistently eliminated. However, the *contributions* of these unphysical degrees of freedom may be consistently eliminated by introducing ghost fields and reducing correspondingly the gauge symmetry to the BRST nilpotent symmetry. This level of technical details is beyond the scope of this book; see, e.g., Refs. [425, 123, 586, 316, 277] and especially the texts [268, 555, 484, 496, 589, 590].

[6] In traditional normalization, just as *halves* of Pauli matrices generate the $SU(2)$ group, so do *halves* of Gell-Mann matrices (A.71) generate the $SU(3)$ group.

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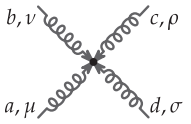
This factor corresponds to the terms in the Lagrangian (6.23) that are 3-linear in A^a_μ, and represents the elementary interaction of three gluons.

(c) *4-gluon vertex:*



$$\rightarrow \quad \begin{aligned} -ig_c^2[&f^{abe}f^{cd}{}_e(\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\sigma}) \\ &+f^{ace}f^{db}{}_e(\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma}) \\ &+f^{ade}f^{bc}{}_e(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\rho}\eta_{\nu\sigma})]. \end{aligned} \tag{6.44}$$

This factor corresponds to the terms in the Lagrangian (6.23) that are 4-linear in A^a_μ, and represents the elementary interaction of four gluons.

3. ***Propagators*** *To each internal line with the jth 4-momentum assign the factor*

$$\textit{quark:}\quad \underset{n,\alpha \quad \mathrm{q}_j \quad n',\beta}{\longrightarrow} \quad \rightarrow \quad \frac{i\delta^{n,n'}\delta^\beta_\alpha}{\not{\mathrm{q}}_j - m_j c} = i\delta^{n,n'}\delta^\beta_\alpha \frac{\not{\mathrm{q}}_j + m_j c\mathbb{1}}{\mathrm{q}_j^2 - m_j^2 c^2}, \tag{6.45}$$

$$\textit{gluon:}\quad \underset{\mu,a \quad \mathrm{q}_g \quad \nu,b}{\sim\!\!\sim\!\!\sim\!\!\rightarrow} \quad \rightarrow \quad -i\frac{\eta_{\mu\nu}}{\mathrm{q}_g^2}\delta^{ab}. \tag{6.46}$$

As internal lines depict virtual particles, $\not{\mathrm{q}}_j \neq m_j c$ and $\mathrm{q}_g^2 \neq 0$, respectively [☞ Tables C.7 on p. 529 and C.8 on p. 529]. Up to multiplicative coefficients, these factors also stem from the Lagrangian (6.23); these are Fourier transforms of the Green functions for the differential operators $\not{\partial}$ and $D^{\mu\nu}_{ab}$ in $\overline{\Psi}\not{\partial}\Psi := -\sum_n \overline{\Psi}_{n,\alpha}[i\hbar c\not{\partial} - mc^2]\Psi^\alpha_n$ and $A^a_\mu D^{\mu\nu}_{ab} A^b_\nu :\simeq -\frac{1}{4}\mathring{F}^a_{\mu\nu}\mathring{F}^{\mu\nu}_a$, respectively, where $\mathring{F}^a_{\mu\nu} := (\partial_\mu A^a_\nu - \partial_n A^a_\nu)$ is the so-called linearization of the field $F^a_{\mu\nu}$.

4. ***Energy–momentum conservation*** *To each vertex assign a factor $(2\pi)^4\delta^4(\sum_j \mathrm{k}_j)$, where k_j are 4-momenta that enter the vertex. 4-momenta that* ***leave*** *the vertex have a negative sign except for external spin-$\frac{1}{2}$ antiparticles, since they are equivalent to particles that move backwards in time.*
5. ***Integration over 4-momenta*** *Internal lines correspond to virtual particles and their 4-momenta are unknown; these variables must be integrated: $\int \frac{d^4\mathrm{q}_j}{(2\pi)^4}$.*
6. ***Reading off the amplitude*** *The foregoing procedure yields the result*

$$-i\,\mathfrak{M}\,(2\pi)^4\delta^4(\sum_j \mathrm{p}_j), \tag{6.47}$$

where the factor $(2\pi)^4\delta^4(\sum_j \mathrm{p}_j)$ represents the 4-momentum conservation for the entire process, and where the amplitude (matrix element) $\mathfrak{M}$ is read off.

7. ***Fermion loops*** *To each fermion loop (closed line) assign a factor -1. A mathematically rigorous derivation of this rule follows from Feynman's approach using path integrals, which is far beyond the scope of this book [☞ booklet [166] for an intuitive albeit not entirely rigorous explanation, [434, Vol. 1, Appendix A] for a serious introduction, and [165] for the original reference].*
8. ***Antisymmetrization*** *Since the amplitude of the process must be antisymmetric in pairs of identical (external) fermions, the partial amplitudes that differ only in the exchange of two identical external fermions must have the relative sign -1.*

As in Section 3.3.4, one draws all Feynman diagrams that contribute at the desired order in g_c, and then computes the (partial) amplitudes for each of the diagrams. The algebraic sum of these contributions yields the total amplitude, which is then inserted in formulae (3.112) and (3.114) for decays and scatterings, respectively.

Fermion loops (closed lines) will be discussed at the end of this section.

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Digression 6.3 Fermion wave-functions, as a whole, must mutually anticommute. When "factorizing" (4.123), $\Psi = \sum_i \Psi_i(\vec{r},t)\,\chi_i(\text{spin})\,\chi_i(\text{flavor})\,\chi_i(\text{color})$, an odd number of factors is anticommuting, and the choice is in principle arbitrary. However, because of the spin-statistics theorem, herein we consistently choose $\chi_i(\text{spin})$ to be anticommuting for half-integral spin, and the other factors to be commutative functions [☞ also Digression 10.2 on p. 360].

Unlike the computations in Sections 3.3.4 and 5.3.1–5.3.3 where the ultimate goal was to compute the lifetime for decays or the differential and total effective cross-section for collisions, for chromodynamics interactions we cannot finalize the computation. Since quarks cannot be extracted from hadrons, detectors cannot register individual quarks, so that, e.g., an elastic collision $u + d \;\to\; u + d$ cannot be detected independently of the hadronic bound states of which these quarks are the building blocks. Thus, the (differential) effective cross-section for this collision does not have a physical meaning, as it cannot be compared with experiments.

However, the amplitude for chromodynamics processes does have a physical meaning and may easily be used to compare with concrete experiments, somewhat akin to familiar application of the Wigner–Eckart theorem A.3 on p. 475. That is, in a hadronic process such as the elastic collision

$$\big[p^+ = (u,u,d)\big] + \big[n^0 = (u,d,d)\big] \to \big[p^+ = (u,u,d)\big] + \big[n^0 = (u,d,d)\big], \tag{6.48}$$

the dominant, $O(g_s^2)$ contribution is chromodynamical and stems from the quark–quark interaction: the dominant contributions to the amplitude of hadronic processes stem from the interaction of one quark from each of the two hadrons; these contributions then add algebraically, depending on the symmetries of the bound states (6.48). Since the u- and d-quark have approximately the same mass, and may have any of the same spin states ($|\frac{1}{2},+\frac{1}{2}\rangle$ and $|\frac{1}{2},-\frac{1}{2}\rangle$) and any of the same colors (red, yellow, blue), the chromodynamics interaction does not distinguish between u–u, u–d and d–d interactions. For the purely QCD contributions, we have

$$\mathfrak{M}^{(\text{QCD})}_{u+u\to u+u} \approx \mathfrak{M}^{(\text{QCD})}_{u+d\to u+d} \approx \mathfrak{M}^{(\text{QCD})}_{d+d\to d+d} \tag{6.49a}$$

up to $O(\frac{m_u-m_d}{m_u+m_d})$ and $O(g_c^4)$ corrections and up to non-QCD contributions such as were discussed in the computations (4.82)–(4.92). Also,

$$\mathfrak{M}^{(\text{QCD})}_{p^++p^+\to p^++p^+} \approx \mathfrak{M}^{(\text{QCD})}_{p^++n^0\to p^++n^0} \approx \mathfrak{M}^{(\text{QCD})}_{n^0+n^0\to n^0+n^0}\,, \tag{6.49b}$$

in the same approximation. The chromodynamic interaction (up to corrections of the next order in magnitude) thus does not differentiate between protons and neutrons, and the result (6.49) is in excellent agreement with concrete experimental data in nuclear physics. Thus, the differences in the binding energy of protons and neutrons within a nucleus may be reduced to differences in spin values,[7] in the spatial factors that also include the orbital angular moments, as well as in the isospin factors [☞ Section 4.3.1].

[7] The (anti)symmetrization is fairly complex in baryons: recall the argument for relations (4.123) and (4.125).

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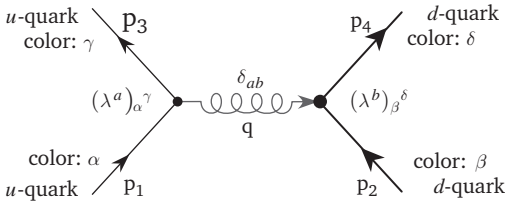
6.2.1 Quark–quark interaction

To describe the interaction between two quarks, assume that they are different, so that only one $O(g_s^2)$ diagram exists:



(6.50)

for which the amplitude is obtained following Procedure 6.1 on p. 232,

$$\mathfrak{M}_{u+d\to u+d} = -\frac{g_s^2}{2}\frac{1}{\mathrm{q}^2}\left[\overline{u}_3\,\boldsymbol{\gamma}^\mu\,u_1\right]\left[\overline{u}_4\,\boldsymbol{\gamma}_\mu\,u_1\right](\chi_3^\dagger\,\boldsymbol{\lambda}^a\,\chi_1)\,(\chi_4^\dagger\,\boldsymbol{\lambda}_a\,\chi_2), \tag{6.51}$$

which is analogous to amplitude (5.131), except that:

1. g_e is replaced with g_c,
2. the color factor, $f_c(3,4|1,2) = \frac{1}{4}(\chi_3^\dagger\boldsymbol{\lambda}^a\chi_1)(\chi_4^\dagger\boldsymbol{\lambda}_a\chi_2)$, is inserted.

The fact that Feynman calculus from quantum electrodynamics is fairly easy to adapt to quantum chromodynamics as well as other kinds of non-abelian gauge interactions has contributed to the popularity of the technique.[8]

Since the electromagnetic interaction of two charged particles of the type (5.129) is known to lead to the Coulomb potential $\frac{\alpha_e\hbar c}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$ and since the result (6.51) differs from (5.131) only in $g_e \to g_c$ and the inserted factor f_c, we conclude that the quantum-chromodynamics interaction of the type (6.50) also leads to a Coulomb-like potential:

$$V_{qq}(r) = f_c\frac{\alpha_s\hbar c}{r}, \tag{6.52}$$

and it only remains to determine the color factor:

$$f_c(3,4|1,2) = \tfrac{1}{4}(\chi_3^\dagger\boldsymbol{\lambda}^a\chi_1)(\chi_4^\dagger\boldsymbol{\lambda}_a\chi_2) = \tfrac{1}{4}\,\chi_{3\gamma}^\dagger\,\chi_{4\delta}^\dagger\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,\chi_1^\alpha\,\chi_2^\beta. \tag{6.53}$$

Use the correspondence between the index- and matrix-notation:

$$\chi^r \leftrightarrow \delta_1^\alpha \leftrightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \chi^y \leftrightarrow \delta_2^\alpha \leftrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \chi^b \leftrightarrow \delta_3^\alpha \leftrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{6.54}$$

From the $SU(3)$ group representation theory result (A.76a) [☞ Appendix A.4.2], we know that the color factors for two (incoming) quarks, $\chi_1^\alpha\chi_2^\beta$, must belong to one of two vector spaces:

1. The antisymmetric triplet (**3***) of states, i.e., the 3-dimensional vector space spanned by two-quark color factors:

$$\begin{aligned}&\{\chi_{12}{}^{[\alpha\beta]} := \tfrac{1}{\sqrt{2}}(\chi_1^\alpha\chi_2^\beta - \chi_1^\beta\chi_2^\alpha),\ \ \alpha,\beta = \mathit{red}, \mathit{yellow}, \mathit{blue} = 1,2,3\}\\ &= \left\{\tfrac{1}{\sqrt{2}}(\delta_1^\alpha\delta_2^\beta - \delta_1^\beta\delta_2^\alpha),\ \tfrac{1}{\sqrt{2}}(\delta_1^\alpha\delta_3^\beta - \delta_1^\beta\delta_3^\alpha),\ \tfrac{1}{\sqrt{2}}(\delta_2^\alpha\delta_3^\beta - \delta_2^\beta\delta_3^\alpha)\right\},\end{aligned} \tag{6.55}$$

[8] The insolubility of quantum chromodynamics stems from the fact that α_s varies with energy much faster than the electrodynamics fine structure parameter, α_e, and oppositely, α_s *diminishes* with energy [☞ Section 6.2.4]. Moreover, perturbative computations indicate that below about 200 MeV, α_s becomes larger than 1, so the perturbative approach to quantum chromodynamics where α_s is the perturbative parameter makes neither practical nor conceptual sense when the interaction energy is less than about 200 MeV [☞ Section 6.3]. Here we then focus on sufficiently high energies.

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where the black subscripts 1 and 2 in the first row indicate the first and second quark, respectively. In the second row, we dispensed with these subscripts,[9] so as not to confuse them with the color labels $1 = r$, $2 = y$ and $3 = b$, which were needed in the second row.

2. The symmetric 6-tuplet (**6**) of states, i.e., the 6-dimensional vector space spanned by two-quark color factors:

$$\begin{aligned}\{\chi_{12}{}^{(\alpha\beta)} &:= \tfrac{1}{\sqrt{(1+\delta_{\alpha\beta})}}(\chi_1^\alpha\chi_2^\beta + \chi_1^\beta\chi_2^\alpha),\ \ \alpha,\beta = r,y,b = 1,2,3\} \\ &= \Big\{(\delta_1^\alpha\delta_1^\beta),\ (\delta_2^\alpha\delta_2^\beta),\ (\delta_3^\alpha\delta_3^\beta),\ \tfrac{1}{\sqrt{2}}(\delta_1^\alpha\delta_2^\beta + \delta_1^\beta\delta_2^\alpha),\ \tfrac{1}{\sqrt{2}}(\delta_1^\alpha\delta_3^\beta + \delta_1^\beta\delta_3^\alpha),\ \tfrac{1}{\sqrt{2}}(\delta_2^\alpha\delta_3^\beta + \delta_2^\beta\delta_3^\alpha)\Big\}.\end{aligned} \tag{6.56}$$

The quantum-mechanical normalization[10] of the color factors was used (so that $\|\chi_{12}{}^{[\alpha\beta]}\|^2 = 1$ as well as $\|\chi_{12}{}^{(\alpha\beta)}\|^2 = 1$ for every choice of α,β) and the numerical identification $\alpha,\beta = r,y,b = 1,2,3$ for the basis in which the Gell-Mann matrices (A.71) are given. For outgoing quarks the Hermitian conjugate factors (6.55)–(6.56) must be used, but note that Hermitian conjugation preserves the (anti)symmetry of the two-particle color factors.

That is, in the process (6.50), the color factor for the incoming quarks (with colors α and β) may be in any linear combination of either the antisymmetrized elements (6.55), or the symmetrized elements (6.56). The color factor for the outgoing quarks (with colors γ and δ) may be – independently of the incoming quarks – in any one of the Hermitian conjugates of those states. In principle then, one must compute the color factors (6.53) for each of the combinations

$$f_c(3,4|1,2) \ = \ f_c(\mathbf{3}_A^*|\mathbf{3}_A^*),\ f_c(\mathbf{3}_A^{*\prime}|\mathbf{3}_A^*),\ f_c(\mathbf{6}_S|\mathbf{3}_A^*),\ f_c(\mathbf{3}_A^*|\mathbf{6}_S),\ f_c(\mathbf{6}_S|\mathbf{6}_S),\ f_c(\mathbf{6}_S'|\mathbf{6}_S), \tag{6.57}$$

where $\mathbf{3}_A^*$ denotes some concrete antisymmetrized state, $\mathbf{6}_S$ denotes some concrete symmetrized state, and prime simply indicates some *other* such concrete state.

Example 6.2 A concrete computation of the first type (6.57), i.e., $f_c(\mathbf{3}_A^*|\mathbf{3}_A^*)$ is done taking, e.g., the red–blue $\frac{1}{\sqrt{2}}(\delta_\gamma^1\delta_\delta^3 - \delta_\delta^1\delta_\gamma^3) \in \mathbf{3}^*$ element:

$$\begin{aligned}\{\tfrac{1}{4}\,(\chi^\dagger_{3\gamma}\,\chi^\dagger_{4\delta})_{\mathbf{3}}\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,(\chi_1^\alpha\,\chi_2^\beta)_{\mathbf{3}^*}\} &\supset \tfrac{1}{4}\,\tfrac{1}{\sqrt{2}}(\delta_\gamma^1\delta_\delta^3 - \delta_\delta^1\delta_\gamma^3)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,\tfrac{1}{\sqrt{2}}(\delta_1^\alpha\delta_3^\beta - \delta_1^\beta\delta_3^\alpha) \\ &= \tfrac{1}{8}\big[\lambda^a{}_1{}^1\,\lambda_{a3}{}^3 - \lambda^a{}_3{}^1\,\lambda_{a1}{}^3 - \lambda^a{}_1{}^3\,\lambda_{a3}{}^1 + \lambda^a{}_3{}^3\,\lambda_{a1}{}^1\big] = \tfrac{1}{4}\big[\lambda^a{}_1{}^1\,\lambda_{a3}{}^3 - \lambda^a{}_3{}^1\,\lambda_{a1}{}^3\big].\end{aligned} \tag{6.58a}$$

The sums over Gell-Mann matrices, $a = 1,\ldots,8$ simplify, as there is only one matrix for which $(\lambda_{a1}{}^1 \neq 0 \neq \lambda_{a3}{}^3)$, and only two matrices for which $(\lambda_{a3}{}^1 \neq 0 \neq \lambda_{a1}{}^3)$:

$$= \tfrac{1}{4}\big[\lambda^8{}_1{}^1\,\lambda_{83}{}^3 - \lambda^4{}_3{}^1\,\lambda_{41}{}^3 - \lambda^5{}_3{}^1\,\lambda_{51}{}^3\big] = \tfrac{1}{4}\big[\tfrac{1}{\sqrt{3}}\cdot\tfrac{-2}{\sqrt{3}} - 1\cdot 1 - i\cdot(-i)\big] = -\tfrac{2}{3}. \tag{6.58b}$$

---

[9] We imply that the first factor in every monomial – whether formally χ^α or the Kronecker symbol – refers to the first quark, and the second factor to the second quark.

[10] In mathematical sources, if such explicit constructions are given at all, one mostly finds combinatorial functions, the normalization of which refers to their use in probability theory. However, wave-functions are not probabilities but probability amplitudes, so the desired normalization mostly requires factors of the type $\frac{1}{\sqrt{2}}$ (for a probability amplitude) instead of $\frac{1}{2}$ (for a probability), etc.

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Similarly, the type $f_c(\mathbf{3}_A^{*\prime}|\mathbf{3}_A^*)$ computation yields

$$\begin{aligned}\{\tfrac{1}{4}\,(\chi_{3\gamma}^\dagger\,\chi_{4\delta}^\dagger)_{\mathbf{3}'}\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,(\chi_1^\alpha\,\chi_2^\beta)_{\mathbf{3}^*}\} &\supset \tfrac{1}{4}\,\tfrac{1}{\sqrt{2}}\big(\delta_\gamma^1\delta_\delta^2-\delta_\delta^1\delta_\gamma^2\big)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,\tfrac{1}{\sqrt{2}}\big(\delta_1^\alpha\delta_3^\beta-\delta_1^\beta\delta_3^\alpha\big)\\ &= \tfrac{1}{8}\big[\lambda^a{}_1{}^1\,\lambda_{a3}{}^2-\lambda^a{}_3{}^1\,\lambda_{a1}{}^2-\lambda^a{}_1{}^2\,\lambda_{a3}{}^1+\lambda^a{}_3{}^2\,\lambda_{a1}{}^1\big]\\ &= \tfrac{1}{4}\big[\lambda^a{}_1{}^1\,\lambda_{a3}{}^2-\lambda^a{}_3{}^1\,\lambda_{a1}{}^2\big]=0,\end{aligned}\tag{6.58c}$$

since there is no Gell-Mann matrix for which $(\lambda_{a1}{}^1\neq 0\neq\lambda_{a3}{}^2)$ or $(\lambda_{a3}{}^1\neq 0\neq\lambda_{a1}{}^2)$. Also, for the $f_c(\mathbf{6}_S|\mathbf{3}_A^*)$ type, one checks

$$\{\tfrac{1}{4}\,(\chi_{3\gamma}^\dagger\,\chi_{4\delta}^\dagger)_{\mathbf{6}}\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,(\chi_1^\alpha\,\chi_2^\beta)_{\mathbf{3}^*}\}\supset \text{ four characteristic cases:}$$

$$\left\{\begin{aligned}&\tfrac{1}{4}\,\big(\delta_\gamma^1\delta_\delta^1\big)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,\tfrac{1}{\sqrt{2}}\big(\delta_1^\alpha\delta_2^\beta-\delta_1^\beta\delta_2^\alpha\big)=\tfrac{1}{4\sqrt{2}}\big[\lambda^a{}_1{}^1\,\lambda_{a2}{}^1-\lambda^a{}_2{}^1\,\lambda_{a1}{}^1\big]=0,&(6.58d)\\ &\tfrac{1}{4}\,\big(\delta_\gamma^3\delta_\delta^3\big)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,\tfrac{1}{\sqrt{2}}\big(\delta_1^\alpha\delta_2^\beta-\delta_1^\beta\delta_2^\alpha\big)=\tfrac{1}{4\sqrt{2}}\big[\lambda^a{}_1{}^3\,\lambda_{a2}{}^3-\lambda^a{}_2{}^3\,\lambda_{a1}{}^3\big]=0,&(6.58e)\\ &\tfrac{1}{4}\,\tfrac{1}{\sqrt{2}}\big(\delta_\gamma^1\delta_\delta^3+\delta_\delta^1\delta_\gamma^3\big)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,\tfrac{1}{\sqrt{2}}\big(\delta_1^\alpha\delta_3^\beta-\delta_1^\beta\delta_3^\alpha\big)\\ &\quad=\tfrac{1}{8}\big[\lambda^a{}_1{}^1\,\lambda_{a3}{}^3-\lambda^a{}_3{}^1\,\lambda_{a1}{}^3+\lambda^a{}_1{}^3\,\lambda_{a3}{}^1-\lambda^a{}_3{}^3\,\lambda_{a1}{}^1\big]=0,&(6.58f)\\ &\tfrac{1}{4}\,\tfrac{1}{\sqrt{2}}\big(\delta_\gamma^1\delta_\delta^2+\delta_\delta^1\delta_\gamma^2\big)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\beta{}^\delta\,\tfrac{1}{\sqrt{2}}\big(\delta_1^\alpha\delta_3^\beta-\delta_1^\beta\delta_3^\alpha\big)\\ &\quad=\tfrac{1}{8}\big[\lambda^a{}_1{}^1\,\lambda_{a3}{}^2-\lambda^a{}_3{}^1\,\lambda_{a1}{}^2+\lambda^a{}_1{}^2\,\lambda_{a3}{}^1-\lambda^a{}_3{}^2\,\lambda_{a1}{}^1\big]=0\,\Big\}.&(6.58g)\end{aligned}\right.$$

The complete collection of values of the function $f_c(\mathbf{6}_S|\mathbf{3}_A^*)$ of course consists of $6\times 3=18$ cases, but these may all be obtained from the above four concrete cases by permuting the values $\alpha,\beta,\gamma,\delta=1,2,3$. It follows that the outgoing pair of quarks is always in the same concrete antisymmetric state as was the incoming pair. (That also follows from the $SU(3)_c$ color conservation, but it is reassuring to confirm this by direct computation.)

Direct computation [☞ Example 6.2 on p. 236, and Exercise 6.2.1] confirms that

$$f_c(\mathbf{3}_A^*|\mathbf{3}_A^*)=-\tfrac{2}{3}\qquad\text{and}\qquad f_c(\mathbf{6}_S|\mathbf{6}_S)=+\tfrac{1}{3},\tag{6.59}$$

while $f_c(\mathbf{3}_A^{*\prime}|\mathbf{3}_A^*)$, $f_c(\mathbf{6}_S|\mathbf{3}_A^*)$, $f_c(\mathbf{3}_A^*|\mathbf{6}_S)$ and $f_c(\mathbf{6}_S'|\mathbf{6}_S)$ vanish for all cases.

Conclusion 6.4 *These results indicate that a gluon exchange between two quarks does not change the color combination for two-quark states.*[11] *Besides, the sign in the result (6.59) indicates the one-gluon exchange chromodynamics force, computed in the standard fashion as* $\vec{F}_{qq}=-\vec{\nabla}(V_{qq})$ *from the relation (6.52), to be:*

1. *attractive if the quark colors are antisymmetrized,*
2. *repulsive if the quark colors are symmetrized.*

Comment 6.3 *The emphasis that this amounts to only a single-gluon exchange contribution to the chromodynamics force is very important: It does not follow that the exchange of*

[11] The computation is of course shown only for the exchange of a single gluon, but its direct iteration is applicable to the exchange of an arbitrary finite number of gluons. Extending this to a formally infinite number of exchanged gluons, including gluon condensation, remains an open issue✍.

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more gluons follows the same regularity, and so it does not follow that the total chromodynamics force follows the same regularity. Several further contributions, however, have been computed and they preserve the qualitative character of the result (6.59).

A baryon, of course, has three quarks, and the options for the color factor are (A.78):

1. totally symmetric, so-called "**10**" (10-dimensional) representation,
2. mixed symmetric, so-called "**8**" (8-dimensional) representation (in two distinct ways),
3. totally antisymmetric, so-called "**1**" (1-dimensional) representation

of the $SU(3)_c$ group, where only the last one is $SU(3)_c$-invariant. Also, only in the last case is the system antisymmetric (i.e., the wave-function of the baryon as a three-particle bound state is antisymmetric) with respect to the exchange of any two quarks. Conclusion 6.4 then indicates that this is the only case in which the chromodynamics force between all quarks in the baryon is attractive.

Also, since $O(g_c^2)$ computations indicate that the chromodynamic interaction is binding (attractive) only when the factor $\chi(\text{color})$ in the factorization (4.123) is totally antisymmetric, it follows that the bound state (i.e., its wave-function) for every baryon must be totally symmetric in the remaining three factors:

$$\Psi(\text{baryon}) = \big[\Psi(\vec{r},t)\,\chi(\text{spin})\,\chi(\text{flavor})\big]_S\,\chi_A(\text{color}).$$

Since the factor $\chi(\text{flavor})$ is determined by the choice of the hadron [☞ Section 4.4] as totally symmetric for the **10**-plet of flavors and mixed symmetric [☞ relation (4.125)] for the **8**-plet, for each of these baryons the symmetries determine the correlation between spin and orbital angular momentum. In ground states, the angular momenta in the three-quark system all vanish, so the spin factor is unambiguously determined to be:

1. spin-$\frac{3}{2}$ and totally symmetric for the decuplet $\{\Delta, \Sigma^*, \Xi^*, \Omega\}$,
2. spin-$\frac{1}{2}$, with a rather more complicated symmetry (4.125) with (4.119)–(4.120) for the octet $\{p^+, n^0, \Lambda, \Sigma, \Xi\}$ of baryons.

Conclusion 6.5 *Furthermore, the chromodynamics interaction between two $SU(3)_c$-invariant bound states cannot happen via the exchange of a single gluon [☞ Example 6.3 on p. 240], but must involve a simultaneous exchange of at least two gluons, and so is of the order of at least $O(g_s^4)$, or a gluon and a quark pair; see process (6.77). Indeed, if the baryon that emits any particle is to remain $SU(3)_c$-invariant both before and after emitting, it follows that the emitted intermediary itself must be $SU(3)_c$-invariant. As none of the eight gluons are $SU(3)_c$-invariant, the intermediary must be an $SU(3)_c$-invariant state composed of at least two gluons or a quark–antiquark pair.*

It then follows that the simplest chromodynamics interaction between two nucleons within an atomic nucleus is about $O(g_c^2)$ times weaker than the strong interaction between two quarks.[12] (This reminds us a little of the fact that the dipole–dipole interaction between two neutral hydrogen atoms is weaker than the Coulomb interaction between the electron and the proton within one atom.)

[12] At this introductory level, we have no means of assessing the contribution to the effective strength of interaction provided by the exchange of quarks between two hadrons. However, the $SU(3)_c$-invariance requirement on the particle mediating the strong interaction between two hadrons clearly forces it to be of higher order than the direct, $SU(3)_c$-variant one-gluon-mediated interaction between quarks.

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Digression 6.4 $SU(3)_c$-invariant states composed entirely of gluons are called "*glueballs*" and in principle may be observed, but no such state has so far been reliably detected. However, all quantum numbers of such purely gluon $SU(3)_c$-invariant bound states are identical to quantum numbers of electrically neutral mesons such as π^0, ρ^0, etc., with which they mix. This mixing makes experimental differentiation of "glueballs" from ordinary mesons extremely difficult, and no "glueball" state has yet been conclusively detected.

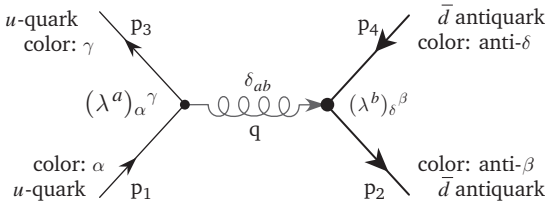
6.2.2 Quark–antiquark interaction

Mesons are much easier to study than baryons, as they are bound states of a quark and an antiquark. However, with this simplification also comes a complication – at least when the meson is neutral with respect to all interactions, so the bound state is of the type

$$\overline{u}\,u + \overline{d}\,d + \overline{s}\,s + \cdots . \tag{6.60}$$

Indeed, now the quark and the antiquark may mutually annihilate. We first consider differently flavored quark–antiquark mesons, where not even virtual annihilation can happen; the next section will consider the possible annihilation in a type (6.60) system.

The amplitude of a single-gluon exchange has a contribution only from one Feynman diagram:



(6.61)

Following Procedure 6.1 on p. 232, and analogously to the result for the first part of (5.147b), we have

$$\mathfrak{M}_{u+\overline{d}\to u+\overline{d}} = -\frac{g_c^2}{4q^2}[\overline{u}_3\boldsymbol{\gamma}^\mu u_1][\overline{v}_2\boldsymbol{\gamma}_\mu v_4]\,(\chi_3^\dagger\,\boldsymbol{\lambda}^a\,\chi_1)\,(\chi_2^\dagger\,\boldsymbol{\lambda}_a\,\chi_4), \tag{6.62}$$

where $q = (p_1 - p_3)$ is the 4-momentum exchange, and the result differs from the electrodynamics one only in that:

1. g_e is replaced by g_c,
2. the color factor, $f_c(3,\overline{4}|1,\overline{2}) = \frac{1}{4}(\chi_3^\dagger\boldsymbol{\lambda}^a\chi_1)(\chi_2^\dagger\boldsymbol{\lambda}_a\chi_4)$, is inserted.

The color factor for the incoming quark–antiquark pair again must belong to one of the two vector spaces:

1. The Hermitian octet (**8**) of states, i.e., the 8-dimensional vector space spanned by the color factors,

$$\begin{aligned}\{\chi_{12}{}^\alpha{}_\beta &= \sqrt{1+\tfrac{1}{2}\delta^\alpha_\beta}(\chi_1^\alpha\chi_{2\beta}^\dagger - \tfrac{1}{\sqrt{3}}\delta^\alpha_\beta\,\mathring{\boldsymbol{\chi}}),\ \ \alpha,\beta = red, yellow, blue = 1,2,3\}\\ &= \Big\{\sqrt{\tfrac{3}{2}}(\delta_1^\alpha\delta_\beta^1 - \mathring{\boldsymbol{\chi}}),\ \sqrt{\tfrac{3}{2}}(\delta_2^\alpha\delta_\beta^2 - \mathring{\boldsymbol{\chi}}),\ \sqrt{\tfrac{3}{2}}(\delta_3^\alpha\delta_\beta^3 - \mathring{\boldsymbol{\chi}}),\\ &\qquad (\delta_1^\alpha\delta_\beta^2),\ (\delta_1^\alpha\delta_\beta^3),\ (\delta_2^\alpha\delta_\beta^1),\ (\delta_2^\alpha\delta_\beta^3),\ (\delta_3^\alpha\delta_\beta^1),\ (\delta_3^\alpha\delta_\beta^2)\Big\},\end{aligned} \tag{6.63}$$

which form a traceless Hermitian matrix, where $\mathring{\boldsymbol{\chi}} := \frac{1}{\sqrt{3}}\,\mathrm{Tr}(\chi_1\chi_2^\dagger) = \frac{1}{\sqrt{3}}(\chi_1^\alpha\chi_{2\alpha}^\dagger)$.

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2. The $SU(3)_c$-invariant $(\mathbf{1})$, where $\chi_{12}{}^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta}\,\mathring{\boldsymbol{\chi}}$ is a multiple of the unit matrix.

Normalization is again quantum mechanical, so $\|\chi_{12}{}^{\alpha}{}_{\beta}\|^2 = 1$ for every choice α, β.

Similarly to the result (6.57), for $u + \overline{d} \to u + \overline{d}$ we have

$$f_c(3,\overline{4}|1,\overline{2}) \;=\; f_c(\mathbf{8}|\mathbf{8}),\; f_c(\mathbf{8}'|\mathbf{8}),\; f_c(\mathbf{8}|\mathbf{1}),\; f_c(\mathbf{1}|\mathbf{8}),\; f_c(\mathbf{1}|\mathbf{1}). \tag{6.64}$$

Also, just as in electrodynamics, the gluon exchange gives rise to a potential of the form

$$V_{q\overline{q}}(r) = -f_c\frac{\alpha_c\hbar c}{r}, \tag{6.65}$$

where the sign is now negative, since the color charges of a quark and an antiquark are "opposite": one is the (chromodynamics) "color" the other the "anticolor."[13]

Example 6.3 To compute the functions $f_c(\mathbf{8}|\mathbf{8})$, $f_c(\mathbf{8}'|\mathbf{8})$ and $f_c(\mathbf{1}|\mathbf{1})$, we pick the simplest particular cases for each; the diligent Student will convince themselves by direct computation that all cases give the same quantitative results.

For $f_c(\mathbf{8}|\mathbf{8})$, the incoming and the outgoing quark–antiquark pair have the same combination of color–anticolor; take, e.g., the red–antiblue $(\delta^1_\gamma\delta^\delta_3) \in \mathbf{8}$ element:

$$\begin{aligned}\{\tfrac{1}{4}\,(\chi^\dagger_{3\gamma}\,\chi^\delta_4)_{\mathbf{8}}\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\delta{}^\beta\,(\chi^\alpha_1\,\chi^\dagger_{2\beta})_{\mathbf{8}}\} &\supset \tfrac{1}{4}\,(\delta^1_\gamma\delta^\delta_3)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\delta{}^\beta\,(\delta^\alpha_1\delta^3_\beta)\\ &= \tfrac{1}{4}\,\lambda^a{}_1{}^1\,\lambda_{a3}{}^3 = \tfrac{1}{4}\,\lambda^8{}_1{}^1\,\lambda_{83}{}^3 = \tfrac{1}{4}\,\tfrac{1}{\sqrt{3}}\,\tfrac{-2}{\sqrt{3}} = -\tfrac{1}{6},\end{aligned} \tag{6.66}$$

since only the eighth Gell-Mann matrix has $(\lambda^a{}_1{}^1 \neq 0 \neq \lambda_{a3}{}^3)$. For $f_c(\mathbf{8}'|\mathbf{8})$ we take, e.g., $(\delta^1_\gamma\delta^\delta_3) \in \mathbf{8}$ and $(\delta^3_\gamma\delta^\delta_1) \in \mathbf{8}'$:

$$\begin{aligned}\{\tfrac{1}{4}\,(\chi^\dagger_{3\gamma}\,\chi^\delta_4)_{\mathbf{8}'}\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\delta{}^\beta\,(\chi^\alpha_1\,\chi^\dagger_{2\beta})_{\mathbf{8}}\} &\supset \tfrac{1}{4}\,(\delta^3_\gamma\delta^\delta_1)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\delta{}^\beta\,(\delta^\alpha_1\delta^3_\beta)\\ &= \tfrac{1}{4}\,\lambda^a{}_1{}^3\,\lambda_{a1}{}^3 = \tfrac{1}{4}(\lambda^4{}_1{}^3\,\lambda_{41}{}^3 + \lambda^5{}_1{}^3\,\lambda_{51}{}^3) = \tfrac{1}{4}\,(1\cdot 1 + (-i)\cdot(-i)) = 0.\end{aligned} \tag{6.67}$$

Since the representation $\mathbf{1}$ has only one dimension, for $f_c(\mathbf{1}|\mathbf{1})$ there is a single contribution:

$$\begin{aligned}&\tfrac{1}{4}\,(\chi^\dagger_{3\gamma}\,\chi^\delta_4)_{\mathbf{1}}\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\delta{}^\beta\,(\chi^\alpha_1\,\chi^\dagger_{2\beta})_{\mathbf{1}}\\ &\quad= \tfrac{1}{4}\,\tfrac{1}{\sqrt{3}}(\delta^1_\gamma\delta^\delta_1 + \delta^2_\gamma\delta^\delta_2 + \delta^3_\gamma\delta^\delta_3)\,(\lambda^a)_\alpha{}^\gamma\,(\lambda_a)_\delta{}^\beta\,\tfrac{1}{\sqrt{3}}(\delta^\alpha_1\delta^1_\beta + \delta^\alpha_2\delta^2_\beta + \delta^\alpha_3\delta^3_\beta)\\ &\quad= \tfrac{1}{12}\,\lambda^a{}_\alpha{}^\gamma\,\lambda_a{}_\gamma{}^\alpha = \tfrac{1}{12}\,\delta_{ab}\,\mathrm{Tr}(\boldsymbol{\lambda}^a\,\boldsymbol{\lambda}^b) = \tfrac{1}{12}\,\delta_{ab}\,2\delta^{ab} = \tfrac{1}{6}\,8 = \tfrac{4}{3},\end{aligned} \tag{6.68}$$

where we used the relation (A.72). This coefficient, $f_c(\mathbf{1}|\mathbf{1}) = \frac{4}{3}$, has shown up in the relation (4.102).

[13] In electrodynamics, of course, there is only one kind of charge – electric – and the opposite charge is simply the negative charge. For chromodynamics colors, "anticolor" is not simply negative "color," but the opposite "color"; i.e., the color that together with the original one produces a colorless, i.e., an $SU(3)_c$-invariant whole. This we may write, e.g., $(\chi_{\alpha\,(\mathrm{red})})^\dagger = (\chi^\dagger)^{\alpha\,(\mathrm{green})}$. We will not use this notational possibility, as it additionally complicates the tensor algebra rules and necessitates printing in color; with the current convention, computations may be followed even in monochromatic printout.

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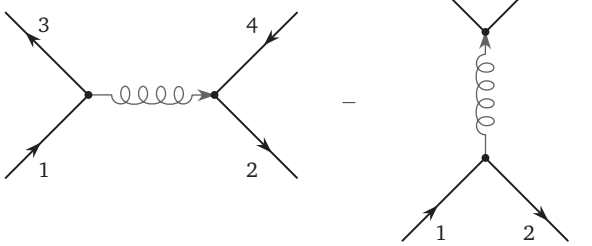
Direct computation shows also that $f_c(\mathbf{8}|\mathbf{1}), f_c(\mathbf{1}|\mathbf{8}) = 0$, and we have:

Conclusion 6.6 *These results show that the single-gluon exchange[14] between a quark and an antiquark preserves the color state: incoming and outgoing quark–antiquark pairs have the same color combination. Besides, the chromodynamics force (6.65) between a quark and an antiquark is*

1. *attractive when both the incoming and the outgoing pair are in the $SU(3)_c$-invariant state, and*
2. *repulsive otherwise.*

6.2.3 Quark–antiquark annihilation

The single-gluon exchange amplitude now has two contributions, corresponding to the two Feynman diagrams:

$\mathfrak{M}_{u+\overline{u}\to u+\overline{u}} =$ 

(6.69)

where the relative minus sign follows from the fact that the amplitude for the second sub-process (the virtual annihilation and re-creation of the $u\,\overline{u}$ pair) equals the first, upon exchanging the incoming antiquark, 2, and the outgoing quark 3 [☞ discussion of the Bhabha scattering and procedure (5.145)]. Adapting the result (5.147b), we have that the amplitude of this process equals

$$\begin{aligned}\mathfrak{M}_{u+\overline{u}\to u+\overline{u}} = &-\frac{g_c^2}{4(\mathrm{p}_1-\mathrm{p}_3)^2}[\overline{u}_3\boldsymbol{\gamma}^\mu u_1][\overline{v}_2\boldsymbol{\gamma}_\mu v_4](\chi_3^\dagger\boldsymbol{\lambda}^a\chi_1)(\chi_2^\dagger\boldsymbol{\lambda}_a\chi_4)\\ &+\frac{g_c^2}{4(\mathrm{p}_1+\mathrm{p}_2)^2}[\overline{v}_2\boldsymbol{\gamma}^\mu u_1][\overline{u}_3\boldsymbol{\gamma}_\mu v_4](\chi_2^\dagger\boldsymbol{\lambda}^a\chi_1)(\chi_3^\dagger\boldsymbol{\lambda}_a\chi_4),\end{aligned} \tag{6.70}$$

where we used that the color factor, f_c, for the first diagram is identical to the factor in the result (6.62), and the factor for the second diagram, $\widetilde{f}_c$, is obtained by swapping $2 \leftrightarrow 3$.

Example 6.4 We will compute one sample value of each of $\widetilde{f}_c(\mathbf{8}|\mathbf{8})$, $\widetilde{f}_c(\mathbf{8}'|\mathbf{8})$ and $\widetilde{f}_c(\mathbf{1}|\mathbf{1})$, and we choose the simplest cases to this end; the diligent Student should verify by direct computation that all cases produce quantitatively the same results. Alternatively, this may also be proven by $SU(3)_c$ group action from the results presented here [☞ Exercise 6.2.2].

For $\widetilde{f}_c(\mathbf{8}|\mathbf{8})$, the incoming and outgoing quark–antiquark pair have the same color–anticolor combination; fix this to be the red–antiblue combination:

$$\begin{aligned}&\left\{\tfrac{1}{4}\,(\chi_{3\gamma}^\dagger\,\chi_4^\delta)_{\mathbf{8}}\,(\lambda^a)_\alpha{}^\beta\,(\lambda_a)_\delta{}^\gamma\,(\chi_1^\alpha\,\chi_{2\beta}^\dagger)_{\mathbf{8}}\right\} \supset \tfrac{1}{4}\,(\delta_\gamma^1\delta_3^\delta)\,(\lambda^a)_\alpha{}^\beta\,(\lambda_a)_\delta{}^\gamma\,(\delta_1^\alpha\delta_\beta^3)\\ &\quad=\tfrac{1}{4}\,\lambda^a{}_1{}^3\,\lambda_{a3}{}^1=\tfrac{1}{4}\,(\lambda^4{}_1{}^3\,\lambda_{43}{}^1+\lambda^5{}_1{}^3\,\lambda_{53}{}^1)=\tfrac{1}{4}\,(1\cdot 1+(-i)\cdot(i))=\tfrac{1}{2},\end{aligned} \tag{6.71}$$

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[14] See Footnote 11 on p. 237.

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since only $\boldsymbol{\lambda}^4$ and $\boldsymbol{\lambda}^5$ have $(\lambda^a{}_1{}^3 \neq 0 \neq \lambda_{a3}{}^1)$. For $\widetilde{f}_c(\mathbf{8}'|\mathbf{8})$ we have, e.g.,

$$\begin{aligned}\{\tfrac{1}{4}\,(\chi^\dagger_{3\gamma}\,\chi^\delta_4)_{\mathbf{8}'}\,(\lambda^a)_\alpha{}^\beta\,(\lambda_a)_\delta{}^\gamma\,(\chi^\alpha_1\,\chi^\dagger_{2\beta})_{\mathbf{8}}\} &\supset \tfrac{1}{4}\,(\delta^3_\gamma\delta^\delta_1)\,(\lambda^a)_\alpha{}^\beta\,(\lambda_a)_\delta{}^\gamma\,(\delta^\alpha_1\delta^3_\beta)\\ &= \tfrac{1}{4}\,\lambda^a{}_1{}^3\,\lambda_{a1}{}^3 = \tfrac{1}{4}(\lambda^4{}_1{}^3\,\lambda_{41}{}^3 + \lambda^5{}_1{}^3\,\lambda_{51}{}^3) = \tfrac{1}{4}\,(1{\cdot}1 + (-i){\cdot}(-i)) = 0,\end{aligned} \tag{6.72}$$

or, e.g.,

$$\begin{aligned}\{\tfrac{1}{4}\,(\chi^\dagger_{3\gamma}\,\chi^\delta_4)_{\mathbf{8}'}\,(\lambda^a)_\alpha{}^\beta\,(\lambda_a)_\delta{}^\gamma\,(\chi^\alpha_1\,\chi^\dagger_{2\beta})_{\mathbf{8}}\} &\supset \tfrac{1}{4}\,(\delta^2_\gamma\delta^\delta_1)\,(\lambda^a)_\alpha{}^\beta\,(\lambda_a)_\delta{}^\gamma\,(\delta^\alpha_1\delta^3_\beta)\\ &= \tfrac{1}{4}\,\lambda^a{}_1{}^3\,\lambda_{a1}{}^2 = 0,\end{aligned} \tag{6.73}$$

as no Gell-Mann matrix has a nonzero 1st entry in both the 2nd and 3rd row (or column). In turn, since the representation $\mathbf{1}$ has only one dimension, for $\widetilde{f}_c(\mathbf{1}|\mathbf{1})$ there is a single case,

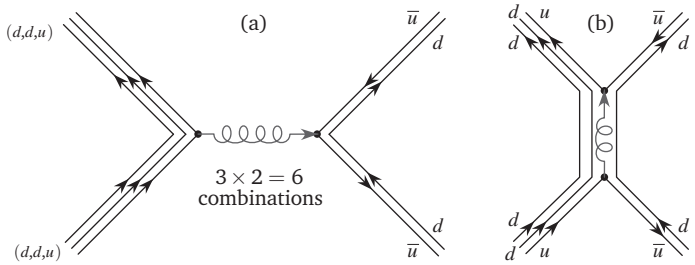
$$\begin{aligned}&\tfrac{1}{4}\,(\chi^\dagger_{3\gamma}\,\chi^\delta_4)_{\mathbf{1}}\,(\lambda^a)_\alpha{}^\beta\,(\lambda_a)_\delta{}^\gamma\,(\chi^\alpha_1\,\chi^\dagger_{2\beta})_{\mathbf{1}}\\ &\quad= \tfrac{1}{4}\,\tfrac{1}{\sqrt{3}}(\delta^1_\gamma\delta^\delta_1 + \delta^2_\gamma\delta^\delta_2 + \delta^3_\gamma\delta^\delta_3)\,(\lambda^a)_\alpha{}^\beta\,(\lambda_a)_\delta{}^\gamma\,\tfrac{1}{\sqrt{3}}(\delta^\alpha_1\delta^1_\beta + \delta^\alpha_2\delta^2_\beta + \delta^\alpha_3\delta^3_\beta)\\ &\quad= \tfrac{1}{12}\,\lambda^a{}_\alpha{}^\alpha\,\lambda_{a\gamma}{}^\gamma = \tfrac{1}{12}\,\mathrm{Tr}(\boldsymbol{\lambda}^a)\,\mathrm{Tr}(\boldsymbol{\lambda}_a) = 0,\end{aligned} \tag{6.74}$$

which is very similar to the reasoning in Conclusion 6.5, at the end of Section 6.2.1: an $SU(3)_c$-invariant state cannot turn into a single gluon, as $SU(3)_c$-invariant gluons do not exist.

Using the direct computations from Example 6.3, we have that

$$\begin{aligned}\mathfrak{M}_{u+\overline{u}\to u+\overline{u}} = &-\frac{g_c^2}{(\mathrm{p}_1-\mathrm{p}_3)^2}\left\{\begin{smallmatrix}-\frac{1}{6}\\ +\frac{4}{3}\end{smallmatrix}\right\}[\overline{u}_3\boldsymbol{\gamma}^\mu u_1][\overline{v}_2\boldsymbol{\gamma}_\mu v_4]\\ &+\frac{g_c^2}{(\mathrm{p}_1+\mathrm{p}_2)^2}\left\{\begin{smallmatrix}\frac{1}{2}\\ 0\end{smallmatrix}\right\}[\overline{v}_2\boldsymbol{\gamma}^\mu u_1][\overline{u}_3\boldsymbol{\gamma}_\mu v_4], \qquad \text{if} \quad \begin{cases}\boldsymbol{\chi}_{12}\subset \mathbf{8},\\ \boldsymbol{\chi}_{12}=\mathbf{1}.\end{cases}\end{aligned} \tag{6.75}$$

If a concrete incoming quark–antiquark pair in fact form a meson, the color factors $\boldsymbol{\chi}_{12} = \boldsymbol{\chi}_{34}$ must be $SU(3)_c$-invariant, so that the second diagram (6.69) in fact contributes nothing because of color conservation. However, in hadronic elastic collisions of the type $n^0 + \pi^- \to n^0 + \pi^-$, both diagrams contribute:



(6.76)

The diagram (a) contributes in six ways (either of the three quarks in the neutron may exchange a gluon with either the $\overline{u}$ antiquark, or the d quark within the pion); the diagram (b) contributes in

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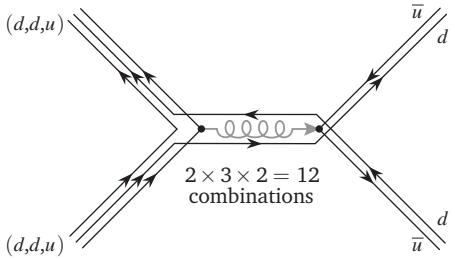
only one way. Except, the processes depicted in diagram (a) are prohibited by Conclusion 6.5, i.e., either at least one more gluon and/or a d quark (being common to both incoming hadrons) must be exchanged, as for example in



(6.77)

This then is still $O(g_c^2)$, up to the undetermined d-quark exchange factor; additional gluon exchange would increase the order. Note that the exchanged state (propagating in the horizontal direction in this Feynman diagram) may well be interpreted as the exchange of a virtual pion, π^0 – vindicating in part Yukawa's original proposal for strong interactions.

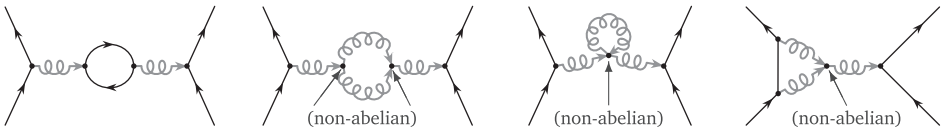
6.2.4 Renormalization and asymptotic freedom

In Section 5.3.3, we obtained the relation (5.202),

$$\alpha_{e,R}(|\mathrm{q}^2|) \approx \frac{\alpha_{e,R}(0)}{1 - \frac{\alpha_{e,R}(0)}{3\pi} \ln\left(\frac{|\mathrm{q}^2|}{m_e^2 c^2}\right)}, \qquad |\mathrm{q}^2| \gg m_e^2 c^2, \tag{5.202}$$

which indicates the electromagnetic fine structure constant to in fact be a variable, and to depend on the transfer 4-momentum q at which the measurement takes place.

In the analogous analysis of $O(g_s^4)$ corrections to the amplitude of the collision (6.50) new diagrams appear, precisely because of the non-abelian (non-commutative) nature of the chromodynamics interaction. Ignoring diagrams that only correspond to renormalizing the parameters of the incoming and outgoing particles, for $O(g_s^4)$ contributions we have



(6.78)

The computation of the contributions depicted by the last three diagrams requires additional rules that involve the introduction of ghost fields and the so-called BRST nilpotent symmetry [☞ Footnote *5* on p. 232]. That level of technical details is beyond the scope of this book, and we simply cite [445] the final result:

$$\alpha_{s,R}(|\mathrm{q}^2|) \approx \frac{\alpha_{s,R}(\mu^2 c^2)}{1 + \frac{\alpha_{s,R}(\mu^2 c^2)}{3\pi} \frac{11n - 2n_f}{4} \ln\left(\frac{|\mathrm{q}^2|}{\mu^2 c^2}\right)}, \qquad |\mathrm{q}^2| \gg \mu^2 c^2. \tag{6.79}$$

This holds for all $SU(n)$-gauge interactions, where n_f is the total number of Dirac spin-$\frac{1}{2}$ fermions that possess such an n-dimensional $SU(n)$ charge. The fermion loop (6.78) contributions are the opposite of the gauge boson loop contributions.[15] The precise computation produces the coefficient

[15] Recall that fermion loops require an additional -1 factor in the amplitude, as well as that both quarks and gluons contribute to the chromodynamics color charge (6.32) [☞ Conclusion 6.3 on p. 229].

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$\frac{11n-2n_f}{4}$, which in our case is $+5\frac{1}{4}$: we have $n=3$ colors and $n_f=6$ quark flavors. Since the relative sign in the denominator (6.79) is opposite from the relative sign in the denominator (5.202), it follows that $\alpha_s(|q^2|)$ *diminishes* as $|q^2|$ grows.

Example 6.5 Effectively, the opposite contributions from the quarks and the gluons in the relation (6.79) imply that virtual quark–antiquark pairs *screen*, and virtual gluons *enhance* the chromodynamics color charge. The example of quantum electrodynamics [☞ Section 5.3.3] has already explained the first part of this phenomenon. For the second part – except, of course, detailed computation – there also exists a qualitative argument [425], depicted in Figure 6.1. Suppose we have a chromodynamic charge source

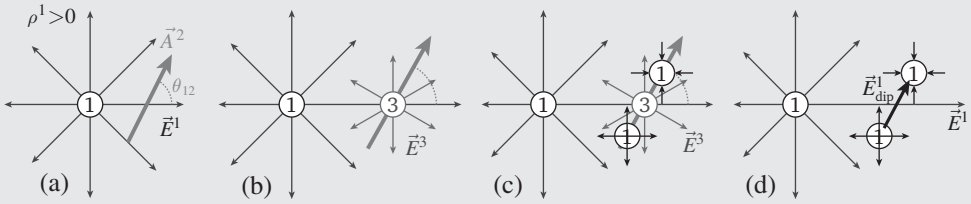

Figure 6.1 A qualitative depiction of the mechanism by which virtual gluons enhance the chromodynamic charge

of the color ρ^1, depicted by the "central" circle labeled "1" in Figures 6.1(a)–(d). By Gauss's law (6.35), this creates a chromo-electric field $\vec{E}^1$; see Figure 6.1(a). Let somewhere nearby a virtual quantum of the chromodynamics field appear; in Figures 6.1(a)–(c), this is depicted by the vector $\vec{A}^2$, at an angle of $\theta_{12} = 60^\circ$ from the positive direction of $\vec{E}^1$. This virtual quantum $\vec{A}^2$ couples to the pre-existing field $\vec{E}^1$ and produces via the non-abelian (non-commutative) part of equation (6.35) a virtual source for the field $\vec{E}^3$:

$$\vec{\nabla}\cdot\vec{E}^3 = -\tfrac{g_c}{\hbar c} f^3{}_{21}\vec{A}^2\cdot\vec{E}^1 = -\tfrac{g_c}{\hbar c}(-1)|\vec{A}^2||\vec{E}^1|(\cos\theta_{12} = +\tfrac{1}{2}) = +\tfrac{g_c}{2\hbar c}|\vec{A}^2||\vec{E}^1|. \qquad (6.80)$$

This virtual source ρ^3 is localized at the position of the virtual potential $\vec{A}^2$, i.e., somewhat removed from the real source ρ^1. It is depicted by a circle labeled "3" in Figures 6.1(b)–(c). By Gauss's law (6.35) again, the virtual source ρ^3 creates a virtual chromo-electric field $\vec{E}^3$, depicted in Figures 6.1(b)–(c). Iterating, the coupling of this virtual field $\vec{E}^3$ and the virtual potential $\vec{A}^2$ serves as an *additional* source (or sink) for the field $\vec{E}^1$. Indeed, just outside the location of the "bare" source ρ^1 and near the virtual source ρ^3, we have

$$\vec{\nabla}\cdot\vec{E}^1 = -\tfrac{g_c}{\hbar c} f^1{}_{23}\vec{A}^2\cdot\vec{E}^3 = -\tfrac{g_c}{\hbar c}(+1)|\vec{A}^2||\vec{E}^3|\cos\theta_{32}, \qquad (6.81)$$

where θ_{32} is the angle between the virtual potential $\vec{A}^2$ and the virtual field $\vec{E}^3$. In Figure 6.1(c), p. 244, we see that:

1. $\cos\theta_{32} > 0$ north-east from the virtual source ρ^3, and
2. $\cos\theta_{32} < 0$ south-west from the virtual source ρ^3.

Thus, the coupling of the virtual field $\vec{E}^3$ and the virtual potential $\vec{A}^2$ serves as an *additional sink* for $\vec{E}^1$ near the virtual source ρ^3 and a little further away from the "bare"

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source ρ^1, and as an *additional source* for $\vec{E}^1$ near the virtual source ρ^3 and a little closer to the "bare" source ρ^1. This additional source-and-sink form a small dipole of the $\vec{E}^1_{\text{dip}}$ field, at the location of the virtual potential $\vec{A}^2$. Such additional dipoles result in a vacuum polarization owing to the nonlinear coupling of chromodynamics fields and potentials.

For clarity, Figure 6.1(d), p. 244, depicts the contributions of only the chromoelectric field $\vec{E}^1$, where we see that the coupling of the virtual potentials $\vec{A}^2$ with the induced virtual field $\vec{E}^3$ has produced the additional field $\vec{E}^1_{\text{dip}}$, and just so that the "bare" source "1" is effectively *enhanced* (anti-screened) rather than screened, i.e., diminished: In the induced virtual dipole, the source is closer to the "bare" source, and the sink is further away.

Repeating the analysis with any other combination of distribution and value of the initial "bare" source and the virtual potential, as well as the further iterations of this nonlinear coupling, confirms this qualitative conclusion. The virtual quanta of the chromodynamics field of course appear with a random distribution around the "bare" source, but the so-induced vacuum polarization uniformly enhances the "bare" source.

The transfer 4-momentum $\mathrm{q} := (\mathrm{p}_1 - \mathrm{p}_2)$ between the left- and the right-hand particles in each diagram (6.78) is

$$\mathrm{q}^2 = \mathrm{p}_1{}^2 + \mathrm{p}_2{}^2 - 2\mathrm{p}_1\cdot\mathrm{p}_2 = (m_1^2+m_2^2)c^2 + 2\vec{p}_1\cdot\vec{p}_2 - 2\frac{E_1E_2}{c^2} = \frac{(E_1-E_2)^2}{c^2} - (\vec{p}_1-\vec{p}_2)^2. \tag{6.82}$$

The distance covered by the virtual particles, which occurs predominantly in the horizontal, mediating portion of the diagram (6.78), is inversely proportional to this transfer momentum. Thus, $\alpha_{s,R}(|\,q^2|)$ *grows* with the distance at which the interaction occurs, which confirms earlier given qualitative arguments and is in full accord with experimental observations; see Section 2.3.14.

Digression 6.5 The careful Reader will have noticed that the relation (6.79) gives the chromodynamics fine structure parameter at the energy $c\sqrt{|\mathrm{q}^2|}$ as a function of two quantities: the mass μ and the value of the chromodynamics fine structure parameter at the transfer momentum μc. These two quantities may be "collected," by defining

$$\Lambda_{QCD} \;:\quad \ln(\Lambda_{QCD}^2) := \ln(\mu^2c^2) - \frac{12\pi}{(11n-2n_f)\alpha_{s,R}(\mu^2c^2)}, \tag{6.83}$$

the substitution of which into the result (6.79) yields

$$\alpha_{s,R}(|\mathrm{q}^2|) \approx \frac{12\pi}{(11n-2n_f)\ln\left(\frac{|\mathrm{q}^2|}{\Lambda_{QCD}^2}\right)}, \tag{6.84}$$

where Λ_{QCD} is the magnitude of the transfer 4-momentum at which $\alpha_{s,R}(|\mathrm{q}^2|)$ diverges; this divergence is called the Landau pole, after L. D. Landau. The importance of this divergence is only formal, since perturbative computations fail to make sense before that, when $\alpha_{s,R}(|\mathrm{q}^2|) \lesssim 1$. Experimental estimates give only an approximate region $100\,\mathrm{MeV}/c < \Lambda_{QCD} < 500\,\mathrm{MeV}/c$, and one typically uses $\Lambda_{QCD} \approx 220\,\mathrm{MeV}/c$ as the approximate value of the geometric mean of the experimental bounds.

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Finally, it is worth noting that for quantum electrodynamics, in relation (5.202), the reference value of $\alpha_{e,R}(0) \approx \frac{1}{137}$ is an excellent choice. That is the value of the fine structure parameter – and so also the intensity of the electromagnetic interaction (5.122), $g_e = \sqrt{4\pi\alpha_e} = |e|/\sqrt{\epsilon_0 \hbar c}$ – that is measurable in experiments where the interacting electric charges are at a distance much larger than the typical (sub-)atomic distances. Those are, of course, all "classical" experiments with electric charges.

By contrast, in quantum chromodynamics, $\alpha_{s,R}(0)$ makes no sense. Perturbative computations wherein the parameter $\alpha_{s,R}(|q^2|)$ and the relation (6.79) are defined fails to be valid at 4-momenta *below* $\sim$200 MeV/c, i.e., at distances bigger than $\sim 10^{-15}$ m. Perturbative computations in quantum chromodynamics make sense only at distances smaller than $\sim 10^{-15}$ m, i.e., at energies larger than $\sim$200 MeV. That makes the introduction of an arbitrary reference value, such as $\alpha_{s,R}(\mu^2 c^2)$, necessary. For sufficiently large μ, $\alpha_{s,R}(\mu^2 c^2)$ can even be measured, whereupon the relation (6.79) is of better practical use than the simpler relation (6.84).

A moment's thought reveals that this striking difference between $\alpha_{s,R}(|q^2|)$ and $\alpha_{e,R}(|q^2|)$ in their dependence on the transfer 4-momentum fully supports two of the experimentally noted properties of quarks:

Asymptotic freedom the limit $\lim_{|q^2|\to\infty} \alpha_{s,R}(|q^2|) = 0$ agrees with the experimentally observed fact that the strong interaction between quarks is vanishingly small at vanishingly small distances.

Confinement the limit $\lim_{|q^2|\to\Lambda_{QCD}} \alpha_{s,R}(|q^2|) = \infty$ agrees with the experimentally observed fact that the strong interaction between quarks grows as the distance between two quarks is being increased, e.g., so as to separate them.

Note that this is *not* a proof of confinement, since perturbation theory, used to compute $\alpha_{s,R}(|q^2|)$, breaks down as $|q^2| \to \Lambda_{QCD}$; nevertheless, this perturbative result is encouraging and gives good hope that other methods will eventually provide a rigorous proof.

6.2.5 Exercises for Section 6.2

✎ ***6.2.1*** *Following Example 6.2 on p. 236, compute all possible cases of the color factors* $f_c(\mathbf{3}_A^*|\mathbf{6}_S)$, $f_c(\mathbf{6}_S|\mathbf{6}_S)$ *and* $f_c(\mathbf{6}_S'|\mathbf{6}_S)$.

✎ ***6.2.2*** *Using all elements (6.63) and by explicit computation – or using the* $SU(3)_c$ *action – show that the results of Example 6.3 on p. 240 are independent of the choice of the concrete case(s).*

✎ ***6.2.3*** *Following Example 6.3 on p. 240, compute all possible cases of* $f_c(\mathbf{8}|\mathbf{1})$.

✎ ***6.2.4*** *For all six elastic collisions of a nucleon* (p^+, n^0) *and a pion* $(\pi^\pm, n^0)$*, determine the relative contribution of the diagrams of type (a) and type (b) in the display (6.76).*

✎ ***6.2.5*** *Redoing the analysis of Example 6.5 on p. 244, verify that a virtual gauge vector* $\vec{A}^2$ *oriented, however, at an angle* 120° *will produce the same effect of anti-screening of the initial source* ρ^1.

6.3 Non-perturbative comments

Field theory is – in practice – a perturbative discipline, and most of the detailed work in quantum chromodynamics indeed relies on perturbative computations. Because the fine structure parameter

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and the interaction intensity depend on the distance at which the interaction takes place (mediated by exchange of gluons) [☞ result (6.79)–(6.84)], perturbative computations do not suffice. A complete solution of quantum chromodynamics must include essentially non-perturbative effects. Here, we mention a few themes that appear in attempts at non-perturbative analysis.

6.3.1 Strong CP-violation, "topological" solutions and the ϑ-vacuum

The chromodynamics analogue of the question at the end of Digression 5.7 on p. 183, about the expression (5.80c), is as follows: In the most general (both gauge- and Lorentz-invariant) Lagrangian density for quantum chromodynamics,

$$\begin{aligned}\mathscr{L}_{QCD+} = &-\sum_n \mathrm{Tr}\left[\overline{\mathbf{\Psi}}_n(\mathrm{x})\left[i\hbar c\not{\partial} - m_n e^{i\vartheta'\hat{\gamma}_5 c^2}\right]\mathbf{\Psi}_n(\mathrm{x})\right] \\ &- \tfrac{1}{4}\,\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}\right] - \tfrac{n_f g_s^2 \vartheta}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}_{\rho\sigma}\right],\end{aligned} \tag{6.85}$$

why are the parameters $\vartheta' \sim \vartheta < 3\times 10^{-10}$? ☝ The most reliable bound follows from the fact that the presence of the ϑ, ϑ'-dependent terms would provide the neutron with an electric dipole momentum. Experimentally, the electric dipole momentum of the neutron vanishes, and the bounds then follow from the limits on the experimental error in that measurement. Unlike the *CP*-violation as discussed in Section 4.2.3, *CP*-violation that follows from this so-called "ϑ_{QCD}-problem" is also-called the "strong *CP*-violation."

The additional term, $\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[\mathbb{F}_{\mu\nu}\mathbb{F}_{\rho\sigma}\right]$ is the 4-divergence of the so-called Loos–Chern–Simons "current" [555],

$$\mathcal{K}^\mu = \tfrac{n_f g_s^2}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}(\delta_{ab}\,A^a_\nu\,F^b_{\rho\sigma} - \tfrac{1}{3}g_s f_{abc}\,A^a_\nu\,A^b_\rho\,A^c_\sigma). \tag{6.86}$$

Then, a formal ϑ-transformation $\exp\{i\vartheta\mathcal{Q}\}$ exists with $\mathcal{Q} := \int \mathrm{d}^3\vec{r}\,\mathcal{K}^0$ that transforms the vacuum $|0\rangle$ with $\vartheta = 0$ into the vacuum $|\vartheta\rangle = e^{i\vartheta\mathcal{Q}}|0\rangle$ with the $\vartheta \neq 0$ value. Since all operators transform as $H(\vartheta) = e^{i\vartheta\mathcal{Q}}H(0)e^{-i\vartheta\mathcal{Q}}$, it follows that all physics with $\vartheta \neq 0$ is identical to the physics with $\vartheta = 0$. The vacua with distinct values of ϑ define "sectors" in the Hilbert space of quantum chromodynamics, and "our World" could easily be one such sector, which is physically indistinguishable from the sector with $\vartheta = 0$.

On the other hand, the equations of motion for quantum chromodynamics, derived from the Lagrangian density (6.23), are nonlinear equations, and have solutions that cannot be obtained by perturbative methods. To a large degree, such solutions are similar to magnetic monopoles that were discussed in Section 5.2.3; because of the nonlinear nature of the coupled system of equations of motion (6.37)–(6.40), one expects the set of solutions to be more complex and varied than in the case of electrodynamics. Suffice it to mention here the fact that such solutions are often determined by "global geometry," i.e., by boundary conditions at infinity, which often includes (but is not limited to) topology.

In the physics jargon, such solutions are often called "topological." This typically implies that the solutions are parametrized (also) by some characteristic integers. As such integer characterization cannot continuously vary, this provides a degree of stability to such solutions. With the benefit of hindsight, we see that the stationary states of the hydrogen atom – counted by the "quantum numbers" $n, \ell, m \in \mathbb{Z}$ and $m_s = \pm\frac{1}{2}$ – are also stable precisely because of the (half-)integrality of these numbers. In spite of this qualitative similarity, it is important to note that such "topological" solutions – which also include the Dirac monopole from Section 5.2.3 – also exist for the gauge field alone, i.e., for the electromagnetic field without charged particles, the chromodynamics field without quarks, and so on.

Using geometrical and topological methods that are beyond our present scope, it may be shown that non-perturbative solutions of the system (6.37)–(6.40) may be counted by an integer

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index. These solutions are similar to the vacua of the various ϑ-sectors as discussed in the previous paragraph. However, Alexander Belavin, Alexander Polyakov, Albert S. Schwartz and Yuri Tyupkin [484, 555] showed in 1975 that there is "tunneling" (via so-called BPST *instantons*[16]) from one sector into another, and that the true vacuum is a linear combination $|\vartheta\rangle := \sum_N e^{i\vartheta N}|N\rangle$, for $N \in \mathbb{Z}$. This effectively cancels the conclusion of the discussion about the result (6.86), as it proves that different ϑ-sectors are not independent.

In gauge theories with the Higgs field [☞ Chapter 7] the same role is played by the so-called 't Hooft–Polyakov monopole, and Polyakov also showed that instanton effects in quantum electrodynamics where photons interact with a scalar field (e.g., with the Higgs field) provide the photon with a mass – which is simply unacceptable.

The question why $\vartheta, \vartheta' = 0$, therefore remains unanswered.

— ❦ —

On the other hand, the discussion of the Dirac monopole and its Wu–Yang construction (5.101)–(5.105) as well as 't Hooft and Polyakov's constructions for non-abelian Yang–Mills theories with a Higgs field extends the gauge principle, which originated from the observation that the phases of complex wave-functions are fundamentally unmeasurable quantities, just as are the generalized, matrix-valued phases of wave-function n-tuples such as the chromodynamics triples of quarks (6.1).

> ***Conclusion 6.7*** *Since the gauge 4-vectors $\mathbb{A}_\mu$ in all Yang–Mills gauge theories are themselves fundamentally unobservable quantities, they may well be multi-valued or otherwise ambiguously defined as functions over spacetime. It is necessary and sufficient only that the gauge fields, the tensor components $\mathbb{F}_{\mu\nu} := [D_\mu, D_\nu]$ (up to a conventional multiplicative constant), are well-defined functions over spacetime.*

As an immediate corollary of this conclusion and the Wu–Yang and then the t'Hoof–Polyakov construction where gauge transformations connect differently specified gauge 4-vector potentials into a class, it follows that Yang–Mills theories, even without appropriately charged matter, may have a class of nontrivial "topological" solutions to their equations of motion. Here "topological" refers to the fact that the existence and the counting of such solutions may be determined by topological methods, depending on the gauge symmetry groups and boundary conditions [☞ also the nontrivial geometries of "empty spacetime" in Section 9.3].

In quantum theories, all (and so also the topologically nontrivial) solutions of the equations of motion may be used as "vacua." Particles – the quanta of all fields, including the gauge fields the nontrivial classical solution of which defines the vacuum – then move through this vacuum, to a first approximation without disturbing it. We thus have:

> ***Conclusion 6.8 (background fields)*** *Each (and so also the topologically nontrivial) solution of a system of classical equations of motion for all fields defines a "vacuum" in which the quanta of those fields move, to a first approximation, without changing these classical,* ***background fields****.*

6.3.2 The Weinberg–Witten theorem

On the heels of the quark model success, theories of preons and of technicolor became popular in the 1980s. At least some particles among the quarks, leptons, gauge and Higgs bosons were

[16] Instantons in general denote special particle-like objects in field theory, which are well localized not only in (position) space but also in time. That is, instantons are particles that exist but for an instant in time. They were first discovered in non-abelian Yang–Mills theory, but can appear generally in all nonlinearly coupled field theories.

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represented as composite states in these models. In an attempt to disqualify such models with a general argument, Stephen Weinberg and Edward Witten [564] proved a theorem now bearing their names:

> ***Theorem 6.1 (Weinberg–Witten)*** *No quantum field theory in $(3+1)$-dimensional spacetime with a Poincaré-covariant and gauge-invariant 4-vector current J^μ that satisfies a continuity equation may have a massless particle with a helicity bigger than $\frac{1}{2}$ and a non-vanishing charge of $\int d^3\vec{r}\, J^0$.*
>
> *No quantum field theory in $(3+1)$-dimensional spacetime with a Poincaré-covariant and gauge-invariant rank-2 tensor that satisfies a continuity equation may have a massless particle with a helicity bigger than 1.*

> ***Comment 6.4*** *The expression "Poincaré-covariant" means that it transforms properly with respect to the Lorentz transformations and translations in spacetime, regardless of whether co- or contra-variant and how many times; the continuity equation for a 4-vector is the usual $\partial_\mu J^\mu = 0$, and for a rank-2 tensor, $T^{\mu\nu}$, it is $\partial_\mu T^{\mu\nu} = 0$.*

The proof of the theorem is non-perturbative and very general, but the assumptions of the theorem are very stringent. Indeed, it turns out that the theorem in fact does not apply precisely in the models that were meant to be disqualified. For example, in at least several preonic models and in the technicolor theory, there exists an additional non-abelian gauge interaction, the purpose of which is to bind the states that in such models replace some of the particles that are regarded as elementary in the Standard Model. As shown in relations (6.24)–(6.32), the non-abelian (non-commutative) current that is conserved, i.e., satisfies a continuity equation (6.30), is not gauge-invariant, whereby the (prerequisite) assumption of the Weinberg–Witten theorem is not satisfied and the theorem does not apply.

This is related to another unresolved question. Indeed, in a regime where the quark masses are negligible, the chromodynamics Lagrangian density has a doubly larger symmetry: the Dirac spinors representing quarks may be projected into the left- and the right-handed $\Psi_\pm$ (5.58). This Lagrangian density is invariant with respect to an *independent* and global (constant in spacetime) $SU(n_f)$ flavor transformation of the left- and right-handed quarks, so that the full symmetry of this Lagrangian density is $SU(n_f)_L \times SU(n_f)_R \times U(1)_L \times U(1)_R$. Quantum effects in quantum chromodynamics break this symmetry into the "diagonal"[17] $SU(n_f) \times U(1)_B$, where the $U(1)_B$ charge is the baryon number, and which has two significant consequences for the complete understanding of which one also needs material from Section 7.1:

1. The quantum (not spontaneous) breaking of the classical $U(1)_A$ symmetry (the complement of $U(1)_B$ in the product $U(1)_L \times U(1)_R$) is an *anomaly*; instanton solutions from the previous section contribute to this effect as well as to the "strong" *CP*-violation and connect these two unexplained characteristics of quantum chromodynamics. Generally, anomalies are an indicator of an inconsistency in the model, but as $U(1)_A$ is an *approximate* symmetry, anomalies indicate an inconsistency in the model only in the unphysical limiting case when the quark masses vanish.
2. The eight spin-0 mesons ($\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$) could be identified as the Goldstone bosons [☞ Section 7.1.2] of the symmetry breaking $SU(3)_L \times SU(3)_R \to SU(3)_f$. Of course, quark

[17] For groups of the form $G_L \times G_R$, where the two factors have the same structure but act upon different objects or different aspects of a given object, the "diagonal" subgroup $G \subset G_L \times G_R$ again has the same structure but acts simultaneously both as G_L and as G_R. Only when G_L and G_R are abelian (commutative) does there also exist an "anti-diagonal" complement. Thus $U(1)_L \times U(1)_R = U(1)_D \times U(1)_A$, where the first factor is the diagonal subgroup, so the $U(1)_D$ charge is the sum of $U(1)_L$ and $U(1)_R$ charges; the $U(1)_A$ charge is their difference.

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masses are not zero, the broken symmetry was never exact, and neither are the masses of these spin-0 mesons zero, but they are significantly lower than the $SU(3)_f$ singlet meson η'.

3. The eight spin-1 mesons ($\rho^{\pm}, \rho^0, K^{*\pm}, K^{*0}, \bar{K}^{*0}, \phi$) could be identified as the gauge bosons of the remaining symmetry $SU(3)_f$. Of course, quark masses are not zero, this symmetry is not exact, and neither are the masses of these spin-1 mesons zero. It is not clear, however, if the masses of these mesons (and especially their lightness) may be explained completely as an *explicit* $SU(n_f)$ symmetry-breaking effect, or if there exists a generalization of the Weinberg–Witten theorem that would apply.

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