A Level Mathematics for OCR A
Student Book 2 (Year 2)
Vesna Kadelburg and Ben Woolley

This resource has been submitted to OCR’s endorsement process
Contents

Introduction ........................................................................................................................................ v
How to use this book ......................................................................................................................... vi

1 Proof and mathematical communication .................................................................................... 1
   Section 1: A reminder of methods of proof .................................................................................... 1
   Section 2: Proof by contradiction ................................................................................................. 3
   Section 3: Criticising proofs ......................................................................................................... 5

2 Functions ...................................................................................................................................... 10
   Section 1: Mappings and functions ............................................................................................. 11
   Section 2: Domain and range ....................................................................................................... 16
   Section 3: Composite functions .................................................................................................. 21
   Section 4: Inverse functions ........................................................................................................ 25

3 Further transformations of graphs ............................................................................................... 40
   Section 1: Combined transformations ......................................................................................... 41
   Section 2: Modulus function ........................................................................................................ 50
   Section 3: Modulus equations and inequalities ............................................................................ 55

4 Sequences and series ..................................................................................................................... 63
   Section 1: General sequences ....................................................................................................... 64
   Section 2: General series and sigma notation .............................................................................. 68
   Section 3: Arithmetic sequences ................................................................................................ 70
   Section 4: Arithmetic series ....................................................................................................... 72
   Section 5: Geometric sequences .................................................................................................. 75
   Section 6: Geometric series ....................................................................................................... 79
   Section 7: Infinite geometric series ............................................................................................. 81
   Section 8: Using sequences and series to solve problems .......................................................... 85

5 Rational functions and partial fractions ....................................................................................... 93
   Section 1: Review of the factor theorem ..................................................................................... 93
   Section 2: Simplifying rational expressions .............................................................................. 96
   Section 3: Partial fractions with distinct factors ....................................................................... 99
   Section 4: Partial fractions with a repeated factor ................................................................. 102

6 General binomial expansion ......................................................................................................... 107
   Section 1: General binomial expansion ..................................................................................... 107
   Section 2: Binomial expansions of compound expressions ..................................................... 111

Focus on … Proof 1 ......................................................................................................................... 117
Focus on … Problem-solving 1 ...................................................................................................... 119
Focus on … Modelling 1 ................................................................................................................. 121
Cross-topic review exercise 1 ....................................................................................................... 124

7 Radian measure .......................................................................................................................... 127
   Section 1: Introducing radian measure ....................................................................................... 127
   Section 2: Inverse trigonometric functions and solving trigonometric equations .................. 133
   Section 3: Modelling with trigonometric functions .................................................................. 138
   Section 4: Arcs and sectors ....................................................................................................... 146
   Section 5: Triangles and circles ............................................................................................... 150
   Section 6: Small angle approximations ..................................................................................... 154

8 Further trigonometry ................................................................................................................... 163
   Section 1: Compound angle identities ....................................................................................... 164
   Section 2: Double angle identities ............................................................................................ 167
   Section 3: Functions of the form a sin x + b cos x ..................................................................... 172
   Section 4: Reciprocal trigonometric functions ........................................................................ 177

9 Calculus of exponential and trigonometric functions ................................................................. 184
   Section 1: Differentiation ........................................................................................................ 184
   Section 2: Integration .............................................................................................................. 189

10 Further differentiation ................................................................................................................ 196
   Section 1: The chain rule ........................................................................................................ 197
   Section 2: The product rule ...................................................................................................... 202
   Section 3: The quotient rule ..................................................................................................... 205
   Section 4: Implicit differentiation ........................................................................................... 208
   Section 5: Differentiating inverse functions ............................................................................. 213

11 Further integration techniques .................................................................................................... 218
   Section 1: Reversing standard derivatives .............................................................................. 218
   Section 2: Integration by substitution ....................................................................................... 222
   Section 3: Integration by parts ................................................................................................ 229
   Section 4: Using trigonometric identities in integration .......................................................... 233
   Section 5: Integrating rational functions .................................................................................. 238

12 Further applications of calculus ................................................................................................ 245
   Section 1: Properties of curves ................................................................................................ 246
   Section 2: Parametric equations ............................................................................................. 252
   Section 3: Related rates of change ........................................................................................... 264
   Section 4: More complicated areas ......................................................................................... 267

13 Differential equations ................................................................................................................ 280
   Section 1: Introduction to differential equations ...................................................................... 281
   Section 2: Separable differential equations ............................................................................ 283
   Section 3: Modelling with differential equations .................................................................... 287

© Cambridge University Press 2017
The third party copyright material that appears in this sample may still be pending clearance and may be subject to change.
Before you start…

<table>
<thead>
<tr>
<th>Student Book 1, Chapters 2 and 7</th>
<th>You should be able to use rules of indices and logarithms.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Book 1, Chapter 13</td>
<td>You should be able to differentiate $x^n$.</td>
</tr>
<tr>
<td>Student Book 1, Chapter 14</td>
<td>You should be able to integrate $x^n$ for $n \neq -1$.</td>
</tr>
<tr>
<td>Chapters 7 and 8</td>
<td>You should be able to use compound angle formulae and small angle approximations.</td>
</tr>
</tbody>
</table>

1. Write $\ln(3^x)$ in the form $A + B \ln x$.  
2. Differentiate $y = \left(3x - \frac{1}{x}\right)\left(x + \frac{2}{3x}\right)$.  
3. Find the exact value of $\int_1^2 \frac{x^3 + 3}{2x^2} \, dx$.  
4. Use small angle approximations to find the approximate value of $\cos\left(\frac{\pi}{3} + \frac{\pi}{100}\right)$.  

Extending differentiation and integration

You have already seen many situations that can be modelled using trigonometric, exponential and logarithm functions. For example, in Chapter 8 of Student Book 1 you studied exponential models for population growth, and in Chapters 7 and 8 of this book you saw an example of modelling water waves using sine and cosine functions.

We are often interested in the rate of change of quantities in such models; for example, at what rate is the population increasing after three years? Finding rates of change involves differentiation. Integration reverses this process – given the rate of change you can find the equation for the original quantity.

So far you have learnt how to differentiate and integrate functions of the form $ax^n$. In this section you will extend rules of differentiation and integration to include a wider variety of functions.

Section 1: Differentiation

You already know from Student Book 1, Chapter 8 that the rate of growth of an exponential function is proportional to the value of the function.
In particular, for $y = e^x$ the rate of growth, which is the same as the derivative, equals the $y$ value.

The derivative of the natural logarithm function is perhaps surprising, but you will see how it follows from the fact that $y = \ln x$ is the inverse function of $y = e^x$.

### Key point 9.1

- If $y = e^x$ then $\frac{dy}{dx} = e^x$
- If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$

## PROOF 4

Let point $A$ on the graph of $y = e^x$ have $y$-coordinate $a$.

Let point $B$ be the point on the graph of $y = \ln x$ which is the reflection of $A$ in the line $y = x$.

The reflection swaps $x$ and $y$ coordinates, so $B$ has $x$-coordinate $a$.

The gradient of $y = e^x$ at $A$ is $a$.

You know that the gradient of $y = e^x$ at $A$ equals the $y$ value.

The gradient of the reflected line:

gradient $a^{-1} = \frac{1}{a}$

To see what happens to the gradient on reflection in $y = x$, consider reflecting a right-angled triangle.

This shows that, if a line with gradient $a$ is reflected in $y = x$, the gradient of the reflected line is $\frac{1}{a}$.

The gradient of the graph $y = \ln x$ at $B$ is $\frac{1}{a}$.

This is $1$ divided by the $x$-coordinate of $B$, so the gradient of $y = \ln x$ is $\frac{1}{x}$. 

© Cambridge University Press 2017

The third party copyright material that appears in this sample may still be pending clearance and may be subject to change.
The rules for differentiating sums and constant multiples of expressions still apply. Sometimes an expression needs to be simplified before it can be differentiated.

WORKED EXAMPLE 9.1

Differentiate $y = e^{3x} - 3 \ln (4x^2)$.

You can only differentiate $e^x$ and $\ln x$, so you need to simplify the expression using rules of indices and logarithms.

$e^x$ is multiplied by a constant ($e^3$).

$3 \ln 4$ is a constant, so its derivative is $0$.

The derivative of $6 \ln x$ is $6 \times \frac{1}{x}$.

To find out how to differentiate $\sin x$ and $\cos x$ you need to remember differentiation from first principles (Student Book 1, Chapter 13). You also need to use compound angle formulae from Chapter 8, Section 1 and small angle approximations from Chapter 7, Section 6.

WORKED EXAMPLE 9.2

Use differentiation from first principles to prove that the derivative of $\sin x$ is $\cos x$, where $x$ is measured in radians.

Let $f(x) = \sin x$

Use $\sin (A + B) = \sin A \cos B + \sin B \cos A$

Use small angle approximations:

$\cos h = 1 - \frac{1}{2} h^2$

$\sin h = h$

Let $h$ tend to $0$. 

Tip

One you have derived the rules, as summarised in Key Point 9.2 below, you can use them without proof.
Differentiating $\cos x$ follows the same method. The result for $\tan x$ can also be proved in this way, as well as by using the quotient rule which you will meet in Chapter 10, Section 3.

**Key point 9.2**

- If $y = \sin x$ then $\frac{dy}{dx} = \cos x$
- If $y = \cos x$ then $\frac{dy}{dx} = -\sin x$
- If $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$

The following two examples should help you remember two applications of differentiation: finding equations of tangents and normals and finding stationary points.

**WORKED EXAMPLE 9.3**

Find the equations of the tangent and the normal to the graph of the function $f(x) = \cos x + e^x$ at the point where $x = 0$. Give your answer in the form $ax + by + c = 0$, where $a$, $b$, $c$ are integers.

\[ f'(x) = -\sin x + e^x \]
\[ \therefore f'(0) = -\sin 0 + e^0 = 1 \]

When $x = 0$,
\[ y = f(0) = \cos 0 + e^0 = 1 + 1 = 2 \]

Tangent:
\[ y - y_1 = m(x - x_1) \]
\[ y - 2 = \frac{1}{1}(x - 0) \]
\[ y = x + 2 \]
\[ y - x - 2 = 0 \]

Normal:
\[ y - 2 = -1(x - 0) \]
\[ y = -x + 2 \]
\[ y + x - 2 = 0 \]

You need the gradient, which is $f'(0)$.

To find the equation of a straight line you also need coordinates of one point.

The tangent passes through the point on the graph where $x = 0$. Its $y$-coordinate is $f(0)$.

Put all the information into the equation of a line.

The gradient of the normal is $\frac{1}{m}$ and it passes through the same point.
WORKED EXAMPLE 9.4

Find the coordinates of the stationary point on the graph of \( y = \ln x + \frac{1}{x^2} \) and determine its nature.

\[
\frac{dy}{dx} = \frac{1}{x} - 2x^{-3}
\]

You need to find where the gradient equals 0.

\[
\frac{1}{x} - \frac{2}{x^3} = 0
\]

Write \( \frac{1}{x^2} \) as \( x^{-2} \).

\[
x^2 - 2 = 0
\]

\[
x > 0, \text{ so } x = \sqrt{2}
\]

\[
y = \ln \sqrt{2} + \frac{1}{2}
\]

\[
The \text{ stationary point is } \left( \sqrt{2}, \frac{1}{2} \ln 2 + \frac{1}{2} \right)
\]

Use rules of logs to rewrite the \( y \)-coordinate.

\[
d^2y
\]

To determine the nature of the stationary point you need to evaluate the second derivative at \( x = \sqrt{2} \).

\[
d^2y = \frac{1}{x^2} + 6x^{-4}
\]

\[
= \frac{1}{2 \sqrt{2}^2} + \frac{6}{2 \sqrt{2}^4}
\]

\[
= \frac{1}{4} + \frac{3}{4} = 1 > 0
\]

The stationary point is a minimum.

EXERCISE 9A

1. Differentiate the following:

   a. i. \( y = 3e^x \)
   
   b. i. \( y = -2 \ln x \)
   
   c. i. \( y = \frac{\ln x}{5} - 3x + 4e^x \)
   
   d. i. \( y = 3 \sin x \)
   
   e. i. \( y = 2x - 5 \cos x \)
   
   f. i. \( y = \frac{\sin x + 2 \cos x}{5} \)
   
   ii. \( y = \frac{2e^x}{5} \)
   
   ii. \( y = \frac{1}{3} \ln x \)
   
   ii. \( y = 4 - \frac{e^x}{2} + 3 \ln x \)
   
   ii. \( y = 2 \cos x \)
   
   ii. \( y = \tan x + 5 \)
   
   ii. \( y = \frac{1}{2} \tan x - \frac{1}{3} \sin x \)

2. Differentiate after simplifying first:

   a. i. \( y = \ln x^3 \)
   
   b. i. \( y = 3 \ln 2x \)
   
   c. i. \( y = e^{x+3} \)
   
   d. i. \( y = e^{2 \ln x} \)
   
   e. i. \( y = \frac{3 \sin x - \cos x}{\cos x} \)
   
   f. i. \( y = \frac{e^{2x} - 2e^x}{e^x} \)
   
   ii. \( y = 2 \ln x^5 \)
   
   ii. \( y = \ln 5x \)
   
   ii. \( y = e^{x-3} \)
   
   ii. \( y = e^{3 \ln x^2} \)
   
   ii. \( y = \frac{2 \cos x + 4 \sin x}{\cos x} \)
   
   ii. \( y = \frac{4e^x - e^{2x}}{e^x} \)

© Cambridge University Press 2017 

The third party copyright material that appears in this sample may still be pending clearance and may be subject to change.
Find the exact value of the gradient of the graph of \( f(x) = \frac{1}{2}e^x - 7\ln x \) at the point \( x = \ln 4 \).

Find the exact value of the gradient of the graph \( f(x) = e^x - \frac{\ln x}{2} \) when \( x = \ln 3 \).

Find the value of \( x \) where the gradient of \( f(x) = \frac{1}{2}e^x - 7\ln x \) is \(-6\).

Find the value of \( x \) where the gradient of \( g(x) = x^2 - 12\ln x \) is \(2\).

Find the exact value of the gradient of the graph \( f(x) = e^x + x \) which is parallel to \( y = 3x \).

Find the equation of the tangent to the curve \( y = e^x + x \) which is parallel to \( 2x - y = 6 \).

Given that \( h(x) = \sin x + \cos x \), \( 0 \leq x < 2\pi \), find the values of \( x \) for which \( h'(x) = 0 \).

Find the equations of the tangent and the normal to the graph of \( y = e^x + x \) at \( x = \frac{\pi}{4} \). Give all the coefficients in an exact form.

Given that \( y = \frac{1}{4}\tan x + \frac{1}{x^2} \) for \( 0 < x < 2\pi \), solve the equation \( \frac{dy}{dx} = 1 - \frac{2}{x^3} \).

Find and classify the stationary points on the curve \( y = 3\tan x - 2\sqrt{2} \sin x \) at \( x = \frac{\pi}{4} \). Give all the coefficients in an exact form.

Find and classify the stationary points of:

\[ a \quad y = \ln x - \sqrt{x} \]
\[ b \quad y = 2e^x - 5x \]

The volume of water in millions of litres \((V)\) in a tidal lake is modelled by \( V = 60\cos t + 100 \) where \( t \) is the time in days after a hydroelectric plant was switched on.

\[ a \quad \text{What is the smallest volume of the lake?} \]

\[ b \quad \text{The hydroelectric plant produces an amount of electricity proportional to the rate of flow of water (measured as volume per unit time) through a tidal dam. Assuming all flow is through the dam, find the time, in the first 6 days, when the plant is producing maximum electricity.} \]

Section 2: Integration

You can reverse all the differentiation results from the previous section to integrate several new functions.

Key point 9.3

\[ \int e^x \, dx = e^x + c \]
\[ \int \frac{1}{x} \, dx = \ln|x| + c \]
\[ \int \sin x \, dx = -\cos x + c \]
\[ \int \cos x \, dx = \sin x + c \]
Note the modulus sign in \( \int \frac{1}{x} \, dx = \ln|x| + c \). \( \ln(x) \) is not defined for negative \( x \). However, you can see from the diagram that the area between the \( x \)-axis and the negative part of the graph of \( \frac{1}{x} \) is the same as the corresponding area between the \( x \)-axis and the positive part of the graph.

\[
\int_{-b}^{-a} \frac{1}{x} \, dx = \ln(-a) - \ln(-b) = \ln3.
\]

However, splitting the area in two doesn’t give the same answer:
\[
\int_{-b}^{0} \frac{1}{x} \, dx + \int_{0}^{a} \frac{1}{x} \, dx
\]

has no value, since \( \ln(0) \) is undefined. It turns out that the shaded area does not have a finite value.

Explore

Is it possible to find the shaded area between the graph of \( y = \frac{1}{x} \) and the \( x \)-axis? Using definite integration gives
\[
\int_{-2}^{6} \frac{1}{x} \, dx = \ln(6) - \ln(-2) = \ln3.
\]

However, splitting the area in two doesn’t give the same answer:
\[
\int_{-2}^{6} \frac{1}{x} \, dx = \int_{-2}^{0} \frac{1}{x} \, dx + \int_{0}^{6} \frac{1}{x} \, dx
\]

has no value, since \( \ln(0) \) is undefined. It turns out that the shaded area does not have a finite value.

However, some graphs with asymptotes can still enclose a finite area; find out about improper integrals.

You can combine these facts with rules of integration you already know. Sometimes an expression needs to be rewritten in a different form before integrating.

**WORKED EXAMPLE 9.5**

Find \( \int \frac{5x - 3x^3}{2x^2} \, dx \).

\[
\int \frac{5x - 3x^3}{2x^2} \, dx = \int \frac{5}{2x} - \frac{3}{2}x \, dx = \frac{5}{2} \ln|x| - \frac{3}{4}x^2 + c
\]

You don’t know how to integrate a quotient, but you can split the fraction and integrate each term separately.
Which of the following statements is correct? Identify the mistake in the other three.

<table>
<thead>
<tr>
<th>Statement 1</th>
<th>Statement 2</th>
<th>Statement 3</th>
<th>Statement 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \frac{1}{x} )</td>
<td>( \int \frac{1}{x^2} , dx = \ln(x^3) + c )</td>
<td>( y = \ln(3x) )</td>
<td>( y = \ln(3x) )</td>
</tr>
<tr>
<td>( \Rightarrow \frac{dy}{dx} = \ln x )</td>
<td>( \Rightarrow \frac{dy}{dx} = \frac{1}{3x} )</td>
<td>( \Rightarrow \frac{dy}{dx} = \frac{1}{x} )</td>
<td>( \Rightarrow \frac{dy}{dx} = \frac{1}{x} )</td>
</tr>
</tbody>
</table>

Remember that integration is the reverse of differentiation, so if you know the derivative of a function you can use integration to find the function itself.

**WORKED EXAMPLE 9.6**

A curve passes through the point \((-1, 3)\) and its gradient is given by \( \frac{dy}{dx} = 3e^x - 1 \). Find the equation of the curve.

\[
\begin{align*}
y &= \int (3e^x - 1) \, dx \\
&= 3e^x - x + c
\end{align*}
\]

Integrate \( \frac{dy}{dx} \) to get \( y \). Don’t forget the constant of integration.

Using \( x = -1, y = 3 \):

\[
\begin{align*}
3 &= 3e^{-1} - (-1) + c \\
\Rightarrow c &= 2 - \frac{3}{e} \\
\therefore y &= 3e^x - x + 2 - \frac{3}{e}
\end{align*}
\]

Use given values of \( x \) and \( y \) to find the constant of integration.

You can also use integration to find the area between the curve and the \( x \)-axis. This involves evaluating **definite integrals**. It is always a good idea to draw a diagram to make sure you are finding the correct area.

**Rewind**

You know from Student Book 1 Chapter 15 that when a curve is below the \( x \)-axis the integral is negative.
Find the following integrals.

1

\( \int 5e^x \, dx \)

\( \int 9e^x \, dx \)

\( \int 2e^x \, dx \)

\( \int 7e^x \, dx \)

\( \int (e^x + 3x) \, dx \)

\( \int (e^x + x^2) \, dx \)

\( \int 3 \cos x \, dx \)

\( \int 4 \sin x \, dx \)

\( \int 2 \cos x - 3x \, dx \)

\( \int 2 \cos x - \sin x \, dx \)

\( \int \sqrt{x} + \sin x \, dx \)

\( \int \cos x - \frac{1}{\sqrt{x}} \, dx \)

2

Find the following integrals

\( \int x^2 \, dx \)

\( \int 3x \, dx \)

\( \int \frac{1}{2x} \, dx \)

\( \int \frac{1}{3x} \, dx \)

\( \int \frac{5}{2x} \, dx \)

\( \int \frac{2}{3x} \, dx \)
3 Find the exact value of these definite integrals:

\[
\begin{align*}
\text{a i} & \quad \int_0^3 3e^x \, dx & \text{ii} & \quad \int_1^3 2e^x \, dx \\
\text{b i} & \quad \int_0^\ln 3 e^x \, dx & \text{ii} & \quad \int_0^\ln 5 e^x \, dx \\
\text{c i} & \quad \int_1^\ln 2 (3e^x + 2) \, dx & \text{ii} & \quad \int_1^\ln 3 (4 - 3e^x) \, dx \\
\text{d i} & \quad \int_1^3 \frac{3}{2x} \, dx & \text{ii} & \quad \int_2^3 \frac{4}{3x} \, dx \\
\text{e i} & \quad \int_0^{\pi/2} \cos x \, dx & \text{ii} & \quad \int_0^{\pi/2} \frac{3}{2x} \, dx \\
\text{f i} & \quad \int_0^{\pi/2} \cos x \, dx & \text{ii} & \quad \int_0^{2\pi} \sin x \, dx \\
\text{g i} & \quad \int_\pi^2 3\sin x - 4\cos x \, dx & \text{ii} & \quad \int_0^{2\pi} 2 \cos x - \sin x \, dx \\
\end{align*}
\]

4 Find the exact value of the area enclosed by the curve \( y = \frac{2}{3x} \), the \( x \)-axis and the lines \( x = 2 \) and \( x = 6 \). Give your answer in the form \( a \ln b \).

5 Find the area enclosed by the curve \( y = 3 \sin x \), the \( x \)-axis, and the line \( x = \frac{\pi}{3} \).

6 Find the exact value of \( \int_0^{\pi/3} (\sin x + 2\cos x) \, dx \).

7 Find the exact value of \( \int_0^{\pi} 3 \, dx \).

8 a Evaluate \( \int_{-2}^2 \frac{2}{x} \, dx \).
   
   b State the value of the area between the graph of \( y = \frac{1}{x} \), the \( x \)-axis and the lines \( x = -9 \) and \( x = -3 \).

9 Find the equation of the curve given that \( \frac{dy}{dx} = \cos x + \sin x \) and the curve passes through the point \((\pi,1)\).

10 Find \( \int \frac{\sin x + \cos x}{2 \cos x} \, dx \).

11 The derivative of the function \( f(x) \) is \( \frac{1}{2x^2} \).
   
   a Find an expression for all possible functions \( f(x) \).

   b If the curve \( y = f(x) \) passes through the point \((2,7)\), find the equation of the curve.

See Support sheet for a further example of finding the equation of a curve and for more practice questions.
12 The diagram shows part of the curve \( y = e^x - 5\sin x \).

![Diagram of the curve](image)

a Use your calculator to find the \( x \)-intercept of the graph.

b Find the shaded area, enclosed between the curve, the \( x \)-axis and the lines \( x = 0 \) and \( x = 1 \).

13 The diagram shows a part of the curve with equation \( y = \frac{4}{x} + x - 5 \).

![Diagram of the curve](image)

a Show that the curve crosses the \( x \)-axis at \( x = 1 \) and \( x = 4 \).

b Find the exact value of the shaded area. Give your answer in the form \( p - 4\ln q \), where \( p \) and \( q \) are rational numbers.

14 Show that the value of the integral \( \int_{k}^{2k} \frac{1}{x} \, dx \) is independent of \( k \).

15 The gradient of the normal to a curve at any point is equal to the \( x \)-coordinate at that point. If the curve passes through the point \((e^2, 3)\), find the equation of the curve in the form \( y = \ln g(x) \) where \( g(x) \) is a rational function.

### Checklist of learning and understanding

- You can now differentiate basic trigonometric, exponential and logarithm functions:
  \[
  \frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \sec^2 x, \quad \frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}
  \]
- The results for \( \sin x \) and \( \cos x \) can be derived using differentiation from first principles.
- You can also integrate some new functions:
  \[
  \int \sin x \, dx = -\cos x + c, \quad \int \cos x \, dx = \sin x + c, \quad \int e^x \, dx = e^x + c, \quad \int \frac{1}{x} \, dx = \ln|x| + c
  \]
- You can combine these with the rules of differentiation and integration you already know to find:
  - Rates of change
  - Equations of tangents and normals
  - Stationary points
  - Equation of a curve with a given gradient
  - Area between a curve and the \( x \)-axis
Mixed practice 9

1. Find the equation of the tangent to the curve \( y = e^x + 2\sin x \) at the point where \( x = \frac{\pi}{2} \).

2. If \( f'(x) = \sin x \) and \( f\left( \frac{\pi}{3} \right) = 0 \) find \( f(x) \).

3. Find the exact value of the gradient of the graph \( f(x) = e^x - \frac{\ln x}{2} \) when \( x = \ln 4 \).

4. Find the exact value of the integral \( \int_0^\pi e^x + \sin x + 1 \, dx \).

5. The diagram shows the curve with equation \( y = x - 7 + \frac{10}{x} \), which crosses the \( x \)-axis at 2 and 5. Find the exact value of the shaded area.

6. Find the indefinite integral \( \int \frac{1 + x^2}{x} \, dx \).

7. Find and classify the stationary points on the curve \( y = \sin x + 4\cos x \) in the interval \( 0 < x < 2\pi \).

8. Find and classify the stationary points on the curve \( y = \tan x - \frac{4x}{3} \) for \( -\pi < x < \pi \). Give only the \( x \)-coordinates, and leave your answers in terms of \( \pi \).

9. Find the equation of the normal to the curve \( y = 3e^x \) at the point \( x = \ln 3 \). Give your answer in the form \( x + ky = p + \ln q \) where \( k, p \) and \( q \) are integers.

10. The population of bacteria \( (P) \) in thousands at a time \( t \) in hours is modelled by \( P = 10 + e^t - 3t \), \( t \geq 0 \).
   a. i. Find the initial population of bacteria.
   ii. At what time does the number of bacteria reach 14 million? (Hint: Use a solver function on your calculator.)
   b. i. Find \( \frac{dP}{dt} \).
   ii. Find the time at which the bacteria are growing at a rate of 6 million per hour.
   c. i. Find \( \frac{d^2P}{dt^2} \) and explain the physical significance of this quantity.
   ii. Find the minimum number of bacteria, justifying that it is a minimum.

11. The diagram shows the graphs of \( y = \sin x \) and \( y = \sqrt{3}\cos x \). Find the exact value of the shaded area.

12. Find \( \int \frac{\cos 2x}{\cos x - \sin x} \, dx \).

Elevate

See Extension sheet for a selection of more challenging problems.