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1 Counting principles and probability

In this chapter you will learn how to:

• break down complicated questions into parts which are easier to count, and then combine them together
• count the number of ways to permute a set of objects
• count the number of ways you can choose objects from a group
• apply these tools to problems involving probabilities.

Before you start…

<table>
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<tr>
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<th>A Level Mathematics Student Book 1, Chapter 17</th>
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Making maths count

Counting is the one of the first things you learn in Mathematics and at first it seems very simple. If you were asked to count how many people there are in your school, this would not be too tricky. If you were asked how many chess matches need to be played if everyone is to play everyone else, this is a little more complicated. If you were asked how many football teams could be chosen, you might find that the numbers become far too large to count without using some mathematical techniques. This chapter will help you to develop strategies for counting in such situations and using these to solve probability problems.

Section 1: The product principle and the addition principle

Counting very small groups is simple. You need to break down more complicated problems into counting small groups. But how do you combine these together to come up with an answer to the overall problem? The answer lies in the product principle and the addition principle, which is illustrated in the following menu.

Anna would like to order a main course and a dessert. She could make six different orders – three choices for the main course and for each choice she makes there, two choices for dessert, so she multiplies the individual possibilities.
Bob would like to order either a main course or a dessert. He could make five different orders – one of the three main courses or one of the two desserts so he adds the individual possibilities.

You can use the notation $n(A)$ to mean the number of ways of making a choice about $A$.

The product principle tells us that when you wish to select one option from $A$ and one option from $B$ you multiply the individual possibilities together.

**Key point 1.1**

Product principle: $n(A \text{ AND } B) = n(A) \times n(B)$

The addition principle tells us that when you wish to select one option from $A$ or one option from $B$ you add the individual possibilities together.

The addition principle has one caveat. You can only use it if there is no overlap between the choices for $A$ and the choices for $B$. For example, you cannot apply the addition principle to counting the number of ways of getting an odd number or a prime number on a dice. If there is no overlap between the choices for $A$ and for $B$ the two events are mutually exclusive.

**Key point 1.2**

Addition principle:

$$n(A \text{ OR } B) = n(A) + n(B)$$

if $A$ and $B$ are mutually exclusive.

The hardest part of applying the addition or product principles is breaking the problem down into relevant parts. It is often useful to rewrite questions to emphasise whether what is required is ‘AND’ or ‘OR’

**WORKED EXAMPLE 1.1**

An examination has ten questions in section A and four questions in section B. Calculate how many different ways there are to choose questions if you must choose:

a) one question from each section
b) a question from either section A or section B.

a Choose one question from A (10 ways) AND one from B (4 ways).

Number of ways = $10 \times 4$

= 40

Describe the problem accurately.

b Choose one question from A (10 ways) OR one from B (4 ways).

Number of ways = $10 + 4$

= 14

Describe the problem accurately.

‘AND’ means you should apply the product principle.

‘OR’ means you should apply the addition principle.
In the context of Worked example 1.1 you cannot have a repeat selection of an object. However, there are situations where you may be able to do so.

**WORKED EXAMPLE 1.2**

In a class there is an award for best mathematician, best sportsperson and nicest person. People can receive more than one award. In how many ways can the awards be distributed if there are twelve people in the class?

- Choose one of 12 people for the best mathematician (12 ways)
- AND one of the 12 for best sportsperson (12 ways)
- AND one of the 12 for nicest person (12 ways).

\[12 \times 12 \times 12 = 1728\]

This suggests a general idea:

**Key point 1.3**

The number of ways of selecting something \( r \) times from \( n \) objects is \( n^r \).

**EXERCISE 1A**

1. If there are ten ways of doing \( A \), three ways of doing \( B \) and 19 ways of doing \( C \) calculate how many ways there are of doing:
   - a i both \( A \) and \( B \) ii both \( B \) and \( C \)
   - b i either \( A \) or \( B \) ii either \( A \) or \( C \).

2. If there are four ways of doing \( A \), seven ways of doing \( B \) and five ways of doing \( C \), calculate how many ways there are of doing:
   - a all of \( A \), \( B \) and \( C \) b exactly one of \( A \), \( B \) or \( C \).

3. In the diagram below state how many different paths there are:
   - a from \( A \) to \( C \)
   - b from \( C \) to \( E \)
   - c from \( A \) to \( E \)

4. John is planting out his garden and needs a new rose bush and some dahlias. There are 12 types of rose and 4 varieties of dahlia in his local nursery. How many possible selections does he have to choose from if he wants exactly one type of rose and one type of dahlia?

5. A lunchtime menu at a restaurant offers five starters, six main courses and three desserts. State how many different choices of meal you can make if you would like:
   - a a starter, a main course and a dessert
   - b a main course and either a starter or a dessert
   - c any two different courses.
6 Five men and three women would like to represent their club in a tennis tournament. In how many ways can one mixed doubles pair be chosen?

7 A Mathematics team consists of one student from each of Years 7, 8, 9 and 10. There are 58 students in Year 7, 68 in Year 8, 61 in Year 9 and 65 in Year 10.
   a How many ways are there of picking the team?
   Year 10 is split into three classes: 10A (21 students), 10B (23 students) and 10C (21 students).
   b If students from 10B cannot participate in the challenge, how many ways are there of picking the team?

8 Student ID codes consist of 3 letters chosen from A to Z, followed by 4 digits chosen from 1–9. Repeated characters are permitted. How many possible ID codes are there?

9 A beetle walks along the struts from the bottom to the top of an octahedral sculpture, visiting exactly two of the middle points (A, B, C or D), How many possible routes are there?

10 Professor Small has 15 different ties (7 blue, 3 red and 5 green), 4 waistcoats (red, black, blue and brown) and 12 different shirts (3 each of red, pink, white and blue). He always wears a shirt, a tie and a waistcoat.
   a How many different outfits can he use until he has to repeat one?
   Professor Small never wears any outfit that combines red with pink.
   b How many different outfits can he make with this limitation?

11 State how many different 3-digit numbers can be formed using the digits 1, 2, 3, 5, 7:
   a at most once only
   b as often as desired.

12 State how many ways:
   a can four toys be put into three boxes
   b can three toys be put into five boxes.

Section 2: Permutations

A permutation is a way of ordering a set of objects. For example, a flag has four horizontal stripes: one each of the colours red, yellow, green and black. If you want to count how many different possible flags you have, you need to find the number of permutations of the colours. You can use this as a generic example to illustrate how to think about permutations.

There are four options for the colour of the top stripe, three options for the colour of the next one (because one of the colours has already been used), two options for the third stripe and one option for the last one. Using the 'product principle,' the number of possible options for each are multiplied together, so the total number of permutations is $4 \times 3 \times 2 \times 1 = 24$. The number of ways $n$ different objects can be...

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permuted is equal the product of all positive integers less than or equal to \( n \); you have already met the notation for this expression: \( n! \)

**Key point 1.4**

\[
n! = n(n - 1)(n - 2) \ldots \times 2 \times 1
\]

The number of ways of arranging \( n \) objects is \( n! \)

**WORKED EXAMPLE 1.3**

A test has 12 questions. How many different arrangements of the questions are possible?

*Permute 12 items.*  

Number of permutations = 479 001 600

In most examination questions you will have to combine the idea of permutations with the product and addition principles:

**WORKED EXAMPLE 1.4**

A 7-digit number is formed by using each of the digits 1 to 7 exactly once. How many such numbers are even?

*Pick the final digit to be even (3 ways).*  

AND  

*then permute the remaining 6 digits (6! ways).*  

\[3 \times 720 = 2160\text{ possible even numbers}\]

Apply the product principle.

The above example shows the very common situation in counting where you are given a constraint – in this case you have to end with an even digit.

**WORKED EXAMPLE 1.5**

How many permutations of the letters of the word SQUARE start with three vowels?

*Permute the three vowels at the beginning (3! ways).*  

AND  

*permute the three consonants at the end (3! ways).*  

Number of ways = 3! \times 3! = 36

Apply the product principle.
EXERCISE 1B

1 Evaluate:
   a i 5!
   ii 6!
   b i 2\times 4!
   ii 3\times 5!
   c i 6! − 5!
   ii 6! − 4 \times 5!

2 Evaluate:
   a i 8!
   ii 11!
   b i 9 \times 5!
   ii 9! \times 5
   c i 12! − 10!
   ii 9! − 7!

3 Find the number of ways of arranging:
   a six CDs
   b eight photographs
   c 26 books.

4 a How many ways are there of arranging seven textbooks on a shelf?
   b In how many of those permutations is the single biggest textbook in the middle?

5 a How many 5-digit numbers can be formed by using each of the digits 1-5 exactly once?
   b How many of those numbers are divisible by 5?

6 A class of 16 pupils and their teacher are queuing outside a cinema.
   a How many different permutations are there?
   b How many different permutations are there if the teacher has to stand at the front?

7 A group of nine pupils (five boys and four girls) are lining up for a photograph, with all the girls in the front row and all the boys at the back. How many different permutations are there?

8 How many permutations of the letters of the word ‘SQUARE’ start with a consonant?

9 a How many 6-digit numbers can be made by using each of the digits 1-6 exactly once?
   b How many of those numbers are smaller than 300 000?

10 A class of 30 pupils is lining up in three rows of ten for a class photograph. How many different arrangements are possible?

11 A baby has nine different toy animals. Five of them are red and four of them are blue. She arranges them in a line so that, in terms of colour, they are symmetrical. How many different arrangements are possible?

Section 3: Combinations

Suppose that three pupils are to be selected out of a class of 11 to attend a meeting with the Head Teacher. How many different groups of three can be chosen? In this example you need to choose three pupils out of 11. They are not to be arranged in any specified order. Therefore, the selection of Ali, Bill and then Cathy is the same as the selection of Bill, Cathy and then Ali. This sort of selection is called a combination. In general the formula for the number of ways of choosing r objects out of n is given the symbol \( \binom{n}{r} \), nCr or "C", said as ‘n choose r’.

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Key point 1.5

The formula for the number of ways of choosing \( r \) objects out of \( n \) is:

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

This is in your formula book.

Tip

You can also calculate \( \binom{n}{r} \) on your calculator, usually via an "C, option.

WORKED EXAMPLE 1.6

A group of 12 friends wants to form a team for a 5-a-side football tournament.

a In how many different ways can a team of five be chosen?

b Rob and Amir are the only goalkeepers and they cannot play in any other position. If a team has to contain exactly one goalkeeper, how many possible teams are there?

a Choosing 5 players from 12.

Number of ways = \( \binom{12}{5} \)

\[ = 792 \]

b Pick a goalkeeper, then fill in the rest of the team.

Amir is in goal (1 way)
AND
choose four other players (\( ? \) ways)
OR
Rob is in goal (1 way)
AND
choose four other players (\( ? \) ways).

Number of ways with Amir in goal = \( \binom{10}{4} \)
Number of ways with Rob in goal = \( \binom{10}{4} \)

Total number of ways = \( 1 \times \binom{10}{4} + 1 \times \binom{10}{4} \)

\[ = 210 + 210 \]
\[ = 420 \]

EXERCISE 1C

Evaluate:

a i \( \binom{7}{2} \)

\[ \text{b i } 3 \times \binom{6}{3} \]

\[ \text{c i } \frac{5}{0} \times \frac{9}{5} \]

\[ \text{d i } \binom{5}{2} + \frac{9}{5} \]

\[ \text{ii } \binom{12}{5} \]

\[ \text{ii } 10 \times \binom{6}{5} \]

\[ \text{ii } \binom{10}{8} \times \binom{3}{1} \]

\[ \text{ii } \binom{6}{0} + \frac{7}{3} \]
2  a  i  In how many ways can 6 objects be selected from 8?
    ii  In how many ways can 5 objects be selected from 9?

    b  i  In how many ways can either 3 objects be selected from 10 or 7 objects be selected from 12?
    ii  In how many ways can either 2 objects be selected from 5 or 3 objects be selected from 4?

    c  i  In how many ways can 5 objects be selected from 7 and 3 objects be selected from 8?
    ii  In how many ways can 6 objects be selected from 8 and 3 objects be selected from 7?

3  An exam paper consists of 15 questions. Students can select any 9 questions to answer. How many different selections can be made?

4  Suppose you are revising for 7 subjects, and you study 3 subjects in one evening. You might study a subject on more than one evening.

    a  In how many ways can you select 3 subjects to do on Monday evening?

    b  If you have to revise Maths on Tuesday, in how many ways can you select the subjects to do on Tuesday evening?

5  In the ‘Pick 'n' Mix' lottery players select 7 numbers out of 39. How many different selections are possible?

6  There are 16 boys and 12 girls in a class. Three boys and two girls are needed to take part in the school play. In how many different ways can they be selected?

7  A football team consists of 1 goalkeeper, 4 defenders, 4 midfielders and 2 forwards. A manager has 3 goalkeepers, 8 defenders, 6 midfielders and 5 forwards in the squad. In how many ways can she pick the team?

8  A school is planning some Maths trips over the summer. There are 12 places on the Greece trip, 10 places on the China trip and 10 places on the Disneyland trip. If there are 140 pupils in the school, and assuming that they are all happy to go on any of the three trips, in how many ways can the spaces be allocated?

9  Out of 26 teachers in a school, 4 are needed to accompany a school theatre trip.

    a  In how many ways can the 4 teachers be chosen?

    b  How many selections are possible if Mr Brown and Mrs Brown cannot both go on the trip?

10  A committee of 3 boys and 3 girls is to be selected from a class of 14 boys and 17 girls. State how many ways the committee can be selected if:

    a  Anna has to be on the committee

    b  the committee has to include Bill or Emma, but not both.

11  Sam’s sweet shop stocks 7 different types of 2p sweets and 5 different types of 5p sweets. If you want at most one of each sweet, state how many different selections of sweets can be made when spending:

    a  exactly 6p  
    b  exactly 7p  
    c  exactly 10p  
    d  at most 5p.

12  An English examination has two sections. Section A has five questions and section B has four questions. Four questions must be answered.

    a  How many different ways are there of selecting four questions to answer if there are no restrictions?

    b  How many different ways are there of selecting four questions if there must be at least one question answered in each section?
10 points are drawn on a sheet of paper so that no three lie in a straight line. By connecting up the points, state:

a how many different triangles can be drawn
b how many different quadrilaterals can be drawn.

A group of 45 students are to be seated in 3 rows of 15 for a school photograph. Within each row, students must sit in alphabetical order according to name, but there is no restriction determining the row in which a student must sit. How many different seating permutations are possible, assuming no students have identical names?

Section 4: The exclusion principle

The exclusion principle is a trick for counting what you are interested in by counting what you are not interested in. This typically is needed when counting a situation where a certain property is prohibited.

Key point 1.6

Exclusion principle:
Count what you are not interested in and subtract it from the total.

As an example, suppose a 5-digit code is formed by using each of the digits 1 to 5 exactly once. If you wanted to count how many such codes do not end in ‘25’ you could consider all possible options for the last two digits.

**WORKED EXAMPLE 1.7A**

How many 5-digit codes formed by using each of the digits 1 to 5 exactly once do not end in ‘25’?

Pick the final digit from \{1,2,3,4\} (4 ways)

AND

permutate the remaining four digits (4! ways)

OR

pick the final digit as 5 (1 way)

AND

pick the penultimate digit from \{1,3,4\} (3 ways)

AND permutate the remaining three digits (3! ways).

\[(4 \times 4!) + (1 \times 3 \times 3!) = 114\]

Describe the problem accurately.

Use the product and addition principles.

An alternative way to solve the same problem is to use the exclusion principle.
WORKED EXAMPLE 1.7B

How many 5-digit codes formed by using each of the digits 1 to 5 exactly once do not end in ‘25’?

- Count permutations of 5-digit codes (5! ways)
- then EXCLUDE cases where the last two digits are ‘25’ (1 way)
- AND
  - permute the remaining three digits (3! ways)

\[ 5! - 1 \times 3! = 114 \]

A very common use of the exclusion principle occurs when you are asked to count a situation containing an ‘at least’ or ‘at most’ restriction.

WORKED EXAMPLE 1.8

Theo has eight different jigsaws and five different bears. He chooses four toys. In how many ways can he make a selection with at least two bears?

- Count combinations of four toys from thirteen \( \binom{13}{4} \) ways
- then EXCLUDE combinations with no bears \( \binom{5}{0} \) ways
- AND four jigsaws \( \binom{8}{4} \) ways
- OR combinations with one bear \( \binom{5}{1} \) ways
- AND three jigsaws \( \binom{8}{3} \) ways

\[
\text{Number of ways} = 715 - (1 \times 70 + 5 \times 56) = 365
\]

EXERCISE 1D

1. How many of the numbers between 101 and 800 inclusive are not divisible by 5?
2. How many permutations of the letters of the word JUMPER do not start with a J?
3. A bag contains 12 different chocolates, 4 different mints and 6 different toffees. Three sweets are chosen. State how many ways there are of choosing:
   a. all not chocolates
   b. not all chocolates.
4. State how many permutations of the letters of KITCHEN
   a. do not begin with KI
   b. do not have K and I in the first two letters.
5. A committee of 5 people is to be selected from a class of 12 boys and 9 girls. How many such committees include at least one girl?

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In a word game there are 26 letter tiles, each with a different letter. How many ways are there of choosing seven tiles so that at least two are vowels?

A committee of six is to be selected from a group of 10 men and 12 women. In how many ways can the committee be chosen if it has to contain at least two men and one woman?

Seven numbers are chosen from the integers between one and nineteen inclusive. State how many selections have:

a at most two even digits

b at least two even digits.

How many permutations of letters DANIEL do not begin with D or do not end with L?

Section 5: Permutations of combinations

Sometimes there are occasions where you want to choose a number of objects from a bigger group but the order they are chosen in matters. For example, finding the possibilities for the first three finishers in a race or forming numbers from a fixed group of digits. The strategy for dealing with these situations is to first choose from the larger group and then permute the chosen objects.

According to the Guinness Book of Records, at the time of writing the largest number which has ever been assigned any meaning comes from the field of counting. It is called ‘Graham’s number’ and it is so large that it is not feasible to write it out in normal notation even using all the paper in the world, so a new system of writing numbers had to be invented to describe it. Find out the topic Graham’s Number is used in – it is called Ramsey Theory. Investigate the notation used – it is called tetration.

A class of 28 pupils needs to select a committee consisting of a class representative, treasurer, secretary and football captain. Each post needs to be taken by a different person. In how many different ways can the four posts be filled?

Choose 4 from 28 \( \binom{28}{4} \) ways

AND permute those 4 (4!\text{ways}).

Number of ways = \( \frac{28}{4} \times 4! \) 491,400 ways

Describe problem accurately. Select four people and then allocate them to different jobs. Note that this is the same as selecting four people and then permuting them.

Apply the product principle.

If you wish to select a subset of size \( r \) from a group of size \( n \) and the order matters, first choose the appropriate size of combination, and then permute this combination. This is given the symbol \(^nP_r\) or \(nPr\) in the formula given in Key point 1.7.
EXERCISE 1E

1. Find the number of permutations of:
   a. 4 objects out of 10
   b. 6 objects out of 7.

2. Find the number of ways of selecting when order matters:
   a. 3 objects out of 5
   b. 2 objects out of 15.

3. In a ‘Magic sequence’ lottery draw there are 39 balls numbered 1–39. Seven balls are drawn at random. The result is a sequence of 7 numbers which must be matched in the correct order to win the grand prize. How many possible sequences can be made?

4. A teacher needs to select 4 pupils from a class of 24 to receive four different prizes. How many possible ways are there to award the prizes?

5. How many 3-digit numbers can be formed from digits 1–9 if no digit may be repeated?

6. Eight athletes are running a race. In how many different ways can the first three places be filled?

7. An identification number consists of two letters followed by four digits chosen from 0 to 9. No digit or letter may appear more than once. How many different identification numbers can be made?

8. Three letters are chosen from the word PICTURE and arranged in order. How many of the possible permutations contain at least one vowel?

9. Eight runners compete in a race. In how many different ways can the three medals be awarded if James wins either a gold or a silver?

10. A class of 18 needs to select a committee consisting of a President, a Secretary and a Treasurer. How many different ways are there to form the committee if one student, Ellie, does not want to be President?

Section 6: Keeping objects together or separated

How many anagrams of the word ‘SQUARE’ have the Q and U next to each other? How many have the vowels all separated? When such constraints are added to the problem you need some clever tricks to deal with them.

The first type of problem you shall look at is where objects are forced to stay together. The trick is to imagine the letters in the word SQUARE as being on tiles. If the Q and the U need to be together, you are really dealing with five tiles, one containing QU:

S  QU  A  R  E

You must also remember that the condition is satisfied if Q and U are in the other order:

S  UQ  A  R  E

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WORKED EXAMPLE 1.10

How many permutations of the word ‘SQUARE’ have the Q and the U next to each other?

Permute the five ‘tiles’ (S, QU, A, R, E) (5! ways)
AND permutate the letters QU on the ‘double tile’ (2! ways).

Number of ways = 5! × 2! = 240

Apply the addition principle.

Key point 1.8

If a group of items have to be kept together, treat them all as one object. Remember that there may be permutations within this group too.

Another type of constraint is that objects have to be kept apart. It is tempting to treat this as the opposite of objects being kept together, and this is indeed the case if you are separating only two objects. However, the opposite of three objects staying together includes having two of them together and the third one apart. So when dealing with this situation you need to focus on the gaps that the critical objects can fit into. Consider the question of how many permutations of the word SQUARE have none of the vowels together. You first permute all of the consonants. One such permutation is

\[ \_Q\_R\_S\_
\]

There are four gaps in which you can put the vowels. You only need to choose three of them, and then decide in what order to insert the vowels.

WORKED EXAMPLE 1.11

How many permutations of the word ‘SQUARE’ have none of the vowels together?

Permute 3 consonants (3! ways)
AND choose 3 out of the four gaps \( \binom{4}{3} \) ways
AND permute the three vowels to put into the gaps (3! ways).

Total number of ways = 3! × 4 × 3! = 144

Apply the product principle.

Key point 1.9

If objects have to be kept apart, permute all the other objects and then find the permutations of gaps for the other objects to be put in.
EXERCISE 1F

1. In how many ways can fourteen people be arranged in a line if two of these people – Jack and Jill – have to stand together?

2. Students from three different classes are standing in the lunch queue. There are six students from 10A, four from 10B and four from 10C. In how many ways can the queue be arranged if students from the same class have to stand together?

3. In how many ways can six Biology books and three Physics books be arranged on the shelf if the three Physics books are always next to each other?

4. In a photo there are three families – six Greens, four Browns and seven Grays – arranged in a row. The Browns have had an argument so they have to stand separately. How many different permutations are permitted?

5. In a cinema there are 15 seats in a row. State how many ways seven friends can be seated in the same row if:
   a. there are no restrictions
   b. they all want to sit together.

6. Five women and four men stand in a line.
   a. In how many arrangements will all four men stand next to each other?
   b. In how many arrangements will all the men stand next to each other and all the women stand next to each other?
   c. In how many arrangements will all the men be apart?
   d. In how many arrangements will all the men be apart and all the women be apart?

Section 7: Permuting objects with repetitions

If you are permuting a group of objects where some are identical you need a different formula to find the number of arrangements.

Key point 1.10

If there are $n$ objects with $r_A$ of type A, $r_B$ of type B, $r_C$ of type C etc. then the number of permutations is:

$$\frac{n!}{r_A! \cdot r_B! \cdot r_C! \cdot \ldots}$$

WORKED EXAMPLE 1.12

Find the number of arrangements of the word MATHEMATICS.

$$\frac{11!}{2!2!2!} = 4989600$$

There are 11 letters, of which there are two Ms, two Ts and two As. All other letters occur once. You could put in a 1! in the denominator to represent these, but it would not change the value.

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Section 8: Counting principles in probability

Counting principles can be used to calculate probabilities in some problems that would be extremely difficult in any other way. To find the probability of an event labelled \( A \) occurring you compare the number of ways in which \( A \) occurs with the total number of ways any event can occur.

\[
P(A) = \frac{\text{number of outcomes in which } A \text{ occurs}}{\text{total number of possible outcomes}}
\]

This only works if all the ways are equally likely.

WORKED EXAMPLE 1.13

A committee of 4 is randomly chosen from 6 girls and 5 boys. What is the probability that the committee contains exactly 3 girls?

First decide how many different committees can be made up. This is a selection where order does not matter.

\[
\text{Total number of committees} = \binom{11}{4} = \frac{11!}{4!(11-4)!} = 330
\]

Then see how many committees have exactly three girls.

Choosing 3 girls from 6 and 1 boy from 5 can be done in \( \binom{6}{3} \times \binom{5}{1} = 100 \) ways.

\[
P(\text{exactly 3 girls}) = \frac{100}{330} = \frac{10}{33}
\]
EXERCISE 1H

1. A set of four alphabet blocks bearing the letters A, R, S and T are dropped in a line at random. What is the probability that they spell out one of the words STAR, RATS or ARTS?

2. Consider the word PARTING. What is the probability that a sequence of four letters chosen from this word contains the letter P?

3. a. A team of 11 is chosen randomly from a squad of 18. What is the probability that both the captain and the vice captain are selected?
   
   b. Two of the squad are goalkeepers, and one of them must be chosen. If neither of the goalkeepers is captain or vice captain, what now is the probability that both the captain and the vice captain are selected?

4. A team of five students is to be chosen at random to take part in a debate. The team is to be chosen from a group of six history students and three philosophy students. Find the probability that:
   
   a. only history students are chosen
   
   b. all three philosophy students are chosen.

5. Six boys sit at random in six seats arranged in a row. Two of the boys are brothers. Find the probability that they:
   
   a. sit at the ends of the row
   
   b. sit next to each other.

6. A rugby team consisting of 8 forwards, 7 backs and 5 substitutes. They all line up at random in one row for a picture. State the probability:
   
   a. that the forwards are all next to each other
   
   b. that no two forwards are next to each other.

\[\text{Checklist of learning and understanding}\]

The product principle states that if you have two choices you can count the total number of outcomes when you are interested in both the first choice \textit{and} the second choice being made by multiplying together the number of outcomes for each choice.

- The addition principle states that the number of outcomes when you are interested in either the first choice \textit{or} the second choice being made is the sum of the number of outcomes for each choice.

- The number of permutations of \(n\) different objects is \(n!\)

If you are only selecting \(r\) objects from \(n\) and the order in which you do so does \textit{not} matter. We describe this as a combination and there are \(\binom{n}{r}\) ways to do this. This has the formula: \(\binom{n}{r} = \frac{n!}{(n-r)!r!}\)

- The exclusion principle says that to find the number of permutations which do \textit{not} have a certain property, find the number of permutation which \textit{do} have the property and subtract it from the number of all possible permutations.

- The number of permutations of a subset of size \(r\) from a set of size \(n\) is:\n\[n_\text{P}_r = \frac{n!}{(n-r)!}\]

- If a group of items have to be kept together, treat them all as one object. Remember that there may be permutations within this group too.

- If objects have to be kept apart, permute all the other objects and then find the permutations of ‘gaps’ for the other objects to be put in.

- If there are \(n\) objects with \(r_A\) of type A, \(r_B\) of type B, \(r_C\) of type C etc. then the number of permutations is \(\frac{n!}{r_A!r_B!r_C!...}\)

- You can use counting ideas to work out a probability: \(P(A) = \frac{\text{number of outcomes in which } A \text{ occurs}}{\text{total number of possible outcomes}}\)
Mixed practice 1

1. Seven athletes take part in the 100 m final of the Olympic games. In how many ways can three medals be awarded?
   a. Usain uses his answer to part a to claim that the probability of the medals going to Ana, Betty and Ciara is 1 in 35.
      i. Explain how Usain found this figure.
      ii. What is the flaw in Usain’s logic?

2. In how many ways can five different letters be put into five different envelopes?
   a. Five addressed letters are placed into five addressed envelopes at random. Find the probability that the letters are all in the correct envelopes.

3. In how many ways can 10 cartoon characters stand in a queue if Mickey, Bugs Bunny and Jerry must occupy the first three places?

4. How many 3-digit numbers contain no zeros?
   a. A random 3-digit number (from 100 to 999) is chosen. Find the probability that it contains no zeroes.

5. What is the probability that a randomly chosen permutation of the word ‘CAROUSEL’ starts and ends in a consonant?

6. What is the probability that a randomly chosen permutation of the word SCHOOL starts with an O?

7. A group of 15 students contains 7 boys and 8 girls.
   a. In how many ways can a committee of 5 be selected if it must contain at least one boy?
   b. What is the probability that a randomly chosen committee contains at least one boy?

8. Abigail, Bahar, Chris, Dasha, Eustace and Franz are sitting next to each other in six seats in a cinema. Bahar and Eustace cannot sit next to each other.
   a. In how many different ways can they permute themselves?
   b. If they were to sit at random, what would be the probability that Bahar and Eustace are not sitting next to each other?

9. A committee of 5 is to be selected from a group of 12 children. The two youngest cannot both be on the committee. In how many ways can the committee be selected?

10. A car registration number consists of 3 different letters followed by 5 digits chosen from 1 to 9 (the digits can be repeated).
    a. How many different registration numbers can be made?
    b. What is the probability that the number plate starts with SUM?

11. A van has 8 seats, 2 at the front a row of 3 in the middle and a row of 3 at the back. If each of 8 people can drive, in how many different ways can they be arranged in the car?

12. Ten people are to travel in one car (taking 4 people) and one van (taking 6 people). Only two of the people can drive. In how many ways can they be allocated to the two vehicles? (The permutation of the passengers in each vehicle is not important.)
13. Five girls – Anna, Beth, Carol, Dasha and Elena – stand in a line.
   
a. State how many permutations are possible in which:
   
   i. Anna is at one end of the line
   
   ii. Anna is not at either end.
   
b. What is the probability that, in a randomly chosen arrangement, Anna is at the left of the line or Elena is on the right, or both?

14. In how many ways can five different sweets be split amongst two people if:
   
a. each person must have at least one sweet
   
b. one person can take all of the sweets
   
c. one of the sweets is split into two to be shared, and each person gets two of the remaining sweets.

15. In a doctor’s waiting room, there are 14 seats in a row. Eight people are waiting to be seen.
   
a. In how many ways can they be seated?
   
b. Three of the people are all in the same family and they want to sit together. How many ways can this now happen?
   
c. The family no longer have to sit together, but there is someone with a very bad cough who must sit at least one seat away from anyone else. If all arrangements are equally likely, what is the probability that this happens?

16. A group of 7 students sit in random order on a bench.
   
a. i. Find the number of orders in which they can sit.
   
   ii. The 7 students include Tom and Jerry. Find the probability that Tom and Jerry sit next to each other.
   
b. The students consist of 3 girls and 4 boys. Find the probability that
   
   i. no two boys sit next to each other,
   
   ii. all three girls sit next to each other.

17. A bag contains 9 discs numbered 1, 2, 3, 4, 5, 6, 7, 8, 9.
   
a. Andrea chooses 4 discs at random, without replacement, and places them in a row.
   
   i. How many different 4-digit numbers can be made?
   
   ii. How many different odd 4-digit numbers can be made?
   
b. Andrea’s 4 discs are put back in the bag. Martin then chooses 4 discs at random, without replacement.
   
   Find the probability that:
   
   i. the 4 digits include at least 3 odd digits
   
   ii. the 4 digits add up to 28.
Discrete variables don’t have to take integer values, but the possible distinct values can be listed (though the list may be infinite). For example, if \( X \) is the standard UK shoe size of a random adult member of the public, \( X \) takes values 2, 2.5, 3, 3.5 up to 15.5 and is a discrete random variable.

If \( Y \) is the exact foot length of a random adult member of the public (in cm), \( Y \) takes values in the interval \([20, 35]\) and is a continuous random variable.
Section 1: Average and spread of a discrete random variable

The most commonly used measure of the average of a random variable is the expectation. It is a value representing the mean result if the variable were to be measured an infinite number of times. The expected value is the mean of the population, represented by $\mu$ (mu). See Key point 2.1.

**Key point 2.1**

The expectation of a random variable $X$ is written $E(X)$ and calculated as

$$\mu = E(X) = \sum x_i p_i$$

The subscript $i$ is just a counter referring to each possible value and its associated probability.

This will appear in your formula book

You do not need to be able to prove this result, but you might find it helpful to see how it is proved below.

**PROOF 1**

The mean of $n$ pieces of discrete data is

$$\bar{x} = \frac{1}{n} \sum f_i x_i$$

$$= \sum \left( \frac{f_i}{n} \right) x_i$$

If $n$ is large, $\frac{f_i}{n}$ will tend towards the probability of $x_i$ happening, therefore

$$\bar{x} = \mu = \sum x_i p_i$$

Start from the definition of the mean.

Since $\frac{1}{n}$ is constant you can take it into the sum.

When the sample size tends to infinity, the sample mean, $\bar{x}$ becomes the true population mean, $\mu$.

**WORKED EXAMPLE 2.1**

The random variable $X$ has a probability distribution as shown in the table. Calculate $E(X)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{10}$</td>
</tr>
</tbody>
</table>

$$E(X) = 1 \times \frac{1}{10} + 2 \times \frac{1}{4} + 3 \times \frac{1}{10} + 4 \times \frac{1}{4} + 5 \times \frac{1}{5} + 6 \times \frac{1}{10} = \frac{7}{2}$$

Use the values from the distribution in the formula.
As well as knowing the expected average, you may also be interested in how far away from the average you can expect an outcome to be. The variance, \( \sigma^2 \), of a random variable is a value representing the degree of variation that would be seen if the variable were to be repeatedly measured an infinite number of times. It is a measure of how spread out the variable is (see Key point 2.2).

Tip

The expectation of a random variable does not need to be a value which the variable can actually be.

As well as knowing the expected average, you may also be interested in how far away from the average you can expect an outcome to be. The variance, \( \sigma^2 \), of a random variable is a value representing the degree of variation that would be seen if the variable were to be repeatedly measured an infinite number of times. It is a measure of how spread out the variable is (see Key point 2.2).

Did you know?

Standard deviation – the square root of variance – is a much more meaningful representation of the spread of the variable. So why do you bother with variance at all? The answer is purely to do with mathematical elegance. It turns out that the algebra of variance is far neater than the algebra of standard deviations.

Key point 2.2

The variance of a random variable \( X \) is written \( \text{Var}(X) \) and calculated as

\[
\sigma^2 = \text{Var}(X) = \sum (x_i - \mu)^2 p_i = \sum x_i^2 p_i - \mu^2
\]

This will appear in your formula book.

The quantity \( \sum x_i^2 p_i \) is the expected value of \( X^2 \), read as ‘the mean of the squares.’ This variance formula is often read as ‘the mean of the squares minus the square of the mean’.

Fast forward

You will see in Section 2 how you find expectations of other functions of \( X \).

WORKED EXAMPLE 2.2

Calculate \( \text{Var}(X) \) for the probability distribution in Worked example 2.1

From Worked example 2.1, \( E(X) = 3.5 \)

\[
E(X^2) = \frac{1^2}{10} + \frac{2^2}{4} + \frac{3^2}{4} + \frac{4^2}{4} + \frac{5^2}{5} + \frac{6^2}{10} = 14.6
\]

\[
\text{Var}(X) = E(X^2) - [E(X)]^2 = 14.6 - 12.25 = 2.35
\]

A probability distribution can also be described by a function.

WORKED EXAMPLE 2.3

\( W \) is a random variable which can take values \(-0.5, 1.5, 2.5 \) and \( k \) where \( k > 0 \).

\[
P(W = w) = \frac{w^3}{29}
\]

a Find the value of \( k \).

b Find the expected mean of \( W \).

c Find the standard deviation of \( W \).
Find the variance of $X$, the random variable defined by the following distribution.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X=x)$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Which is the correct solution? Can you identify the errors made in the incorrect solutions?

<table>
<thead>
<tr>
<th>Solution</th>
<th>$E(X)$</th>
<th>$E(X^2)$</th>
<th>Var($X$)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Solution A</strong></td>
<td>$0 + 1 + 2 = 3$</td>
<td>$0^2 + 1^2 + 2^2 = 5$</td>
<td>$\frac{5}{3} - \frac{1}{3} = \frac{4}{3}$</td>
</tr>
<tr>
<td><strong>Solution B</strong></td>
<td>$1 \times 0.3 + 2 \times 0.5 = 1.3$</td>
<td>$1 \times 0.3 + 4 \times 0.5 = 2.3$</td>
<td>$2.3 - 1.3^2 = 0.61$</td>
</tr>
<tr>
<td><strong>Solution C</strong></td>
<td>$0 \times 0.2 + 1 \times 0.3 + 2 \times 0.5 = 1.3$</td>
<td>$0^2 \times 0.2^2 + 1^2 \times 0.3^2 + 2^2 \times 0.5^2 = 1.09$</td>
<td>$1.09 - 1.3^2 = -0.6$</td>
</tr>
</tbody>
</table>

**WORK IT OUT 2.1**

Use the fact that the total probability must add up to 1.

Using Key point 2.1

To find standard deviation you first need to find variance which means you need to find $E(W)$ and use Key point 2.2.

Although you only write down 3 sf in the working, you have used the full accuracy from the calculator to find the final answer.
EXERCISE 2A

1. Calculate the expectation, variance and standard deviation of each of the following random variables.

   a i
   \[
   \begin{array}{c|cccc}
   x & 1 & 2 & 3 & 4 \\
   P(X = x) & 0.4 & 0.3 & 0.2 & 0.1 \\
   \end{array}
   \]

   ii
   \[
   \begin{array}{c|cccc}
   x & 8 & 9 & 10 & 11 \\
   P(X = x) & 0.4 & 0.3 & 0.2 & 0.1 \\
   \end{array}
   \]

   b i
   \[
   \begin{array}{c|cccc}
   x & 10 & 20 & 30 & 40 \\
   P(X = x) & 0.4 & 0.3 & 0.2 & 0.1 \\
   \end{array}
   \]

   ii
   \[
   \begin{array}{c|cccc}
   x & 80 & 90 & 100 & 110 \\
   P(X = x) & 0.4 & 0.3 & 0.2 & 0.1 \\
   \end{array}
   \]

   c i
   \[
   \begin{array}{c|cccc}
   w & 0.1 & 0.2 & 0.3 & 0.4 \\
   P(W = w) & 0.4 & 0.1 & 0.25 & 0.25 \\
   \end{array}
   \]

   ii
   \[
   \begin{array}{c|cccc}
   v & 0.1 & 0.2 & 0.3 & 0.4 \\
   P(V = v) & 0.5 & 0.3 & 0.1 & 0.1 \\
   \end{array}
   \]

   d i
   \[P(X = x) = \frac{x^2}{14}, \quad x = 1, 2, 3\]

   ii
   \[P(X = x) = \frac{1}{x}, \quad x = 2, 3, 6\]

2. A discrete random variable \(X\) is given by \(P(X = x) = k(x+1)\) for \(x = 2, 3, 4, 5, 6\).

   a Show that \(k = 0.04\)
   b Find \(E(X)\).

3. The discrete random variable \(V\) has the probability distribution shown below and \(E(V) = 5.1\).

   \[
   \begin{array}{c|cccc}
   v & 1 & 2 & 5 & 8 \\
   P(V = v) & 0.2 & 0.3 & 0.1 & 0.1 \\
   \end{array}
   \]

   a Find the value of \(p\) and \(q\)
   b Find \(P(X > \mu)\).

4. A discrete random variable \(X\) has its probability given by

   \[P(X = x) = k(x+3), \text{ where } x \text{ is } 0, 1, 2, 3.\]

   a Show that \(k = \frac{1}{18}\).
   b Find the exact value of \(E(X)\).

5. The probability distribution of a discrete random variable \(X\) is defined by

   \[P(X = x) = kx(4-x), \quad x = 1, 2, 3.\]

   a Find the value of \(k\).
   b Find \(E(X)\).
   c Find the standard deviation of \(X\).

6. A fair six-sided dice, with sides numbered 1, 1, 2, 2, 2, 5 is thrown. Find the mean and variance of the score.

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The table below shows the probability distribution of a discrete random variable $X$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>0.1</td>
<td>$p$</td>
<td>$q$</td>
<td>0.2</td>
</tr>
</tbody>
</table>

a Given that $E(X) = 1.5$, find the values of $p$ and of $q$.

b Find the standard deviation of $X$.

A biased die with four faces is used in a game. A player pays 5 counters to roll the die. The table below shows the possible scores on the die, the probability of each score and the number of counters the player receives in return for each score.

<table>
<thead>
<tr>
<th>Score</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{1}{20}$</td>
</tr>
<tr>
<td>Number of counters player receives</td>
<td>4</td>
<td>5</td>
<td>15</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Find the value of $n$ in order for the player to get an expected return of 3.25 counters per roll.

Two fair dice are thrown. The random variable $D$ is the difference between the larger and the smaller score, or zero if they are the same.

a Copy and complete the following table to show the probability distribution of $D$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{16}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b Find $E(D)$.

c Find $\text{Var}(D)$

d Find $P(D > E(D))$

In a game a player pays an entrance fee of £$n$. He then selects one number from 1, 2, 3 or 4 and rolls three unbiased four-sided dice. If his chosen number appears on all three dice he wins four times his entrance fee. If his number appears on exactly two of the dice he wins three times the entrance fee. If his number appears on exactly one dice he wins £1. If his number does not appear on any of the dice he wins nothing.

Copy and complete the probability table.

<table>
<thead>
<tr>
<th>Profit (£)</th>
<th>$-n$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>$\frac{27}{64}$</td>
<td></td>
</tr>
</tbody>
</table>

b The game organiser wants to make a profit over many plays of the game. Given that he must charge a whole number of pence, what is the minimum amount the organiser must charge?

Section 2: Expectation and variance of transformations of discrete random variables

Linear transformations

You may have noticed a link between question 1 parts a and b in Exercise 2A. The distributions were very similar but in part b all the $x$ values were multiplied by 10. All the averages and the standard deviations were also multiplied by 10 but the variances were multiplied by 100. This is an example of a transformation.
The most common type of transformation is a linear transformation. This is where the new variable \( Y \) is found from the old variable \( X \) by multiplying by a constant and/or adding on a constant. You might do this if you change the units of measurement. This kind of change is also known as ‘linear coding’.

If you know the original mean and variance and how the data was transformed, you can use a shortcut to find the mean and variance of the new distribution, as shown in Key point 2.3.

**Key point 2.3**

If \( Y = aX + b \) then

\[
\begin{align*}
E(Y) &= aE(X) + b \\
\text{Var}(Y) &= a^2 \text{Var}(X)
\end{align*}
\]

This means that the standard deviation of \( Y, \sigma_Y \), is \( |a| \sigma_X \). This makes sense as multiplying the data by \( a \) does change how spread out they are, but adding on \( b \) does not change the spread.

**WORKED EXAMPLE 2.4**

A random variable \( X \) has expectation 7 and variance 100. \( Y \) is a transformation of \( X \) given by \( Y = 100 - 2X \). Find:

- **a** the expectation of \( Y 
- **b** the standard deviation of \( Y 

\[
\begin{align*}
a \quad E(Y) &= 100 - 2 \times E(X) \\
&= 100 - 2 \times 7 \\
&= 86 \\
b \quad \text{Var}(Y) &= (-2)^2 \text{Var}(X) \\
&= 4 \times 100 \\
&= 400 \\
&= 400 = 20
\end{align*}
\]

This is just a direct application of Key point 2.3.

To find the standard deviation you need to first find the variance of \( Y \), using Key point 2.3.

**Tip**

It is easy to get confused with the minus sign in the transformations above. Remember that both variances and standard deviations are always positive.
EXERCISE 2B

1. If $E(X) = 9$ and $Var(X) = 25$ find $E(Y)$ and $Var(Y)$ if:
   a. $Y = 3X$
   b. $Y = X - 1$
   c. $Y = 2X - 1$
   d. $Y = 10 - 3X$
   e. $Y = \frac{X - 1}{4}$

2. If the random $X$ has $\mu = 10$ and $\sigma = 3$. Find:
   a. $E(2X)$
   b. $Var(2X)$

3. Stephen goes on a 30 mile bike ride every weekend. The distance until he stops for a picnic is modelled by $X$, where $E(X) = 20$ and $Var(X) = 16$.
   $Y$ is the amount of distance remaining after his picnic. Find $E(Y)$ and $Var(Y)$.

4. The rule for converting between degrees Celsius ($C$) and degrees Fahrenheit ($F$) is:
   
   $F = 1.8C + 32$.

   When a bread oven is operating it has expected temperature 200°C with standard deviation 5°C. Find the expected temperature and standard deviation in degrees Fahrenheit.

5. The random variable $X$ has expectation 10 and variance 25. If $Y = aX + b$ find the values of $a$ and $b$ so that the expectation of $Y$ is zero and the standard deviation is 1.

6. $X$ is a discrete random variable where $E(X) = 10$ and $E(X^2) = 200$. $Y$ is a transformation of $X$ such that $Y = X - 2$. Find $E(Y)$ and the standard deviation of $Y$.

7. A discrete random variable $X$ has equal expectation and standard deviation. $Y$ is a transformation of $X$ such that $Y = aX - b$. Prove that it is only possible for the expectation of $Y$ to equal the variance of $Y$ if $b < \frac{1}{4}$.

8. The St Petersburg Paradox describes a game where a fair coin is tossed repeatedly until a head is found. You win $2^n$ pounds if the first head occurs on the $n$th toss. How much should you pay to play this game?

Section 3: The discrete uniform distribution

You have already met some special distributions which occur so often that they are named – the binomial and the normal distributions.

Another very common distribution is the **discrete uniform distribution**. This is a distribution where all the whole numbers from 1 to $n$ are equally likely and it is given the symbol $U(n)$ (see Key point 2.4). For example, $U(6)$ gives the distribution of the outcomes on a fair dice.

### Key point 2.4

If $X \sim U(n)$ then $P(X = x) = \frac{1}{n}$ for $x = 1, 2, \ldots, n$.

If you identify a random variable as following a uniform distribution you can immediately write down the expectation and variance (see Key point 2.5).
Key point 2.5

If \( X \sim U(n) \) then \( P(X = x) = \frac{1}{n} \) for \( x = 1, 2, \ldots, n \) and \( E(X) = \frac{n+1}{2} \) and \( \text{Var}(X) = \frac{n^2 - 1}{12} \).

Although you do not need to be able to prove this, you may find it interesting to see the proof of these facts below. It uses the fact that

\[
\sum_{r=1}^{n} r = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}.
\]

PROOF 2

Prove Key point 2.5: If \( X \sim U(n) \) then \( E(X) = \frac{n+1}{2} \) and \( \text{Var}(X) = \frac{n^2 - 1}{12} \).

\[
E(X) = \sum_{r=1}^{n} r \times \frac{1}{n} = \frac{1}{n} \sum_{r=1}^{n} r = \frac{1}{n} \left( \frac{n(n+1)}{2} \right) = \frac{n+1}{2}.
\]

\[
E(X^2) = \sum_{r=1}^{n} r^2 \times \frac{1}{n} = \frac{1}{n} \sum_{r=1}^{n} r^2 = \frac{1}{n} \left( \frac{n(n+1)(2n+1)}{6} \right) = \frac{1}{6} \left( \frac{n(n+1)(2n+1)}{6} \right) = \frac{n+1}{2} \times \frac{(n+1)(2n+1)}{6}.
\]

\[
\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{n+1}{2} \left( \frac{n(n+1)(2n+1)}{6} \right) - \left( \frac{n+1}{2} \right)^2 = \frac{n+1}{2} \left( \frac{n+1}{2} \right) \left( \frac{2n+1}{6} \right).
\]

\[
= \frac{n+1}{2} \left( \frac{2n+1}{6} \right) \left( \frac{n+1}{2} \right) = \frac{n+1}{2} \left( \frac{2n+1}{6} \right) \left( \frac{n+1}{2} \right) = \frac{n^2 - 1}{12}.
\]
In Section 2 you saw how to find the expectation and variance of a linear transformation of a discrete random variable. The expectation and variance of a linear transformation of a discrete uniform distribution can be found in the same way.

**WORKED EXAMPLE 2.5**

The discrete random variable \( Y \) is equally likely to take any even value between 10 and 20 inclusive. Find the variance of \( Y \).

\[
Y = 2X + 8 \text{ where } X \sim U(6)
\]

The values of values of \( Y \) are 10, 12, 14... 20. These can be written as \( Y = 2X + 8 \), where \( X = 1, 2, \ldots, 6 \).

So \( Y \) is a linear transformation of \( X \sim U(6) \).

\[
\text{Var}(X) = \frac{35}{12}
\]

Applying Key point 2.5.

\[
\text{Var}(Y) = 2^2 \text{Var}(X) = \frac{35}{3}
\]

Applying Key point 2.3.

**EXERCISE 2C**

1. Find the mean and variance of the following distributions.
   a i) \( U(5) \)  ii) \( U(8) \)
   b i) \( U(2n) \)  ii) \( U(n-1) \)

2. A fair spinner has sides labelled 2, 4, 6, 8, 10. Find the expected mean and standard deviation of the results of the spinner.

3. A fair die has sides labelled 0, 1, 2, 3, 4, 5. Find the expectation and standard deviation of the outcome of the die.

4. a. The random variable \( Y \) is equally likely to take any integer value between \( -n \) and \( n \). Show that this can be written as \( aX = b \) where \( X \sim U(2n+1) \).
   b. Hence find the variance of \( Y \).

5. A string of 100 Christmas lights starts with a plug then contains a light every 4 cm from the plug. One light is broken. Assuming all bulbs are equally likely to break, what is the expected mean and variance of the distance of the broken light from the plug?

6. The random variable \( X \) is equally likely to take the value of any odd number between 1 and 99 inclusive. Find the variance of \( X \).

7. The discrete random variable \( Y \) takes values \( m, m+1, m+2\ldots m+n \). Find the expectation and variance of \( Y \).

8. \( X \sim U(n) \) and \( \text{Var}(X) = E(X) + 4 \). Find \( n \).

9. A random number, \( X \), is chosen from the fractions: \( \frac{1}{n}, \frac{2}{n}, \frac{3}{n}\ldots 1 \). Prove that \( E(X) > \frac{1}{2} \) but \( \text{Var}(X) < \frac{1}{12} \).

10. \( X \sim U(n) \). Prove that \( 6\text{Var}(X) \) is always divisible by \( E(X) \).
Section 4: Binomial distribution

You have already studied probabilities associated with the binomial distribution, however if you have identified a binomial situation you can immediately write down the expectation and variance (see Key point 2.6).

Key point 2.6

If $X \sim B(n, p)$ then $E(X) = np$ and $\text{Var}(X) = np(1 - p)$

This is in your formula booklet.

WORKED EXAMPLE 2.6

Find the expected value and the standard deviation in the number of 6’s when 20 dice are rolled.

If $X$ is the number of 6’s rolled then $X \sim B(20, \frac{1}{6})$

$E(X) = 20 \times \frac{1}{6} = 3.33$

$\text{Var}(X) = 20 \times \frac{1}{6} \times \frac{5}{6} = \frac{25}{9}$

So the standard deviation will be $\sqrt{\frac{25}{9}} = \frac{5}{3} = 1.67$.

EXERCISE 2D

Remember to round your answer to three significant figures when using the calculator.

1. Find the mean and standard deviation of each of the following variables
   
   a i $Y \sim B\left(100, \frac{1}{10}\right)$
   
   b i $X \sim B(15, 0.3)$
   
   c i $Z \sim B\left(n-1, \frac{1}{n}\right)$
   
   ii $X \sim B\left(16, \frac{1}{2}\right)$
   
   ii $Y \sim B(20, 0.35)$
   
   ii $X \sim B\left(n, \frac{2}{n}\right)$

2. A coin is biased so that when it is tossed the probability of obtaining heads is $\frac{2}{3}$. The coin is tossed 4050 times. Let $X$ be the number of heads obtained. Find:
   
   a the mean of $X$
   
   b the standard deviation of $X$

3. A biology test consists of eight multiple choice questions. Each question has four answers, only one of which is correct. At least five correct answers are required to pass the test. Sheila does not know the answers to any of the questions, so she answers each question at random.

   a What is the probability that Sheila answers exactly five questions correctly?
   
   b What is the expected number of correct answers Sheila will give?
   
   c What is the standard deviation of the number of correct answers Sheila will give?
   
   d What is the probability that Sheila manages to pass the test?
4 Given that \( Y \sim B(12, 0.4) \)
   a Find the expected mean of \( Y \).
   b Find the standard deviation of \( 2Y \).

5 A coin is biased so that the probability of it showing tails is \( p \). The coin is tossed \( n \) times. Let \( X \) be a random variable representing the number of tails. It is known that the mean of \( X \) is 19.5 and the variance is 6.825. Find the values of \( n \) and \( p \).

6 In an experiment, a trial is repeated \( n \) times. The trials are independent and the probability \( p \) of success in each trial is constant. Let \( X \) be the number of successes in the \( n \) trials. The mean of \( X \) is 12 and the standard deviation is 2. Find \( n \) and \( p \).

---

### Section 5: The geometric distribution

If there is a series of independent trials with a constant probability of success, the geometric distribution models the number of trials up to and including the first success. It only depends upon \( p \), the probability of a success. If \( X \) follows a geometric distribution you use the notation \( X \sim \text{Geo}(p) \).

To get the probability of \( X \) taking any particular value, \( x \), you use the fact that there must be \( x - 1 \) consecutive failures (each with probability \( q = 1 - p \)) followed by a single success.

#### Key point 2.7

If \( X \sim \text{Geo}(p) \) then \( P(X = x) = p(1-p)^{x-1} \) for \( x = 1, 2, 3, \ldots \)

This will appear in your formula booklet

It can be quite useful to apply similar ideas to get a result for \( P(X > x) \). For this situation to occur you must have started with \( x \) consecutive failures, therefore:

#### Key point 2.8

\[
P(X > x) = (1-p)^x.
\]

If you identify a variable as following a geometric distribution you can write down the expectation and variance.

#### Key point 2.9

If \( X \sim \text{Geo}(p) \) then:

\[
E(X) = \frac{1}{p},
\]

\[
\text{Var}(X) = \frac{(1-p)}{p^2}.
\]
WORKED EXAMPLE 2.7

a A normal six-sided dice is rolled. What is the probability that the first ‘6’ occurs
i on the fifth throw
ii after the fifth throw
b What is the expected number of throws it will take until a occurs?

\[ X = \text{d.r.v. ‘Number of throws until the first six’} \]
\[ X \sim \text{Geo} \left( \frac{1}{6} \right) \]

\[ P(X = 5) = \frac{1}{6} \times \left( \frac{5}{6} \right)^4 = 0.0804 \text{ (3 sf)} \]

\[ P(X > 5) = \left( \frac{5}{6} \right)^5 = 0.402 \text{ (3 sf)} \]

\[ E(X) = \frac{1}{\frac{1}{6}} = 6 \]

EXERCISE 2E

1 Find the following probabilities:

   a i \( P(X = 5) \) if \( X \sim \text{Geo} \left( \frac{1}{3} \right) \)
   ii \( P(X = 7) \) if \( X \sim \text{Geo} \left( \frac{1}{10} \right) \)

   b i \( P(X \leq 5) \) if \( X \sim \text{Geo} \left( \frac{1}{4} \right) \)
   ii \( P(X < 4) \) if \( X \sim \text{Geo} \left( \frac{2}{3} \right) \)

   c i \( P(X > 10) \) if \( X \sim \text{Geo} \left( \frac{1}{6} \right) \)
   ii \( P(X \geq 20) \) if \( X \sim \text{Geo} \left( 0.06 \right) \)

   d i The first boy born in a hospital on a given day is the 4th baby born (assuming no multiple births)
   ii A prize contained in 1 in 5 crisp packets is first won with the 8th crisp packet

2 Find the expected mean and standard deviation of:

   a i \( \text{Geo} \left( \frac{1}{3} \right) \)
   ii \( \text{Geo} \left( 0.15 \right) \)

   b i The number of attempts to hit a target with an arrow (there is a 1 in 12 chance of hitting the target on any given attempt).
   ii The number of times a die must be rolled before a multiple of three is scored.

3 The probability of passing a driving test on any given attempt is 0.4. It is Solomon assumes that he has this probability of passing his test and that his attempts are independent of each other.

   a Find the probability that Solomon passes the driving test on his third attempt.
   b Find the expected average number of attempts Solomon needs to pass the driving test.
   c How valid is the assumption that the attempts are independent of each other?
There are 12 green and 8 yellow balls in a bag. One ball is drawn from the bag and replaced. This is repeated until a yellow ball is drawn.

a Find the expected mean and variance of the number of balls drawn.

b Find the probability that the number of balls drawn is at most one standard deviation from the mean.

If $X \sim \text{Geo}(p)$ prove that $\sum_{i=1}^{\infty} P(X = i) = 1$

If $X \sim \text{Geo}(p)$ find the mode of $X$.

If $T \sim \text{Geo}(p)$ and $P(T = 4) = 0.0189$ find the value of $p$.

$Y \sim \text{Geo}(p)$ and the variance of $Y$ is 3 times larger than the mean of $Y$. Find the value of $p$.

a If $X \sim \text{Geo}\left(\frac{3}{4}\right)$ find the smallest value of $x$ such that $P(X = x) < 10^{-6}$.

b Find the smallest value of $x$ such that $P(X > x) < 10^{-6}$.

Prove that the standard deviation of a variable following a geometric distribution is always less than the mean.

---

**Checklist of learning and understanding**

- The expectation of a random variable $X$ is written $E(X)$ and calculated as $E(X) = \sum x_i p_i$.

- The variance of a random variable $X$ is written $\text{Var}(X)$ and calculated as $\text{Var}(X) = E(X^2) - [E(X)]^2$

  where $E(X^2) = \sum x_i^2 p_i$.

- If $Y = aX = b$ then $E(Y) = aE(X) + b$, $\text{Var}(Y) = a^2\text{Var}(X)$.

- A uniform distribution models situations where all discrete outcomes are equally likely.

  - If $X \sim \text{U}(n)$ then $P(X = x) = \frac{1}{n}$ for $x = 1, 2, \ldots, n$ and $E(X) = \frac{n+1}{2}$ and $\text{Var}(X) = \frac{n^2 - 1}{12}$.

  - If $X \sim \text{B}(n,p)$ then $E(X) = np$ and $\text{Var}(X) = np(1-p)$

- A geometric distribution $\text{Geo}(p)$, models the number of trials up to and including the first success in a situation with independent, constant probability trials.

  - If $X \sim \text{Geo}(p)$ then $P(X = x) = (1-p)^{x-1}p$, $P(X > x) = (1-p)^x$, $E(X) = \frac{1}{p}$ and $\text{Var}(X) = \frac{(1-p)}{p^2}$.
Mixed practice 2

1. A drawer contains three white socks and five black socks. Two socks are drawn without replacement. B is the number of black socks drawn.
   a. Find the probability distribution of B.
   b. Find E(B).

2. A fair six-sided dice is thrown once. The random variable X is calculated as half the result if the die shows an even number or one more than the result if the die shows an odd number.
   a. Create a table representing the probability distribution of X.
   b. Find E(X).
   c. Find Var(X).

3. a. X \sim U(13). Find the expectation and variance of X.
   b. Y is the discrete random variable which is equally likely to take any integer value between 14 and 26. Find E(Y) and Var(Y).
   c. Z is the discrete random variable which is equally likely to take any even value between 2 and 26. Find E(Z) and Var(Z).

4. The random variable X follows the following distribution:

\[
\begin{array}{c|c|c|c}
   x & 1 & 2 & 3 \\
   \hline
   p & a & b & 0.6 \\
\end{array}
\]

   a. If E(X) = 2.5 find the value of a and b.
   b. Hence find E(X^2) and show that Var(X) = 0.45.

5. X is a discrete random variable with E(X) = 10 and Var(X) = 16. Y = 12 – X. Find E(Y) and the standard deviation of Y.

6. When a four-sided spinner is spun, the number on which it lands is denoted by X, where X is a random variable taking values 2, 4, 6 and 8. The spinner is biased so that P(X = x) = kx, where k is a constant.
   a. Show that P(X = 6) = \frac{3}{10}
   b. Find E(X) and Var(X).

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[Question part reference style adapted]

7. The random variable X has expectation 12 and variance 100. If Y = aX + b find the values of a and b so that the expectation of Y is 10 and the standard deviation is 20.

8. X is a discrete random variable which can take the values 1 or 2. If E(X) = 1.2 find the standard deviation of X.
   a. If E(X) = 1.2 find the standard deviation of X.
   b. Y is also a discrete random variable defined by Y = 3X + 4. Find E(Y) and Var(Y).

9. A fair dice is thrown until a 6 has been thrown or three throws have been made. T is the discrete random variable ‘number of throws’ taken.
   a. Write down, in tabular form, the distribution of T.
   b. Find E(T)
   c. Find the median of T
   d. The number of points awarded in the game, P, is 4 – T. Find the variance of P.
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10 a A four-sided dice is rolled twice. Write down, in a table, the probability distribution of S, the sum of the two rolls.
b Find $E(S)$ and $\text{Var}(S)$
c A four-sided dice is rolled once and the score, $X$, is the twice the result. Find the mean and variance of $X$.

11 The discrete random variable $X$ follows the U(9) distribution. $\mu$ is the expectation of $X$ and $\sigma^2$ is the variance of $X$. Find $P(\mu - \sigma < X < \mu + \sigma)$

12 Approximately 1 in 30 people are carriers of the cystic fibrosis gene. In a sample of 1200 people state:
a the expected number of carriers of the cystic fibrosis gene
b the expected standard deviation in the number of carriers of the cystic fibrosis gene.

13 A squirrel drops nuts until one cracks. The probability of a nut cracking is 0.25. $X$ is the number of drops required up until and including when one cracks.
a State two assumptions required to model $X$ with a geometric distributions. How likely are these assumptions to hold in this context?
b If the required assumptions hold, find:
i $E(X)$
ii $P(X > 2)$

14 Sandra makes repeated, independent attempts to hit a target. On each attempt, the probability that she succeeds is 0.1.
a Find the probability that
i the first time she succeeds is on her 5th attempt
ii the first time she succeeds is after her 5th attempt
iii the second time she succeeds is before her 4th attempt.

Jill also makes repeated attempts to hit the target. Each attempt of either Jill or Sandra is independent. Each time that Jill attempts to hit the target, the probability that she succeeds is 0.2. Sandra and Jill take turns attempting to hit the target, with Sandra going first.
b Find the probability that the first person to hit the target is Sandra, on her
i 2nd attempt
ii 10th attempt.

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[Question part reference style adapted]

15 $X$ is a discrete random variable satisfying $P(X = x) = kx$ for $x = 1, 2, 3 \ldots n$.
Find, in terms of $n$:
a $k$
b $E(X)$
c $\text{Var}(X)$