

EXAMPLE 1.3-2: Absorption in a Lens

A lens is used to focus the illumination energy (i.e., radiation) that is required to develop the resist in a lithographic manufacturing process, as shown in Figure 1. The lens can be modeled as a plane wall with thickness $L = 1.0$ cm and thermal conductivity $k = 1.5$ W/m-K. The lens is not perfectly transparent but rather absorbs some of the illumination energy that is passed through it; the absorption coefficient of the lens is $\alpha = 0.1$ mm⁻¹. The flux of radiant energy that is incident at the lens surface ($x = 0$) is $\dot{q}_{rad}'' = 0.1$ W/cm². The top and bottom surfaces of the lens are exposed to air at $T_\infty = 20^\circ\text{C}$ and the average heat transfer coefficient on these surface is $\bar{h} = 20$ W/m²-K.

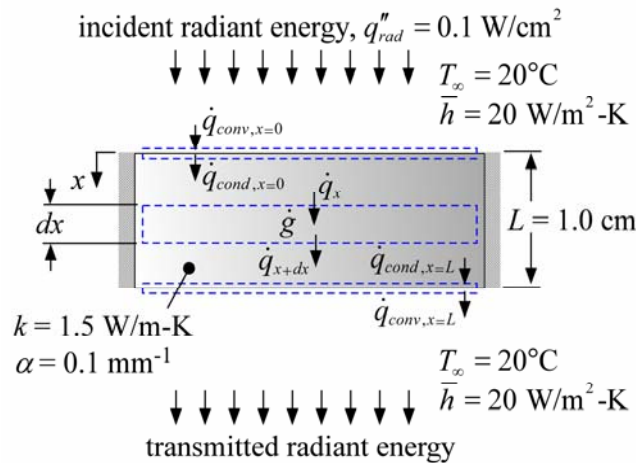


Figure 1: Lens absorbing radiant energy

The volumetric rate at which absorbed radiation is converted to thermal energy in the lens (\dot{g}''') is proportional to the local intensity of the radiant energy flux, which is reduced in the x -direction by absorption. The result is an exponentially distributed volumetric generation that can be expressed as:

$$\dot{g}''' = \dot{q}_{rad}'' \alpha \exp(-\alpha x) \quad (1)$$

a.) Determine and plot the temperature distribution within the lens.

The inputs are entered into EES:

```
"EXAMPLE 1.3-2: Energy absorption in a lens"
```

```
$UnitSystem SI MASS RAD PA K J
```

```
$Tabstops 0.2 0.4 0.6 3.5 in
```

```
k=1.5 [W/m-K]
```

```
L=1 [cm]*convert(cm,m)
```

```
alpha=0.1 [1/mm]*convert(1/mm,1/m)
```

```
qf_dot_rad=0.1 [W/cm^2]*convert(W/cm^2,W/m^2)
```

```
h_bar=20 [W/m^2-K]
```

```
T_infinity=converttemp(C,K,20 [C])
```

```
"conductivity"
```

```
"lens thickness"
```

```
"absorption coefficient"
```

```
"incident energy flux"
```

```
"average heat transfer coefficient"
```

```
"ambient air temperature"
```

$A_c=1 \text{ [m}^2\text{]}$

"carry out the problem on a per unit area basis"

An energy balance on an appropriate, differential control volume (see Figure 1) provides:

$$\dot{q}_x + \dot{g} = \dot{q}_{x+dx}$$

which can be expanded and simplified:

$$\dot{g} = \frac{d\dot{q}}{dx} dx$$

Substituting the rate equations for \dot{q} and \dot{g} leads to:

$$\dot{g}'' A_c = \frac{d}{dx} \left(-k A_c \frac{dT}{dx} \right) \quad (2)$$

where A_c is the cross-sectional area of the lens. Substituting Eq. (1) into Eq. (2) and simplifying leads to the governing differential equation for this problem.

$$\frac{d}{dx} \left(\frac{dT}{dx} \right) = -\frac{\dot{q}_{rad}'' \alpha}{k} \exp(-\alpha x) \quad (3)$$

The governing differential equation is entered in Maple.

> restart;

> GDE:=diff(diff(T(x),x),x)=-qf_dot_rad*alpha*exp(-alpha*x)/k;

$$GDE := \frac{d^2}{dx^2} T(x) = -\frac{qf_dot_rad \alpha e^{(-\alpha x)}}{k}$$

The general solution to the equation is obtained using the dsolve command:

> Ts:=dsolve(GDE);

$$Ts := T(x) = -\frac{qf_dot_rad e^{(-\alpha x)}}{\alpha k} + _C1 x + _C2$$

We can check that this solution is correct by integrating Eq. (3) twice by hand:

$$\int d \left(\frac{dT}{dx} \right) = -\int \frac{\dot{q}_{rad}'' \alpha}{k} \exp(-\alpha x) dx$$

which leads to:

$$\frac{dT}{dx} = \frac{\dot{q}_{rad}''}{k} \exp(-\alpha x) + C_1 \quad (4)$$

Equation (2) is integrated again:

$$\int dT = \int \left[\frac{\dot{q}_{rad}''}{k} \exp(-\alpha x) + C_1 \right] dx$$

which leads to:

$$T = -\frac{\dot{q}_{rad}''}{k \alpha} \exp(-\alpha x) + C_1 x + C_2 \quad (5)$$

Note that Eq. (5) is consistent with the result from Maple. The constants of integration, C_1 and C_2 are obtained by enforcing the boundary conditions. The boundary conditions for this problem are derived from “interface” energy balances at the two edges of the computational domain ($x = 0$ and $x = L$, as shown in Figure 1). It would be correct to include the radiant energy flux in the interface balances; however, because the interface thickness is zero, no radiant energy is absorbed and the amount of radiant energy entering and leaving the interface is the same and these terms would immediately cancel.

$$\dot{q}_{conv,x=0} = \dot{q}_{cond,x=0}$$

$$\dot{q}_{cond,x=L} = \dot{q}_{conv,x=L}$$

Substituting the rate equations for convection and conduction into the interface energy balances leads to boundary conditions for the temperature distribution:

$$\bar{h} A_c (T_\infty - T_{x=0}) = -k A_c \left. \frac{dT}{dx} \right|_{x=0} \quad (6)$$

$$-k A_c \left. \frac{dT}{dx} \right|_{x=L} = \bar{h} A_c (T_{x=L} - T_\infty) \quad (7)$$

Note that it was important to consider the direction of the energy transfers during the substitution of the rate equations; for example, $\dot{q}_{conv,x=0}$ was defined in Figure 1 as being *into* the top surface of the lens and therefore it is driven by $T_\infty - T_{x=0}$ while $\dot{q}_{conv,x=L}$ is defined as being *out of* the bottom surface of the lens and therefore it is driven by $T_{x=L} - T_\infty$. The general solution for the temperature distribution, Eq. (5) must be substituted into the boundary conditions, Eqs. (6) and (7), and solved algebraically to determine the constants C_1 and C_2 . Maple and EES can be used together in order to solve the differential equation, derive the symbolic expressions for the

boundary conditions, carry out the required algebra to obtain C_1 and C_2 , and manipulate the solution.

The temperature gradient can be obtained symbolically from Maple using the diff command.

```
> dTdx:=diff(Ts,x);
```

$$dTdx := \frac{d}{dx} T(x) = \frac{qf_dot_rad e^{(-\alpha x)}}{k} + _C1$$

The first boundary condition, Eq. (6), requires both the temperature and the temperature gradient evaluated at $x = 0$; the eval function in Maple can be used to symbolically determine these quantities (T0 and dTdx0):

```
> T0:=eval(Ts,x=0);
```

$$T0 := T(0) = -\frac{qf_dot_rad}{\alpha k} + _C2$$

Note that the result returned for T0 is almost, but not quite what is needed in Eq. (6). Use the rhs function to redefine T0 to be just the expression on the right hand side of the above result.

```
> T0:=rhs(T0);
```

$$T0 := -\frac{qf_dot_rad}{\alpha k} + _C2$$

The rhs function can be applied directly to the eval function; the statement below determines the symbolic expression for the temperature gradient evaluated at $x = 0$.

```
> dTdx0:=rhs(eval(dTdx,x=0));
```

$$dTdx0 := \frac{qf_dot_rad}{k} + _C1$$

These expressions for T0 and dTdx0 can be copied and pasted into EES in order to specify the first boundary condition. The equation format used by Maple is similar to that used in EES, so only minor modifications are required. Select the desired symbolic expressions in Maple; note that the selected text will appear to be highlighted. Use the Copy command from the Edit menu to place the selection on the Clipboard. When pasted into EES, the equations will appear as:

```
"boundary condition at x=0"
```

```
T0 := -1/alpha*qf_dot_rad/k+_C2
```

```
"temperature at x=0, copied from Maple"
```

```
dTdx0 := qf_dot_rad/k+_C1
```

```
"temperature gradient at x=0, copied from Maple"
```

All that is necessary to use this equation in EES is to change the := to = and change the constants _C1 and _C2 to C_1 and C_2, respectively; for lengthier expressions, the search and replace

feature in EES (select Replace from the Search menu) facilitates this process. After modification, the expressions should be:

$$\begin{aligned} T_0 &= -1/\alpha * q_{\text{dot_rad}}/k + C_2 && \text{"temperature at } x=0\text{"} \\ dT_{dx0} &= q_{\text{dot_rad}}/k + C_1 && \text{"temperature gradient at } x=0\text{"} \end{aligned}$$

The boundary condition at $x = 0$, Eq. (6), is specified in EES:

$$h_{\text{bar}} * A_c * (T_{\text{infinity}} - T_0) = -k * A_c * dT_{dx0} \quad \text{"boundary condition at } x=0\text{"}$$

The same process is used for the boundary condition at $x = L$, Eq. (5). The temperature and temperature gradient at $x = L$ are determined using Maple:

> TL:=rhs(eval(Ts,x=L));

$$TL := -\frac{q_{\text{dot_rad}} e^{(-\alpha L)}}{\alpha k} + C_1 L + C_2$$

> dTdxL:=rhs(eval(dTdx,x=L));

$$dTdxL := \frac{q_{\text{dot_rad}} e^{(-\alpha L)}}{k} + C_1$$

These expressions are copied into EES:

$$\begin{aligned} &\text{"boundary condition at } x=L\text{"} \\ TL &:= -1/\alpha * q_{\text{dot_rad}} * \exp(-\alpha * L)/k + C_1 * L + C_2 && \text{"temperature at } x=L, \text{ copied from Maple"} \\ dTdxL &:= q_{\text{dot_rad}} * \exp(-\alpha * L)/k + C_1 && \text{"temperature gradient at } x=L, \text{ copied from Maple"} \end{aligned}$$

and modified for consistency with EES:

$$\begin{aligned} TL &= -1/\alpha * q_{\text{dot_rad}} * \exp(-\alpha * L)/k + C_1 * L + C_2 && \text{"temperature at } x=L, \text{ copied from Maple"} \\ dTdxL &= q_{\text{dot_rad}} * \exp(-\alpha * L)/k + C_1 && \text{"temperature gradient at } x=L, \text{ copied from Maple"} \end{aligned}$$

The boundary condition at $x = L$, Eq. (7), is specified in EES:

$$-k * A_c * dTdxL = h_{\text{bar}} * A_c * (TL - T_{\text{infinity}}) \quad \text{"boundary condition at } x=L\text{"}$$

Solving the EES program will provide numerical values for both of the constants. The units should be set for each of the variables, including the constants, in order to ensure that the expressions are dimensionally consistent.

The general solution is copied from Maple to EES:

$$T(x) = -1/\alpha * q_{\text{dot_rad}} * \exp(-\alpha * x)/k + C_1 * x + C_2 \quad \text{"general solution, copied from Maple"}$$

and modified for consistency with EES:

$$\begin{aligned} x_{\text{mm}} &= 0 \text{ [mm]} && \text{"x-position, in mm"} \\ x &= x_{\text{mm}} * \text{convert(mm,m)} && \text{"x_position"} \end{aligned}$$

$T = -1/\alpha * qf_dot_rad * \exp(-\alpha * x) / k + C_1 * x + C_2$ "general solution for temperature"
 $T_C = \text{converttemp}(K, C, T)$ "in C"

The temperature as a function of position is shown in Figure 2; note the asymmetry produced by the non-uniform volumetric generation (i.e., more thermal energy is generated towards the top of the lens than the bottom and so the temperature is higher near the top surface of the lens).

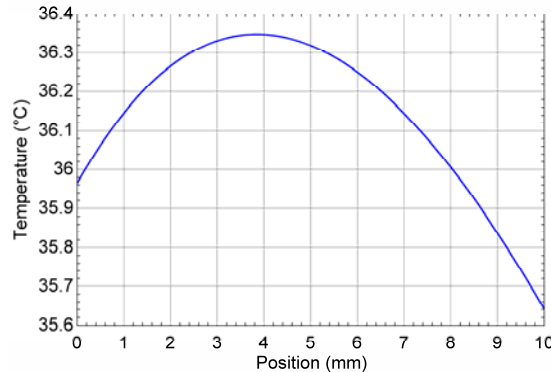


Figure 2: Temperature distribution in the lens.

- b.) Determine the location of the maximum temperature (x_{max}) and the value of the maximum temperature (T_{max}) in the lens.

The location of the maximum temperature in the lens can be determined by setting the temperature gradient, Eq. (2), to zero. The value of the maximum temperature (T_{max}) is obtained by substituting the resulting value of x_{max} into the equation for temperature.

The expression for the temperature gradient is copied from Maple, pasted into EES and modified for consistency.

$qf_dot_rad * \exp(-\alpha * x_max) / k + C_1 = 0$ "temperature gradient is zero at position x_max "
 $x_max_mm = x_max * \text{convert}(m, mm)$ "in mm"

The temperature at x_{max} is determined using the general solution.

$T_max = -1/\alpha * qf_dot_rad * \exp(-\alpha * x_max) / k + C_1 * x_max + C_2$ "maximum temperature in the lens"
 $T_max_C = \text{converttemp}(K, C, T_max)$ "in C"

It is not likely that solving the code above will immediately result in the correct value of either x_{max} or T_{max} . These are non-linear equations and EES must iterate find a solution; the success of this process depends in large part on the initial guesses and bounds used for the unknown variables. In most cases, any reasonable values of the guess and bounds will work. For this problem, set the lower and upper bounds on x_{max} to 0 (the top of the lens) and 0.01 m (10.0 mm, the bottom of the lens), respectively and the guess for x_{max} to 0.005 m (5 mm, the middle of the lens) using the Variable Information dialog (Figure 3). Upon solving, EES will correctly predict that $x_{max} = 0.38$ cm and $T_{max} = 36.35^\circ\text{C}$.

Variable	Guess	Lower	Upper	Display	Units
L	0.01	-infinity	infinity	A 3	N m
qf_dot_rad	1000	-infinity	infinity	A 3	N W/m^2
T	1	-infinity	infinity	A 3	N K
T0	1	-infinity	infinity	A 3	N K
TL	1	-infinity	infinity	A 3	N K
T_C	1	-infinity	infinity	A 3	N C
T_infinity	293.2	-infinity	infinity	A 3	N K
T_max	1	-infinity	infinity	A 3	N K
T_max_C	1	-infinity	infinity	A 3	N C
x	0	-infinity	infinity	A 3	N m
x_max	0.005	0.0000E+00	1.0000E-02	A 3	N m
x_max_mm	1	-infinity	infinity	A 3	N mm
x_mm	5	-infinity	infinity	A 3	N mm

Figure 3: Variable Information window showing the limits and guess value for the variable x_max.

It is also possible to determine the maximum temperature within the lens using EES' built-in optimization routines; there are several sophisticated single- and multi-variable optimization algorithms included with EES, accessed with the Min/Max command in the Calculate menu. The optimization capabilities are explained in Section 1.6.7.