EXAMPLE 1.8-1: Pipe in a Roof

A pipe with outer radius r_p = 5.0 cm emerges from a metal roof carrying hot gas at T_{hot} = 90ºC. The pipe is welded to the roof, as shown in Figure 1. Assume that the temperature at the interface between the pipe and the roof is equal to the gas temperature, T_{hot}. The inside of the roof is well-insulated, but the outside of the roof is exposed to ambient air at T_∞ = 20ºC. The average heat transfer coefficient between the outside of the roof and the ambient air is $\overline{h}$ = 50 W/m^2-K. The outside of the roof is also exposed to a uniform heat flux due to the incident solar radiation, $\dot{q}''_s$ = 800 W/m^2. The spatial extent of the roof is large with respect to the outer radius of the pipe. The metal roof has thickness th = 2.0 cm and thermal conductivity k = 50 W/m-K.

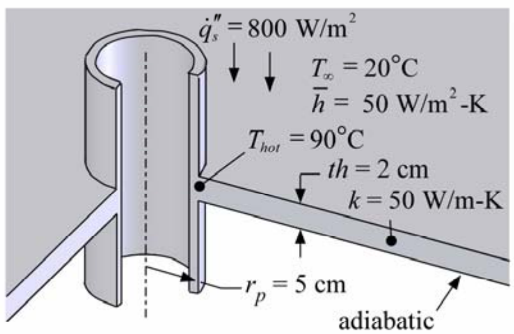

Figure 1: Pipe passing through a roof exposed to solar radiation.

a.) Can the roof be modeled using an extended surface approximation?

The input parameters are entered in EES:

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"EXAMPLE 1.8-1: Pipe in a Roof"
$UnitSystem SI MASS RAD PA K J
$Tabstops 0.2 0.4 0.6 3.5 in

"Input Parameters"
r_p = 5.0 [cm]*convert(cm,m)                    "Pipe radius"
T_hot = converttemp(C,K,90[C])                  "Hot gas temperature"
T_infinity = converttemp(C,K,20[C])             "Air temperature"
h_bar = 50 [W/m^2-K]                            "Heat transfer coefficient"
qf_s=800 [W/m^2]                                "Solar flux"
th = 2.0 [cm]*convert(cm,m)                     "Roof thickness"
k = 50 [W/m-K]                                  "Roof conductivity"
```

The extended surface approximation ignores any temperature gradients across the thickness of the roof. This assumption is equivalent to ignoring the resistance to conduction across the thickness of the roof and accounting only for the resistance associated with convection from the top surface of roof. The ratio of these resistances is calculated using an appropriately defined Biot number:

$$Bi = \frac{th\,\overline{h}}{k}$$

which is calculated in EES:

```
Bi=h_bar*th/k                    "Biot number to check extended surface approximation"
```

The Biot number is 0.02, which is sufficiently less than 1 to justify the extended surface approximation.

b.) Develop an analytical model for the roof that can be used to predict the temperature distribution in the roof and also determine the rate of heat loss from the pipe by conduction to the roof.

Because the roof is large relative to the spatial extent of our problem, the edge of the roof will have no effect on the temperature distribution in the metal around the pipe and the temperature distribution will be axisymmetric; the problem can be solved in radial coordinates.

An energy balance on a differential segment of the roof is shown in Figure 2.

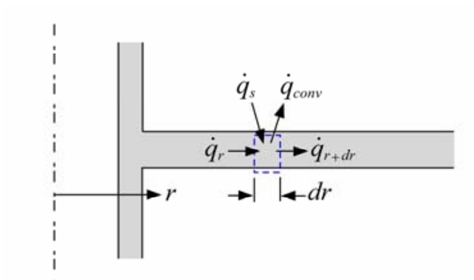

Figure 2: Differential energy balance.

The energy balance includes conduction, convection and solar irradiation:

$$\dot{q}_r + \dot{q}_s = \dot{q}_{r+dr} + \dot{q}_{conv}$$

or

$$\dot{q}_s = \frac{d\dot{q}_r}{dr} dr + \dot{q}_{conv}$$

Substituting the rate equations:

$$\dot{q}_r = -k\, 2\, \pi\, r\, th \frac{dT}{dr}$$

$$\dot{q}_s = \dot{q}_s''\, 2\, \pi\, r\, dr$$

$$\dot{q}_{conv} = 2\, \pi\, r\, dr\, \overline{h} \left(T - T_\infty\right)$$

into the energy balance leads to:

$$\dot{q}_s'' \, 2\pi r \, dr = \frac{d}{dr}\left[-k \, 2\pi r \, th \frac{dT}{dr}\right] dr + 2\pi r \, dr \, \overline{h}\left(T - T_\infty\right)$$

Simplifying leads to:

$$\frac{d}{dr}\left[r\frac{dT}{dr}\right] - \frac{\overline{h}}{k\,th} r\,T = -\frac{\overline{h}}{k\,th} r\,T_\infty - \frac{\dot{q}_s''}{k\,th} r$$

The solution is split into its homogeneous and particular components:

$$T = T_h + T_p$$

which leads to:

$$\underbrace{\frac{d}{dr}\left[r\frac{dT_h}{dr}\right] - \frac{\overline{h}}{k\,th} r\,T_h}_{=0 \text{ for homogeneous differential equation}} + \underbrace{\frac{d}{dr}\left[r\frac{dT_p}{dr}\right] - \frac{\overline{h}}{k\,th} r\,T_p = -\frac{\overline{h}}{k\,th} r\,T_\infty - \frac{\dot{q}_s''}{k\,th} r}_{\text{whatever is left is the particular differential equation}}$$

The solution to the particular differential equation:

$$\frac{d}{dr}\left[r\frac{dT_p}{dr}\right] - \frac{\overline{h}}{k\,th} r\,T_p = -\frac{\overline{h}}{k\,th} r\,T_\infty - \frac{\dot{q}_s''}{k\,th} r$$

is a constant:

$$T_p = T_\infty + \frac{\dot{q}_s''}{\overline{h}}$$

The homogeneous differential equation is:

$$\frac{d}{dr}\left[r\frac{dT_h}{dr}\right] - m^2\,r\,T_h = 0 \qquad (1)$$

where

$$m = \sqrt{\frac{\overline{h}}{k\,th}}$$

Equation (1) is a form of Bessel’s equation:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) \pm c^2 x^s \theta = 0 \tag{2}$$

where p = 1, c = m, and s = 1. Referring to the flow chart presented in Figure 1-54, the value of s - p+2 is equal to 2 and therefore the solution parameters n and a must be computed:

$$n = \frac{1-1}{1-1+2} = 0$$

$$a = \frac{2}{1-1+2} = 1$$

The last term in Eq. (1) is negative and therefore the solution to Eq. (2) is given by:

$$\theta = C_1\, x^{n/a}\, \text{BesselI}\left(n, c\, a\, x^{1/a}\right) + C_2\, x^{n/a}\, \text{BesselK}\left(n, c\, a\, x^{1/a}\right)$$

The homogeneous solution is:

$$T_h = C_1\, \text{BesselI}\left(0, m\, r\right) + C_2\, \text{BesselK}\left(0, m\, r\right)$$

The temperature distribution is the sum of the homogeneous and particular solutions:

$$T = C_1\, \text{BesselI}\left(0, m\, r\right) + C_2\, \text{BesselK}\left(0, m\, r\right) + T_\infty + \frac{\dot{q}_s''}{\bar{h}} \tag{3}$$

Maple can be used to obtain the same result:

```
> restart;
> ODE:=diff(r*diff(T(r),r),r)-m^2*r*T(r)=-m^2*r*T_infinity-qf_s*r/(k*th);
```

$$ODE := \left(\frac{d}{dr}\,\mathrm{T}(r)\right) + r\left(\frac{d^2}{dr^2}\,\mathrm{T}(r)\right) - m^2\, r\, \mathrm{T}(r) = -m^2\, r\, T\_infinity - \frac{qf\_s\; r}{k\; th}$$

```
> Ts:=dsolve(ODE);
```

$$Ts := \mathrm{T}(r) = \mathrm{BesselI}(0,\, m\, r)\; \_C2 + \mathrm{BesselK}(0,\, m\, r)\; \_C1 + \frac{m^2\; T\_infinity\;\; k\; th + qf\_s}{m^2\; k\; th}$$

Note that the constants C_1 and C_2 are interchanged in the Maple solution but it is otherwise the same as Eq. (3).

The boundary conditions must be used to obtain C_1 and C_2. As r approaches ∞, the effect of the pipe disappears. In this limit, the heat gain from the sun exactly balances convection, therefore:

$$\dot{q}''_s = \overline{h}\left(T_{r \to \infty} - T_\infty\right) \tag{4}$$

Substituting Eq. (3) into Eq. (4) leads to:

$$\dot{q}''_s = \overline{h}\left[C_1 \,\text{BesselI}\left(0,\infty\right) + C_2 \,\text{BesselK}\left(0,\infty\right) + T_\infty + \frac{\dot{q}''_s}{\overline{h}} - T_\infty\right]$$

or

$$C_1 \,\text{BesselI}\left(0,\infty\right) + C_2 \,\text{BesselK}\left(0,\infty\right) = 0$$

Figure 1-55 shows that the 0^{th} order modified Bessel function of the 1^{st} kind (i.e., BesselI(0,x)) limits to ∞ as x approaches ∞ while the 0^{th} order modified Bessel function of the 2^{nd} kind (i.e., BesselK(0,x)) approaches 0 as x approaches ∞. This information can be obtained using Maple and the limit command:

```
> limit(BesselI(0,x),x=infinity);
                                        ∞

> limit(BesselK(0,x),x=infinity);
                                        0
```

Therefore, C_1 must be zero while C_2 can be any finite value.

$$T = C_2 \,\text{BesselK}\left(0, m\, r\right) + T_\infty + \frac{\dot{q}''_s}{\overline{h}} \tag{3}$$

The temperature where the roof meets the pipe is specified:

$$T_{r=r_p} = T_{hot}$$

or

$$C_2 \,\text{BesselK}\left(0, m\, r_p\right) + T_\infty + \frac{\dot{q}''_s}{\overline{h}} = T_{hot}$$

The solution is programmed in EES:

```
m=sqrt(h_bar/(k*th))                                     "fin parameter"
C_2*BesselK(0,m*r_p)=T_hot-T_infinity-qf_s/h_bar         "boundary condition"
T=C_2*BesselK(0,m*r)+T_infinity+qf_s/h_bar               "solution"
T_C=converttemp(K,C,T)                                   "in C"
```

The temperature in the roof as a function of position is shown in Figure 3 for $\overline{h}$ = 50 W/m^2-K as specified in the problem statement, and also for $\overline{h}$ = 5 W/m^2-K.

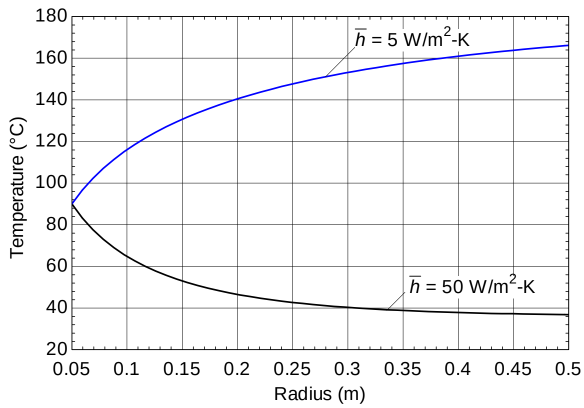

Figure 3: Temperature as a function of radius for $\bar{h}$ = 50 W/m^2-K and $\bar{h}$ = 5 W/m^2-K with $\dot{q}''_s$ = 800 W/m^2.

The heat transfer between the pipe and the roof ($\dot{q}_p$) is evaluated using Fourier's law at $r = r_p$:

$$\dot{q}_p = -k\, th\, 2\, \pi\, r_p \left. \frac{dT}{dr} \right|_{r=r_p} \tag{4}$$

Substituting Eq. (3) into Eq. (4) leads to:

$$\dot{q}_p = -k\, th\, 2\, \pi\, r_p\, C_2 \frac{d}{dr}\left[\text{BesselK}\left(0, m\, r\right)\right]_{r=r_p}$$

which can be evaluated using the differentiation rule provided by Eq. (1-400):

$$\dot{q}_p = k\, th\, 2\, \pi\, r_p\, C_2\, m\, \text{BesselK}\left(1, m\, r_p\right)$$

or using Maple:

```
> q_dot_p:=-k*th*2*pi*r_p*C_2*eval(diff(BesselK(0,m*r),r),r=r_p);
```

$$q\_dot\_p := 2\, k\, th\, \pi\, r\_p\, C\_2\, \text{BesselK}(1, m\, r\_p)\, m$$

The solution is programmed in EES:

```
q_dot_p=k*th*2*pi*r_p*C_2*m*BesselK(1,m*r_p)          "heat transfer into pipe"
```

Figure 4 illustrates the heat transfer into the pipe as a function of the heat transfer coefficient and for various values of the solar flux.

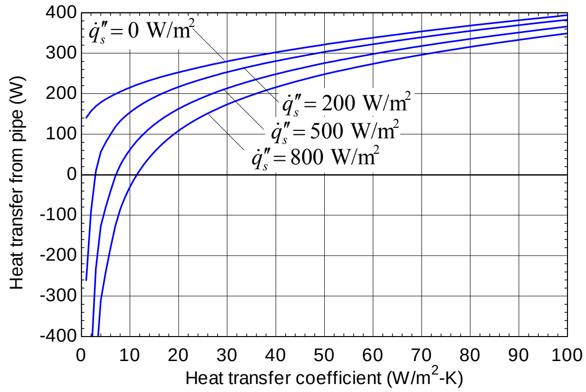

Figure 4: Heat transfer from pipe to roof as a function of the heat transfer coefficient for various values of the solar flux.

It is always important to understand your solution after it has been obtained; notice in Figure 4 that the rate of heat transfer to the roof tends to increase with increasing heat transfer coefficient. This makes sense as the temperature gradient at the interface between the roof and the pipe will increase as the heat transfer coefficient increases. However, when there is a non-zero solar flux, the heat transfer rate will change direction (i.e., become negative) at low values of the heat transfer coefficient indicating that the heat flow is into the pipe under these conditions. This effect occurs when the solar flux elevates the temperature of the roof to the point that it is above the hot gas temperature. Figure 4 shows that we can expect this behavior for $\overline{h} = 5$ W/m^2-K and $\dot{q}_s'' = 800$ W/m^2 and Figure 3 illustrates the temperature distribution under these conditions.