EXAMPLE 1.8-2: Magnetic Ablation with Blood Perfusion

EXAMPLE 1.3-1 examined an ablative technique for heating cancerous tissue locally using small, conducting spheres (thermoseeds) that are embedded at precise locations and exposed to a magnetic field. The thermoseed experiences a volumetric generation of thermal energy that causes their temperature and the temperature of the adjacent tissue to rise. In EXAMPLE 1.3-1, blood perfusion in the tissue was neglected; blood perfusion refers to the volumetric removal of heat in the tissue by the blood flowing in the microvascular structure.

The blood perfusion in the tissue may be modeled as a volumetric heat sink that is proportional to the difference between the local temperature and the normal body temperature (T_b = 37ºC); the constant of proportionality, β, is nominally 20,000 W/m^3-K. The thermoseed has a radius r_{ts} = 1.0 mm and it experiences a total of $\dot{g}_{ts}$ = 1.0 W generation. The temperature far from the thermoseed is the body temperature, T_b. The tissue has thermal conductivity k_t = 0.5 W/m-K.

a.) Determine the steady-state temperature distribution in the tissue associated with a single sphere placed in an infinite medium of tissue considering blood perfusion.

The input parameters are entered in EES:

```
"EXAMPLE 1.8-2: Magnetic Ablation with Blood Perfusion"
$UnitSystem SI MASS  DEG PA C J
$Tabstops 0.2 0.4 0.6 0.8 3.5

"Inputs"
r_ts=1.0 [mm]*convert(mm,m)                                "radius of thermoseed"
T_b=converttemp(C,K,37 [C])                                "blood and body temperature"
g_dot_ts=1.0 [W]                                           "generation in the thermoseed"
beta=50000 [W/m^3-K]                                       "blood perfusion constant"
k_t=0.5 [W/m-K]                                            "tissue conductivity"
```

Figure 1 illustrates a differential control volume in the tissue that balances conduction with blood perfusion. The energy balance on the control volume is:

$$\dot{q}_r = \dot{q}_{r+dr} + \dot{g}$$

where $\dot{q}$ is conduction and $\dot{g}$ is the rate of energy *removed* by blood perfusion.

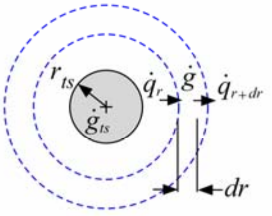

Figure 1: Differential control volume in the tissue

The conduction through the tissue is given by:

$$\dot{q}_r = -k_t \, 4 \pi r^2 \frac{dT}{dr}$$

and the rate of energy removal by blood perfusion is:

$$\dot{g} = 4 \pi r^2 \, dr \, \beta \left(T - T_b \right)$$

Combining these equations leads to:

$$0 = \frac{d}{dr}\left[-k_t \, 4 \pi r^2 \frac{dT}{dr} \right] dr + 4 \pi r^2 \, dr \, \beta \left(T - T_b \right)$$

which can be simplified:

$$\frac{d}{dr}\left[r^2 \frac{dT}{dr} \right] - \frac{\beta}{k_t} r^2 \, T = -\frac{\beta}{k_t} r^2 \, T_b$$

The solution is divided into its homogeneous and particular components:

$$T = T_h + T_p$$

so that:

$$\underbrace{\frac{d}{dr}\left[r^2 \frac{dT_h}{dr} \right] - \frac{\beta}{k_t} r^2 \, T_h}_{=0 \text{ for homogeneous differential equation}} + \underbrace{\frac{d}{dr}\left[r^2 \frac{dT_p}{dr} \right] - \frac{\beta}{k_t} r^2 \, T_p = -\frac{\beta}{k_t} r^2 \, T_b}_{\text{whatever is left is the particular differential equation}}$$

The particular solution is:

$$T_p = T_b$$

The homogeneous differential equation is:

$$\frac{d}{dr}\left[r^2 \frac{dT_h}{dr} \right] - m^2 \, r^2 \, T_h = 0 \qquad (1)$$

where

$$m = \sqrt{\frac{\beta}{k_t}}$$

Equation (1) is a form of Bessel's equation:

$$\frac{d}{dx}\left(x^p\,\frac{d\theta}{dx}\right)\pm c^2\,x^s\,\theta=0$$

where p = 2, c = m, and s = 2. Referring to the flow chart presented in Figure 1-54, the value of s - p+2 is equal to 2 and therefore the solution parameters n and a must be computed:

$$n=\frac{1-2}{2-2+2}=-\frac{1}{2}$$

$$a=\frac{2}{2-2+2}=1$$

The last term in Eq. (1) is negative and therefore the solution is given by:

$$\theta=C_1\,x^{n/a}\,\text{BesselI}\left(n,c\,a\,x^{1/a}\right)+C_2\,x^{n/a}\,\text{BesselK}\left(n,c\,a\,x^{1/a}\right)$$

or

$$T_h=C_1\,r^{-1/2}\,\text{BesselI}\left(-\frac{1}{2},m\,r\right)+C_2\,r^{-1/2}\,\text{BesselK}\left(-\frac{1}{2},m\,r\right)$$

The solution is the sum of the homogeneous and particular solutions:

$$T=C_1\,r^{-1/2}\,\text{BesselI}\left(-\frac{1}{2},m\,r\right)+C_2\,r^{-1/2}\,\text{BesselK}\left(-\frac{1}{2},m\,r\right)+T_b \tag{2}$$

The constants are obtained by applying the boundary conditions. As r approaches ∞, the temperature must approach the body temperature:

$$T_{r\to\infty}=T_b \tag{3}$$

Substituting Eq. (2) into Eq. (3) leads to:

$$C_1\,\frac{\text{BesselI}\left(-\frac{1}{2},\infty\right)}{\sqrt{\infty}}+C_2\,\frac{\text{BesselK}\left(-\frac{1}{2},\infty\right)}{\sqrt{\infty}}=0 \tag{4}$$

At first glance it is unclear how Eq. (4) helps to establish the constants; however, Maple can be used to show that C_1 must be zero because the 1st term limits to ∞ while the 2nd term limits to 0:

```
> limit(BesselI(-1/2,r)/sqrt(r),r=infinity);
                                        ∞

> limit(BesselK(-1/2,r)/sqrt(r),r=infinity);
                                        0
```

The second boundary condition obtained from an interface energy balance at $r = r_{ts}$; the rate of conduction heat transfer into the tissue must equal the rate of generation within the thermoseed:

$$-4\pi r_{ts}^2 k_t \left.\frac{dT}{dr}\right|_{r=r_{ts}} = \dot{g}_{sp} \tag{5}$$

Substituting Eq. (2) with $C_1 = 0$ into Eq. (5) leads to:

$$-4\pi r_{ts}^2 k_t C_2 \frac{d}{dr}\left[r^{-1/2}\,\mathrm{BesselK}\left(-\frac{1}{2}, m\,r\right)\right]_{r=r_{ts}} = \dot{g}_{sp}$$

Using Eq. (1-408) leads to:

$$-C_2\, r_{ts}^{-1/2}\, m\,\mathrm{BesselK}\left(-\frac{3}{2}, m\,r_{ts}\right) = -\frac{\dot{g}_{sp}}{4\pi r_{ts}^2 k_t}$$

The constant C_2 is evaluated in EES:

```
"Determine constant"
m=sqrt(beta/k_t)                                                  "solution parameter"
-C_2*m*BesselK(-3/2,m*r_ts)/sqrt(r_ts)=-g_dot_ts/(4*pi*r_ts^2*k_t)   "determine constant"
```

The solution is programmed in EES and converted to Celsius:

```
"Solution"
T=C_2*BesselK(-0.5,m*r)/sqrt(r)+T_b                  "temperature"
T_C=converttemp(K,C,T)                               "in C"
r_mm=r*convert(m,mm)                                 "radius in mm"
```

Figure 2 illustrates the temperature in the tissue as a function of radial position for various values of blood perfusion. Note that the temperature distribution as $\beta \to 0$ (i.e., in the absence of blood perfusion) agrees exactly with the solution for the tissue temperature obtained in EXAMPLE 1.3-1 (which is overlaid onto Figure 2) although it looks very different. Figure 2 shows that the effect of blood perfusion is to reduce the extent of the elevated temperature region and therefore diminish the amount of tissue killed by the thermoseed.

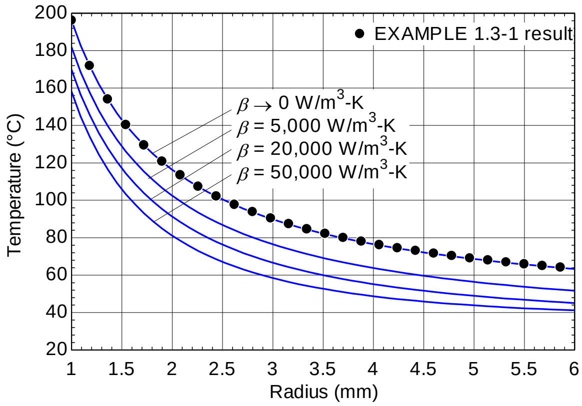

Figure 2: Temperature in the tissue as a function of radius for various values of blood perfusion; also shown is the result from EXAMPLE 1.3-1 which was derived for the same problem in the absence of blood perfusion ($\beta = 0$).