# EXAMPLE 2.2-1: Temperature Distribution in a 2-D Fin

In Section 1.6 the constant cross-section, straight fin shown in Figure 1 was analyzed under the assumption that it could be treated as an extended surface (i.e., temperature gradients in the $y$ direction can be neglected). In this example, the 2-D temperature distribution within the fin will be determined using separation of variables.

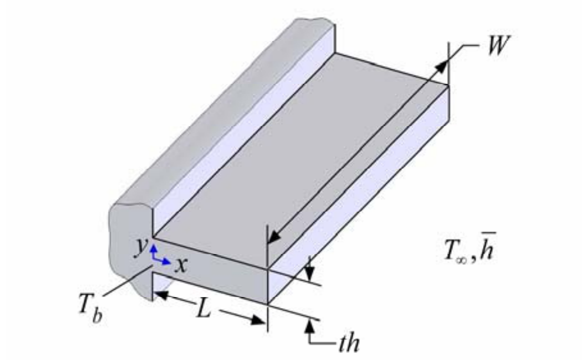
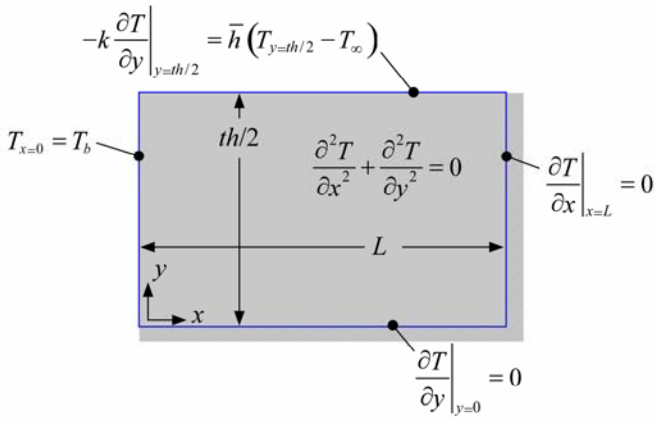


**Figure 1: Straight, constant cross-sectional area fin.**

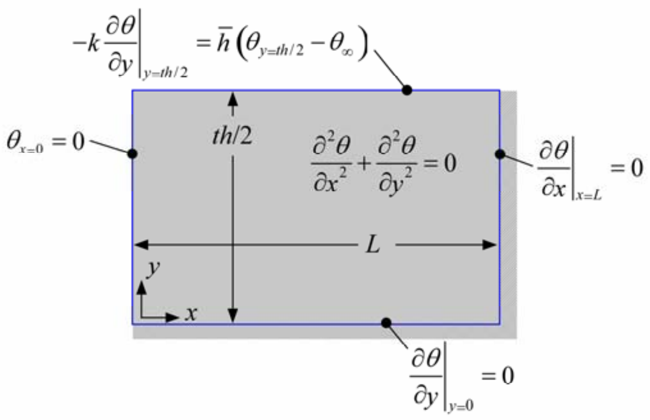
Assume that the tip of the fin is insulated and that the width ($W$) is much larger than the thickness ($th$) so that convection from the edges can be neglected. The length of the fin is $L$. The fin base temperature is $T_b$ and the fin experiences convection with fluid at $T_\infty$ with average heat transfer coefficient, $\overline{h}$ .

a.) Develop an analytical solution for the temperature distribution in the fin using separation of variables.

The upper and lower halves of the fin are symmetric; that is, there is no difference between the upper and lower portions of the fin and therefore there can be no heat transfer across the mid-plane of the fin. The mid-plane of the fin (i.e., the surface at $y$ = 0) can therefore be treated as if it were adiabatic. The computational domain including the boundary conditions is shown in Figure 2(a).



**(a)**



**(b)**

**Figure 2: Problem statement posed in terms of (a) temperature, *T*, and (b) temperature difference, $\theta$.**

The governing equation within the fin can be derived using the process described in Section 2.2.2 and is:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

Figure 2(a) indicates that the problem stated in terms of $T$ has two non-homogeneous boundary conditions (the base and the top surface). However, the boundary condition at the base can be made homogeneous by defining:

$$\theta=T-T_b$$

so that the governing equation becomes:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \tag{1}$$

The boundary conditions for the transformed problem, illustrated in Figure 2(b), are:

$$\theta_{x=0}=0 \tag{2}$$

$$\frac{\partial\theta}{\partial x}\bigg|_{x=L}=0 \tag{3}$$

$$\frac{\partial\theta}{\partial y}\bigg|_{y=0}=0 \tag{4}$$

$$-k\left.\frac{\partial \theta}{\partial y}\right|_{y=th/2} = \overline{h}\left(\theta_{y=th/2} - \theta_{\infty}\right) \tag{5}$$

where

$$\theta_{\infty} = T_{\infty} - T_b$$

The problem stated in terms of $\theta$ satisfies all of the requirements discussed in Section 2.2.2 with $x$ being the homogeneous direction. Therefore, the separation of variables solution proceeds using the steps laid out in Section 2.2.2. The solution for the temperature difference ($\theta$) is expressed as the product of a function only of $x$ ($\theta X$) and a function only of $y$ ($\theta Y$):

$$\theta(x,y) = \theta X(x)\, \theta Y(y) \tag{6}$$

Substitution of Eq. (6) into Eq. (1) leads to two ordinary differential equations, as shown in Section 2.2.1:

$$\frac{d^2 \theta X}{dx^2} \pm \lambda^2\, \theta X = 0$$

$$\frac{d^2 \theta Y}{dy^2} \mp \lambda^2\, \theta Y = 0$$

It is necessary to determine the sign of the constant $\lambda^2$ in the ordinary differential equations. Recall that it is necessary to have the sine/cosine eigenfunctions in the homogeneous direction. Therefore, it is necessary to select the positive sign for the ordinary differential equation for $\theta X$ and the negative sign for the ordinary differential equation for $\theta Y$:

$$\frac{d^2 \theta X}{dx^2} + \lambda^2\, \theta X = 0 \tag{7}$$

$$\frac{d^2 \theta Y}{dy^2} - \lambda^2\, \theta Y = 0 \tag{8}$$

The next step is to solve the eigenproblem (i.e., the problem for $\theta X$); the solution to the ordinary differential equation for $\theta X$, Eq. (7), is:

$$\theta X = C_1 \sin(\lambda\, x) + C_2 \cos(\lambda\, x) \tag{9}$$

The boundary conditions for $\theta X$ are obtained by substituting Eq. (9) into Eqs. (2) and (3):

$$\theta X_{x=0} = 0 \tag{10}$$

$$\left.\frac{d\theta X}{dx}\right|_{x=L} = 0 \tag{11}$$

Substituting Eq. (9) into Eq. (10) leads to:

$$\theta X_{x=0} = C_1 \underbrace{\sin\left(\lambda\, 0\right)}_{0} + C_2 \underbrace{\cos\left(\lambda\, 0\right)}_{1} = 0$$

which can only be true if $C_2 = 0$. Substituting Eq. (9), with $C_2 = 0$, into Eq. (11) leads to:

$$\left.\frac{d\theta X}{dx}\right|_{x=L} = -C_1\, \lambda \cos\left(\lambda\, L\right) = 0$$

which can only be true if the argument of the cosine function is $\pi/2$, $3\pi/2$, $5\pi/2$, etc. Therefore, the argument of the cosine function must be:

$$\lambda_i\, L = \frac{\left(1+2\,i\right)}{2}\pi \quad \text{where } i = 0,1,2,...$$

The eigenfunctions of the problem are:

$$\theta X_i = C_{1,i} \sin\left(\lambda_i\, x\right) \ \text{ where } i = 0,1,2,... \tag{12}$$

and the eigenvalues of the problem are:

$$\lambda_i = \frac{\left(1+2\,i\right)\pi}{2\,L} \tag{13}$$

The next step is to solve the problem in the non-homogeneous direction. The ordinary differential equation in the *y*-direction that is associated with each eigenvalue is:

$$\frac{d^2\theta Y_i}{dy^2} - \lambda_i^2\, \theta Y_i = 0$$

which is solved by either

$$\theta Y_i = C_{3,i} \exp\left(\lambda_i\, y\right) + C_{4,i} \exp\left(-\lambda_i\, y\right)$$

or

$$\theta Y_i = C_{3,i}\cosh\left(\lambda_i\, y\right) + C_{4,i}\sinh\left(\lambda_i\, y\right) \tag{14}$$

The choice of either exponentials or sinh and cosh is arbitrary in that both will lead to the correct solution. However, the proper choice often makes the solution process easier. The plate in Figure 2-3 extended to infinity where the temperature became zero. As a result, the constant multiplying the positive exponential was forced to be zero which made the problem somewhat easier to solve. Looking ahead for this fin problem, we see that the gradient of temperature at $y = 0$ must be 0. This boundary condition would not eliminate either of the exponential terms. On the other hand, the boundary condition will force the constant $C_{4,i}$ in Eq. (14) to be zero and therefore the sinh term will be eliminated. Clearly then Eq. (14) is the better choice; a little insight early in the problem can make the solution process easier.

The next step is to determine the temperature difference solution associated with each eigenvalue:

$$\theta_i = \theta X_i\, \theta Y_i = \sin\left(\lambda_i\, x\right)\left[C_{3,i}\cosh\left(\lambda_i\, y\right) + C_{4,i}\sinh\left(\lambda_i\, y\right)\right]$$

where the constant $C_{2,i}$ was absorbed into the constants $C_{3,i}$ and $C_{4,i}$. This solution should satisfy both of the homogeneous boundary conditions as well as the partial differential equation for all values of $i$; it is worthwhile using Maple to verify that this is true. Specify that $i$ is an integer and enter the definition of the eigenvalues:

```
> restart;
> assume(i,integer);
> lambda:=(1+2*i)*Pi/(2*L);
```

$$\lambda := \frac{(1 + 2\, i\sim)\, \pi}{2\, L}$$

Enter the solution for each eigenvalue:

```
> T:=(x,y)->sin(lambda*x)*(C3*cosh(lambda*y)+C4*sinh(lambda*y));
```

$$T := (x, y) \rightarrow \sin(\lambda\, x)\, (C3\, \cosh(\lambda\, y) + C4\, \sinh(\lambda\, y))$$

Verify that the solution satisfies the two boundary conditions in the $x$-direction, Eqs. (2) and (3):

```
> T(0,y);
```

$$0$$

```
> eval(diff(T(x,y),x),x=L);
```

$$0$$

and the partial differential equation, Eq. (1):

```
> diff(diff(T(x,y),x),x)+diff(diff(T(x,y),y),y);
```

$$-\frac{1}{4}\,\frac{\sin\left(\frac{(1+2\,i\sim)\,\pi\,x}{2\,L}\right)(1+2\,i\sim)^2\,\pi^2\left(C3\,\cosh\left(\frac{(1+2\,i\sim)\,\pi\,y}{2\,L}\right)+C4\,\sinh\left(\frac{(1+2\,i\sim)\,\pi\,y}{2\,L}\right)\right)}{L^2}$$

$$+\sin\left(\frac{(1+2\,i\sim)\,\pi\,x}{2\,L}\right)\left(\frac{1}{4}\,\frac{C3\,\cosh\left(\frac{(1+2\,i\sim)\,\pi\,y}{2\,L}\right)(1+2\,i\sim)^2\,\pi^2}{L^2}+\frac{1}{4}\,\frac{C4\,\sinh\left(\frac{(1+2\,i\sim)\,\pi\,y}{2\,L}\right)(1+2\,i\sim)^2\,\pi^2}{L^2}\right)$$

```
> simplify(%);
```

$$0$$

The sum of the solutions for each eigenvalue becomes the general solution to the problem:

$$\theta=\sum_{i=0}^{\infty}\theta_i=\theta X_i\,\theta Y_i=\sum_{i=0}^{\infty}\sin(\lambda_i\,x)\left[C_{3,i}\cosh(\lambda_i\,y)+C_{4,i}\sinh(\lambda_i\,y)\right] \qquad (15)$$

The boundary conditions in the non-homogeneous directions are enforced. Substituting Eq. (15) into Eq. (4) leads to:

$$\left.\frac{\partial\theta}{\partial y}\right|_{y=0}=\sum_{i=0}^{\infty}\sin(\lambda_i\,x)\left[C_{3,i}\,\lambda_i\underbrace{\sinh(\lambda_i\,0)}_{0}+C_{4,i}\,\lambda_i\underbrace{\cosh(\lambda_i\,0)}_{1}\right]=0$$

The cosh(0) = 1 and the sinh(0) =0 (much like the cos(0) = 1 and the sin(0)=0) and therefore this boundary condition can be written as:

$$\sum_{i=0}^{\infty}\sin(\lambda_i\,x)\,C_{4,i}\,\lambda_i\;=0$$

which can only be true if $C_{4,i} = 0$, therefore:

$$\theta=\sum_{i=0}^{\infty}C_i\sin(\lambda_i\,x)\cosh(\lambda_i\,y) \qquad (16)$$

where the subscript 3 has been removed from $C_{3,i}$ as it is the only remaining undetermined constant. Equation (16) is substituted into the boundary condition at $y = th/2$, Eq. (5):

$$-k\sum_{i=0}^{\infty}C_i\sin(\lambda_i\,x)\,\lambda_i\sinh\left(\lambda_i\frac{th}{2}\right)=\bar{h}\left(\sum_{i=0}^{\infty}C_i\sin(\lambda_i\,x)\cosh\left(\lambda_i\frac{th}{2}\right)-\theta_\infty\right)$$

which can be rearranged:

$$\sum_{i=0}^{\infty} C_i \sin\left(\lambda_i\, x\right)\left[\frac{k\,\lambda_i}{\bar{h}}\sinh\left(\lambda_i\,\frac{th}{2}\right)+\cosh\left(\lambda_i\,\frac{th}{2}\right)\right]=\theta_\infty \tag{17}$$

The eigenfunctions must be orthogonal between $x = 0$ and $x = L$ (it is not necessary to prove this for each problem) and therefore Eq. (17) can be converted into an algebraic equation for each individual constant. Equation (17) is multiplied by one eigenfunction, $\sin(\lambda_j\, x)$, and integrated from $x = 0$ to $x = L$:

$$\sum_{i=0}^{\infty} C_i \left[\frac{k\,\lambda_i}{\bar{h}}\sinh\left(\lambda_i\,\frac{th}{2}\right)+\cosh\left(\lambda_i\,\frac{th}{2}\right)\right]\int_0^L \sin\left(\lambda_i\, x\right)\sin\left(\lambda_j\, x\right)dx=\theta_\infty\int_0^L \sin\left(\lambda_j\, x\right)dx$$

Orthogonality guarantees that the integral on the left side of this equation will be zero for every term in the summation except the one where $i = j$; therefore, the series equation can be rewritten as:

$$C_i\left[\frac{k\,\lambda_i}{\bar{h}}\sinh\left(\lambda_i\,\frac{th}{2}\right)+\cosh\left(\lambda_i\,\frac{th}{2}\right)\right]\int_0^L \sin^2\left(\lambda_i\, x\right)dx=\theta_\infty\int_0^L \sin\left(\lambda_i\, x\right)dx$$

The coefficients can be evaluated according to:

$$C_i=\frac{\theta_\infty\displaystyle\int_0^L \sin\left(\lambda_i\, x\right)dx}{\left[\dfrac{k\,\lambda_i}{\bar{h}}\sinh\left(\lambda_i\,\dfrac{th}{2}\right)+\cosh\left(\lambda_i\,\dfrac{th}{2}\right)\right]\displaystyle\int_0^L \sin^2\left(\lambda_i\, x\right)dx} \tag{18}$$

The integrals in Eq. (18) can be evaluated either using math tables or, more easily, using Maple:

```
> restart;
> assume(i,integer);
> lambda:=(1+2*i)*Pi/(2*L);
```

$$\lambda := \frac{(1+2\,i\sim)\,\pi}{2\,L}$$

```
> int(sin(lambda*x),x=0..L);
```

$$\frac{2\,L}{(1+2\,i\sim)\,\pi}$$

```
> int(sin(lambda*x)*sin(lambda*x),x=0..L);
```

$$\frac{L}{2}$$

The constants can therefore be written as:

$$C_i = \frac{2\,\theta_\infty}{L\,\lambda_i\left[\frac{k\,\lambda_i}{\bar{h}}\sinh\left(\lambda_i\frac{th}{2}\right)+\cosh\left(\lambda_i\frac{th}{2}\right)\right]} \tag{19}$$

Equations (16) and (19) together provide the analytical solution for the temperature distribution within the fin.

b.) Use the analytical solution to predict and plot the temperature distribution in a fin that is $L$ = 5.0 cm long, $th$ = 2.0 cm thick, with conductivity $k$ = 0.5 W/m-K, and $\bar{h}$ = 100 W/m$^2$-K. The base temperature is $T_b$ = 200°C and the fluid temperature is $T_\infty$ = 20°C.

The inputs are entered in EES:

```
"EXAMPLE 2.2-1: 2-D Fin"
$UnitSystem SI MASS  RAD PA C J
$Tabstops 0.2 0.4 0.6 0.8 3.5

"Inputs"
th_cm=4 [cm]                                "thickness of fin in cm"
th = th_cm*convert(cm,m)                    "thickness of fin"
L=5 [cm]*convert(cm,m)                      "length of fin"
th=th_cm*convert(cm,m)                      "width of fin"
k=0.5 [W/m-K]                               "thermal conductivity"
h_bar=100 [W/m^2-K]                         "heat transfer coefficient"
T_b=converttemp(C,K,200[C])                 "base temperature"
T_infinity=converttemp(C,K,20[C])           "fluid temperature"
```

Dimensionless coordinates within the fin are defined in order to facilitate plotting the temperature distribution:

```
y_bar=0.5                                   "dimensionless y-position"
x_bar=0.5                                   "dimensionless x-position"
y=y_bar*th                                  "y-position"
x=x_bar*L                                   "x-position"
```

The solution is implemented using a `duplicate` loop that calculates the first $N$ terms of the series. The number of terms that is required for accuracy should be checked by exploring the sensitivity of the calculation to the number of terms in the same way that a numerical model should be checked for grid convergence.

```
N=100                                       "number of terms in series"
duplicate i=0,N
    lambda[i]=(1+2*i)*pi/(2*L)              "eigenvalues"
    C[i]=2*(T_infinity-T_b)/(L*lambda[i]*(k*lambda[i]*sinh(lambda[i]*th/2)/h_bar+cosh(lambda[i]*th/2)))
        "constants"
    theta[i]=C[i]*sin(lambda[i]*x)*cosh(lambda[i]*y)    "term in summation"
end
theta=sum(theta[0..N])                      "temperature difference"
T=theta+T_b                                 "temperature"
```

```
T_C=converttemp(K,C,T)                                        "in C"
```

Figure 3 shows the temperature distribution as a function of *x*/*L* for various values of *y*/*th*. Notice that for these conditions, an extended surface (i.e., 1-D) model of the fin would not be justified because there is a substantial difference between the temperature at the center of the fin (*y*/*th* = 0) and the edge (*y*/*th* = 0.5). This is evident from the Biot number:

$$Bi = \frac{\overline{h}\,th}{2\,k}$$

```
Bi=h_bar*th/(2*k)                                              "Biot number"
```

which leads to *Bi* = 4.0.

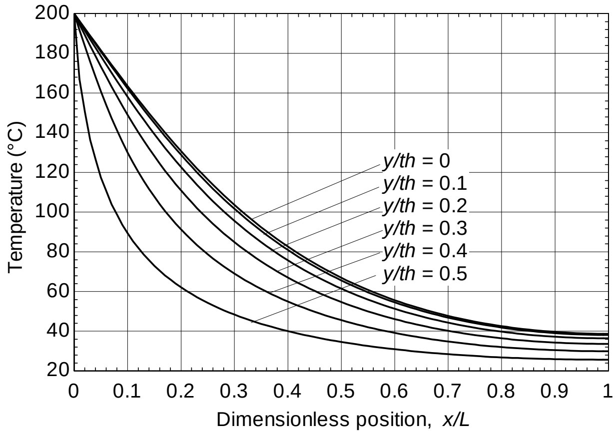


**Figure 3: Temperature as a function of *x*/*L* for various values of *y*/*th*.**

c.) Use the analytical solution to predict the fin efficiency of the 2-D fin.

The rate of conductive heat transfer into the base of fin is:

$$\dot{q}_{fin} = -2\,k\,W \int_0^{th/2} \left.\frac{\partial \theta}{\partial x}\right|_{x=0} dy \qquad (20)$$

Substituting Eqs. (16) and (19) into Eq. (20) leads to:

$$\dot{q}_{fin} = -4\,\theta_\infty \frac{k\,W}{L} \sum_{i=0}^{\infty} \frac{\int_0^{th/2} \cosh\left(\lambda_i\,y\right) dy}{\left[\frac{k\,\lambda_i}{\overline{h}} \sinh\left(\lambda_i \frac{th}{2}\right) + \cosh\left(\lambda_i \frac{th}{2}\right)\right]}$$

The integral can be accomplished using Maple:

```
> restart;
```

```
> int(cosh(lambda*y),y=0..th/2);
```

$$\frac{\sinh\left(\frac{\lambda\, th}{2}\right)}{\lambda}$$

so that the rate of conductive heat transfer to the fin is:

$$\dot{q}_{fin} = -4\,\theta_{\infty}\,\frac{k\,W}{L}\sum_{i=0}^{\infty}\frac{\sinh\left(\lambda_i\,\frac{th}{2}\right)}{\lambda_i\left[\frac{k\,\lambda_i}{\bar{h}}\sinh\left(\lambda_i\,\frac{th}{2}\right)+\cosh\left(\lambda_i\,\frac{th}{2}\right)\right]} \tag{21}$$

The fin efficiency, discussed in Section 1.6.5, is defined as the ratio of the rate of heat transfer to the maximum possible rate of heat transfer rate that is obtained with an infinitely conductive fin:

$$\eta_{fin} = \frac{\dot{q}_{fin}}{2\,\bar{h}\,W\,L\left(T_b - T_{\infty}\right)} \tag{22}$$

Substituting Eq. (21) into Eq. (22) leads to the fin efficiency predicted by the 2-D analytical solution:

$$\eta_{fin,2D} = \frac{2\,k}{\bar{h}\,L^2}\sum_{i=0}^{\infty}\frac{\sinh\left(\lambda_j\,\frac{th}{2}\right)}{\lambda_i\left[\frac{k\,\lambda_i}{\bar{h}}\sinh\left(\lambda_i\,\frac{th}{2}\right)+\cosh\left(\lambda_i\,\frac{th}{2}\right)\right]}$$

which is evaluated in EES according to:

```
duplicate i=0,N
   eta_fin[i]=(2*k/(h_bar*lambda[i]*L^2))*sinh(lambda[i]*th/2)/(k*lambda[i]*sinh(lambda[i]*th/2)/h_bar+&
       cosh(lambda[i]*th/2))
end
eta_fin=sum(eta_fin[0..N])
```

d.) Plot the fin efficiency predicted by the 2-D analytical solution as a function of the fin thickness and overlay on the plot the fin efficiency predicted using the 1-D approximation, developed in Section 1.6.

Figure 4 illustrates the fin efficiency predicted by the 2-D model as a function of fin thickness. Overlaid on Figure 4 is the solution from Section 1.6 that is listed in Table 1-4 for a fin with an adiabatic tip:

$$\eta_{fin,1D} = \frac{\tanh\left(mL\right)}{mL}$$

where

$$mL = \sqrt{\frac{\bar{h}\,2}{k\,th}}L$$

As the fin becomes thicker, the impact of the temperature gradients in the *y*-direction, neglected in the 1-D solution, become larger and therefore the 1-D and 2-D solutions diverge, with the 1-D solution always over-predicting the performance.

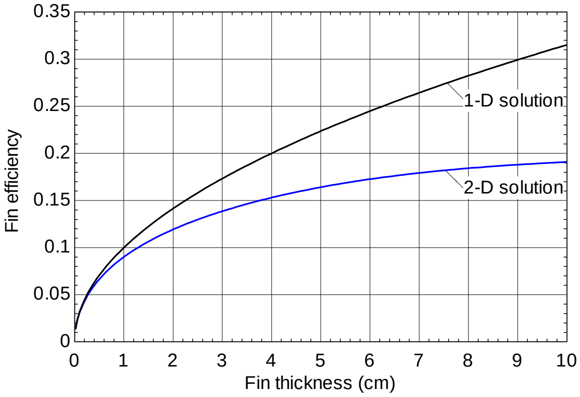


**Figure 4: Fin efficiency as a function of the fin thickness predicted by the 2-D solution and the 1-D solution.**

The ratio $\eta_{fin,2D}/\eta_{fin,1D}$ is shown in Figure 5 as a function of the Biot number; recall that the Biot number was used to justify the extended surface approximation in Section 1.6. Note that 1-D model is quite accurate (better than 2%) provided the Biot number is less than 0.1 and, surprisingly, remains reasonably accurate (10%) even up to a Biot number of 1.0.

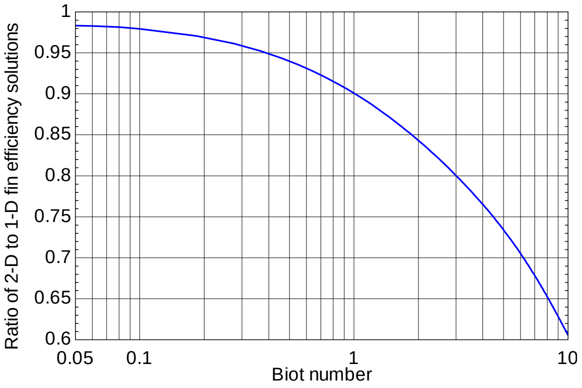


**Figure 5: Ratio of the fin efficiency predicted by the 2-D solution to the fin efficiency predicted by the 1-D solution as a function of the Biot number.**