EXAMPLE 2.2-2: Constriction Resistance

Figure 1 illustrates the situation where energy is transferred by conduction through a structure that suddenly changes cross-sectional area. The conduction resistance associated with this structure can be computed using separation of variables. This problem also illustrates an issue that is often confusing for separation of variables problems; specifically, the 0^{th} term in a cosine series must often be treated separately from the rest of the series and very carefully. The proper methodology for dealing with this situation is demonstrated in this example.

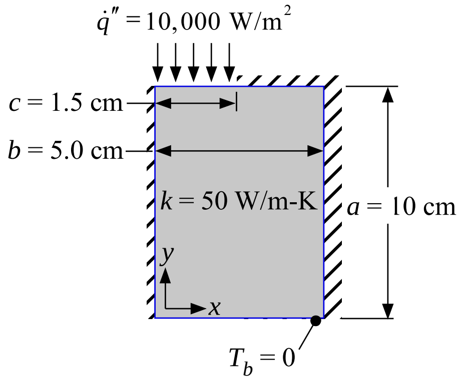

Figure 1: A constriction in a conduction path.

The width of the larger cross-sectional area is b = 5.0 cm and its length is a = 10 cm. The heat flux, $\dot{q}''$ = 10,000 W/m^2, is applied to the upper surface over a smaller width, c = 1.5 cm. The conductivity of the material is k = 50 W/m-K. The bottom surface of the object is maintained at some reference temperature, taken to be 0.

a.) Develop a solution for the temperature distribution in the material.

The partial differential equation for the problem is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

The boundary conditions in the x-direction are:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \tag{2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=b} = 0 \tag{3}$$

and the boundary conditions in the y-direction are:

$$T_{y=0} = 0 \tag{4}$$

$$k\left.\frac{\partial T}{\partial y}\right|_{y=a}=\begin{cases}\dot{q}'' & x<c\\ 0 & x\geq c\end{cases} \tag{5}$$

The 1st step in the solution is to verify that separation of variables can be applied to the problem without transformation or superposition. The governing partial differential equation is linear and homogeneous, all of the boundary conditions are linear, and both boundary conditions in the *x*-direction are homogeneous. Therefore the problem meets all of the requirements discussed in Section 2.2.2 and separation of variables can be applied, with *x* being the homogeneous direction.

The next step in the solution is to assume a separable solution:

$$T\left(x,y\right)=TX\left(x\right)TY\left(y\right) \tag{6}$$

which is substituted into Eq. (1) in order to achieve two ordinary differential equations for *TX* and *TY*, as discussed in Section 2.2.2:

$$\frac{d^2TX}{dx^2}+\lambda^2\,TX=0 \tag{7}$$

$$\frac{d^2TY}{dy^2}-\lambda^2\,TY=0 \tag{8}$$

Notice that the separation process was accomplished so that sine/cosine functions solve the ordinary differential equation for *TX* because *x* is the homogeneous direction. The next step in the solution is to solve the eigenproblem (i.e., the problem in the homogeneous direction). The solution to Eq. (7) is:

$$TX=C_1\sin\left(\lambda\,x\right)+C_2\cos\left(\lambda\,x\right) \tag{9}$$

The *x*-direction boundary conditions, Eqs. (2) and (3), expressed in terms of *TX*, become:

$$\left.\frac{dTX}{dx}\right|_{x=0}=0 \tag{10}$$

$$\left.\frac{dTX}{dx}\right|_{x=b}=0 \tag{11}$$

Substituting Eq. (9) into Eq. (10) leads to:

$$\left.\frac{dTX}{dx}\right|_{x=0} = C_1\,\lambda\,\underbrace{\cos(0)}_{1} - C_2\,\lambda\,\underbrace{\sin(0)}_{0} = 0$$

which can only be true if $C_1 = 0$. Substituting Eq. (9), with $C_1 = 0$, into Eq. (11) leads to:

$$\left.\frac{dTX}{dx}\right|_{x=b} = -C_2\,\lambda\sin(\lambda\,b) = 0$$

which can only be true if the argument of the sine function is an integer multiple of π:

$$\lambda_i\,b = i\,\pi \text{ where } i = 0,1,2,...$$

Therefore, the eigenfunctions for this problem are:

$$TX_i = C_{2,i}\cos(\lambda_i\,x) \text{ where } i = 0,1,2,... \tag{12}$$

and the eigenvalues, λ_i, are:

$$\lambda_i = \frac{i\,\pi}{b} \tag{13}$$

Note that the 0th eigenfunction is retained in Eq. (12) because TX_0 is not zero. The 0th eigenfunction is a constant and it will be necessary to treat this term separately from the others. The next step is to solve the problem in the non-homogeneous direction for each eigenvalue. The solution to Eq. (8) for each eigenvalue is:

$$TY_i = C_{3,i}\sinh(\lambda_i\,y) + C_{4,i}\cosh(\lambda_i\,y) \tag{14}$$

The solution associated with each eigenvalue is the product of Eqs. (12) and (14):

$$T_i = TX_i\,TY_i = C_{2,i}\cos(\lambda_i\,x)\left[C_{3,i}\sinh(\lambda_i\,y) + C_{4,i}\cosh(\lambda_i\,y)\right] \text{ where } i = 0,1,2,...$$

or, absorbing the constant $C_{2,i}$ into the constants $C_{3,i}$ and $C_{4,i}$:

$$T_i = \cos(\lambda_i\,x)\left[C_{3,i}\sinh(\lambda_i\,y) + C_{4,i}\cosh(\lambda_i\,y)\right] \text{ where } i = 0,1,2,... \tag{15}$$

The function T_i provided by Eq. (15) should satisfy both boundary conditions in the x-direction as well as the partial differential equation for any value of i; this should be checked using Maple before continuing.

The general solution is expressed as the sum of the solutions associated with each eigenvalue:

$$T=\sum_{i=0}^{\infty}T_i=\sum_{i=0}^{\infty}\cos\left(\lambda_i\,x\right)\left[C_{3,i}\sinh\left(\lambda_i\,y\right)+C_{4,i}\cosh\left(\lambda_i\,y\right)\right] \tag{16}$$

The final step forces the general solution to satisfy the boundary conditions in the non-homogeneous direction. Equation (16) is substituted into the boundary condition at y = 0, Eq. (4):

$$T_{y=0}=\sum_{i=0}^{\infty}\cos\left(\lambda_i\,x\right)\left[C_{3,i}\underbrace{\sinh\left(0\right)}_{=0}+C_{4,i}\underbrace{\cosh\left(0\right)}_{=1}\right]=0$$

which leads to:

$$\sum_{i=0}^{\infty}\cos\left(\lambda_i\,x\right)C_{4,i}\;=0$$

which can only be true if $C_{4,i}$ =0 for all i, therefore:

$$T=\sum_{i=0}^{\infty}C_i\cos\left(\lambda_i\,x\right)\,\sinh\left(\lambda_i\,y\right) \tag{17}$$

where the subscript 3 has been removed from $C_{3,i}$ since it is the only remaining undetermined constant associated with each eigenvalue. The 0th term in the cosine series is a constant and it must be pulled out and treated carefully; this is generally true for a cosine series where the 0th eigenvalue is 0 (i.e., the 0th eigenfunction is a constant).

$$T=\lim_{i\to 0}\left[C_0\cos\left(\frac{i\,\pi}{b}x\right)\sinh\left(\frac{i\,\pi}{b}y\right)\right]+\sum_{i=1}^{\infty}C_i\cos\left(\lambda_i\,x\right)\sinh\left(\lambda_i\,y\right)$$

or, recognizing that the cos(0) is 1.0:

$$T=\lim_{i\to 0}\left[C_0\sinh\left(\frac{0\,\pi}{b}y\right)\right]+\sum_{i=1}^{\infty}C_i\cos\left(\lambda_i\,x\right)\sinh\left(\lambda_i\,y\right) \tag{18}$$

It is tempting to recognize that the sinh(0)=0 and therefore if $i\to 0$ then the 0th term will not contribute to the solution. This is true provided that C_0 is finite; however, the product C_0 sinh(0) may not be zero and therefore the 0th term must be retained.

The final, non-homogeneous boundary condition, Eq. (5), is used to compute the undetermined coefficients in Eq. (18). Equation (18) is substituted into Eq. (5):

$$k\left.\frac{\partial T}{\partial y}\right|_{y=a} = k\lim_{i\to 0}\left[C_0\frac{i\,\pi}{b}\cosh\left(\frac{i\,\pi}{b}a\right)\right] + k\sum_{i=1}^{\infty} C_i\,\lambda_i\cos\left(\lambda_i\,x\right)\cosh\left(\lambda_i\,a\right) = \begin{cases}\dot{q}'' & x<c\\ 0 & x\ge c\end{cases}$$

or, recognizing that the cosh(0) is 1.0:

$$\frac{k\,\pi}{b}\lim_{i\to 0}\left[C_0\,i\right] + k\sum_{i=1}^{\infty} C_i\,\lambda_i\cos\left(\lambda_i\,x\right)\cosh\left(\lambda_i\,a\right) = \begin{cases}\dot{q}'' & x<c\\ 0 & x\ge c\end{cases} \tag{19}$$

We take advantage of the orthogonality of the eigenfunctions to compute the constants in Eq. (19). First we will deal with the 0th term in the series. Both sides of the equation are multiplied by the 0th eigenfunction, cos($\lambda_0\,x$) which is equal to 1, and the equation is integrated from x=0 to $x = b$:

$$\frac{k\,\pi}{b}\lim_{i\to 0}\left[C_0\;i\right]\int_0^b dx + k\sum_{i=1}^{\infty} C_i\,\lambda_i\cosh\left(\lambda_i\,a\right)\int_0^b \cos\left(\lambda_i\,x\right)dx = \int_0^c \dot{q}''\,dx + \int_c^b 0\,dx$$

The integral of any of the eigenfunctions (other than the 0th one) from 0 to b is 0. Therefore, every term in the summation integrates to zero and we are left with:

$$k\,\pi\lim_{i\to 0}\left[C_0\,i\,\right] = \dot{q}''\,c$$

therefore:

$$\lim_{i\to 0}\left[C_0\,i\right] = \frac{\dot{q}''\,c}{\pi\,k} \tag{20}$$

Substituting into Eq. (20) into Eq. (18) leads to:

$$T = \underbrace{\lim_{i\to 0}\left[\frac{\dot{q}''\,c}{i\,\pi\,k}\sinh\left(\frac{i\,\pi}{b}y\right)\right]}_{0^{\text{th}}\ \text{term in solution}} + \sum_{i=1}^{\infty} C_i\cos\left(\lambda_i\,x\right)\sinh\left(\lambda_i\,y\right)$$

Maple can be used to evaluate the 0th term in the solution:

```
> limit(q_dot_flux*c*sinh(i*Pi*y/b)/(i*Pi*k),i=0);
                                y q_dot_flux c
                                --------------
                                     b k
```

Substituting this result into Eq. (18) leads to:

$$T = \frac{\dot{q}''\,c\,y}{b\,k} + \sum_{i=1}^{\infty} C_i \cos\left(\lambda_i\,x\right)\sinh\left(\lambda_i\,y\right) \tag{21}$$

Substituting Eq. (21) into Eq. (5) leads to:

$$\frac{\dot{q}''\,c}{b} + k\sum_{i=1}^{\infty} C_i\,\lambda_i \cos\left(\lambda_i\,x\right)\cosh\left(\lambda_i\,a\right) = \begin{cases} \dot{q}'' & x < c \\ 0 & x \ge c \end{cases} \tag{22}$$

Next, we will deal with the non-zero terms in the series. Both sides of Eq. (22) are multiplied by cos(λ_j x) and integrated from $x = 0$ to $x = b$.

$$\frac{\dot{q}''\,c}{b}\int_0^b \cos\left(\lambda_j\,x\right)dx + k\sum_{i=1}^{\infty} C_i\,\lambda_i \cosh\left(\lambda_i\,a\right)\int_0^b \cos\left(\lambda_i\,x\right)\cos\left(\lambda_j\,x\right)dx = \int_0^c \dot{q}''\cos\left(\lambda_j\,x\right)dx + \int_c^b 0\,\cos\left(\lambda_j\,x\right)dx$$

The 0^{th} order term integrates to zero for any $j > 0$ and the only term of the summation that does not integrate to zero is the one where $i = j$:

$$k\,C_i\,\lambda_i \cosh\left(\lambda_i\,a\right)\int_0^b \cos^2\left(\lambda_i\,x\right)dx = \dot{q}''\int_0^c \cos\left(\lambda_i\,x\right)dx \tag{23}$$

The integrals in Eq. (23) are computed using Maple:

```
> int((cos(lambda*x))^2,x=0..b);
```

$$\frac{b}{2}$$

```
> int(cos(lambda*x),x=0..c);
```

$$\frac{b\,\sin\left(\frac{i\sim\,\pi\,c}{b}\right)}{i\sim\,\pi}$$

to obtain an equation for the undetermined coefficients:

$$k\,C_i\,\lambda_i \cosh\left(\lambda_i\,a\right)\frac{b}{2} = \dot{q}''\frac{\sin\left(\lambda_i\,c\right)}{\lambda_i}$$

The solution is programmed in EES:

```
"EXAMPLE 2.2-2: Constriction Resistance"
$UnitSystem SI MASS  RAD PA C J
$Tabstops 0.2 0.4 0.6 0.8 3.5

"Inputs"
q_dot_flux=10000 [W/m^2]                                          "Heat flux"
```

```
k = 50 [W/m-K]                                    "Conductivity"
c=1.5 [cm]*convert(cm,m)                          "width of applied flux"
a=10 [cm]*convert(cm,m)                           "length of object"
b = 5.0 [cm]*convert(cm,m)                        "width of object"

x_bar=0.75                                        "dimensionless x-position"
y_bar=1                                           "dimensionless y-position"
x=x_bar*b                                         "x-position"
y=y_bar*a                                         "y-position"

N=400                                             "number of terms"
duplicate i=1,N                                   "evaluate coefficients for N terms"
   lambda[i]=i*pi/b
   k*C[i]*lambda[i]*cosh(lambda[i]*a)*b/2=q_dot_flux*sin(lambda[i]*c)/lambda[i]
   T[i]=C[i]*cos(lambda[i]*x)*sinh(lambda[i]*y)
end
T=q_dot_flux*c*y/(k*b)+sum(T[1..N])
```

A parametric table is generated and used to generate the contour plot of temperature shown in Figure 2.

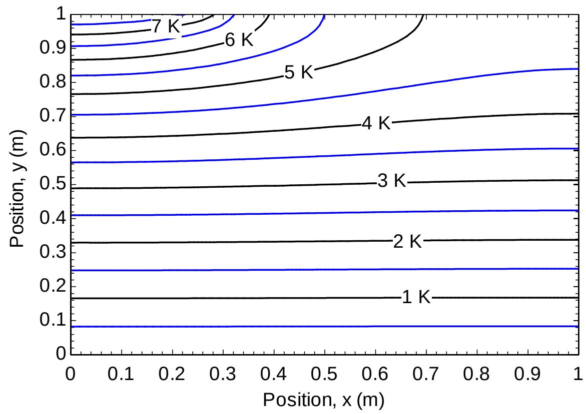

Figure 2: Contour plot of temperature distribution in constriction.

It is worth comparing the answer with physical intuition. The temperature elevation at the constriction relative to the base is approximately 8 K according to Figure 2; does this value make sense? In Section 2.8, methods for estimating the resistance of 2-D problems using 1-D models are discussed. However, clearly the resistance of the constriction cannot be greater than the resistance to conduction through the material if the heat flux is applied uniformly at the top surface:

$$R_{nc} = \frac{a}{b\,L\,k}$$

where L is the length of the material. The temperature elevation at the constriction in this limit is:

$$\Delta T_{nc} = R_{nc}\,\dot{q}''\,c\,L = \frac{a\,\dot{q}''\,c}{b\,k}$$

```
DeltaT_nc=a*q_dot_flux*c/(b*k)                              "temperature rise without constriction"
```

This leads to $\Delta T_{nc} = 6.0$ K, which is has the same magnitude as the observed temperature rise but is smaller, as expected.