2.3: Advanced Separation of Variables Solutions

2.3.1 Introduction

Section 2.2 provides an introduction to the technique of separation of variables and discusses its application in the context of a few examples. The separation of variables method, as it is presented in Section 2.2, is rather limited as it does not allow, for example, non-homogeneous terms that might arise from effects such as volumetric generation or complex boundary conditions or for problems that are in cylindrical coordinates. One technique for solving problems with non-homogeneous terms is discussed in Section 2.3.2. The extension of separation of variables to cylindrical coordinates is presented in Section 2.3.3. A more complete presentation of these topics can be found in Myers (1987).

2.3.2 Non-Homogeneous Terms

This section presents a method in which a problem with non-homogeneous terms (i.e., either a non-homogeneous partial differential equation or non-homogeneous boundary conditions) can be solved by dividing the problem into a 2-D homogeneous problem and 1-D non-homogeneous problems. The homogeneous component can be solved using separation of variables, as discussed in Section 2.2. The non-homogeneous problems may be simple enough that their solution is possible, in which case the technique will work. The technique is straightforward, but there is no guarantee that it will work in general. However, it often works and therefore provides a powerful means of extending the applicability of the separation of variables technique.

The process is demonstrated in the context of a specific example. A heating element, illustrated in Figure 2-12, is mounted between two walls that are each maintained at T_b = 20ºC. The sides of the element are exposed to fluid at T_∞ = 50ºC. The average heat transfer coefficient between the side of the element and the fluid is $\overline{h}$ = 100 W/m^2-K. The heating element is rectangular; the half-width of the element is W = 4 cm and the half-height of the element is H = 10 cm. You may assume that the temperature varies only in the *x*- and *y*-directions. The heating element experiences a uniform rate of volumetric thermal energy generation due to ohmic heating of $\dot{g}''' = 1\text{x}10^5$ W/m^3. The conductivity of the material is k = 1 W/m-K.

The known information is entered into EES:

```
$UnitSystem SI MASS  RAD PA C J
$Tabstops 0.2 0.4 0.6 0.8 3.5

"Inputs"
W=4 [cm]*convert(cm,m)                      "width of element"
H=10 [cm]*convert(cm,m)                     "height of element"
gv_dot=1e5 [W/m^3]                          "volumetric generation"
k=1 [W/m-K]                                 "conductivity"
h_bar=100 [W/m^2-K]                         "heat transfer coefficient"
T_b=ConvertTemp(C,K,20 [C])                 "bottom temperature"
T_infinity=ConvertTemp(C,K,50 [C])          "fluid temperature"
```

A Biot number is computed in order to determine whether the element can be treated using the extended surface approximation in which temperature gradients in the *x*-direction are ignored:

$$Bi = \frac{\overline{h}\,W}{k} \tag{2-76}$$

```
Biot=h_bar*W/k                                         "Biot number"
```

The Biot number is 2.0 and so the extended surface approximation is not justified; the problem is inherently a 2-D problem with temperature varying with position in both the *x* and *y* directions.

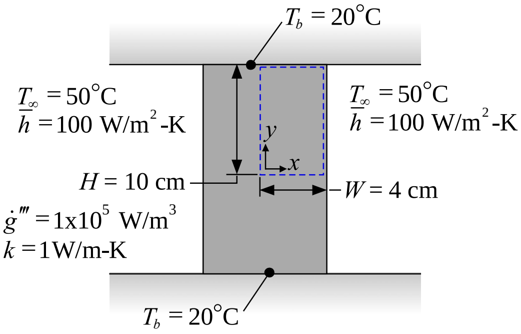
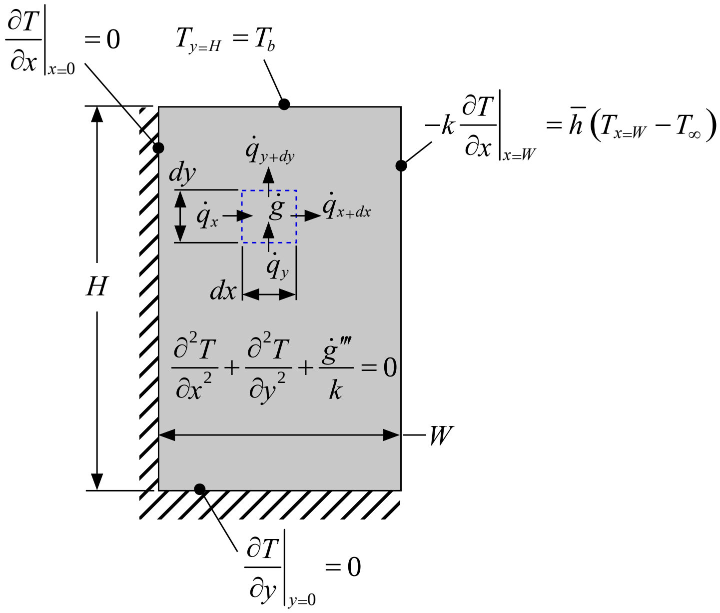
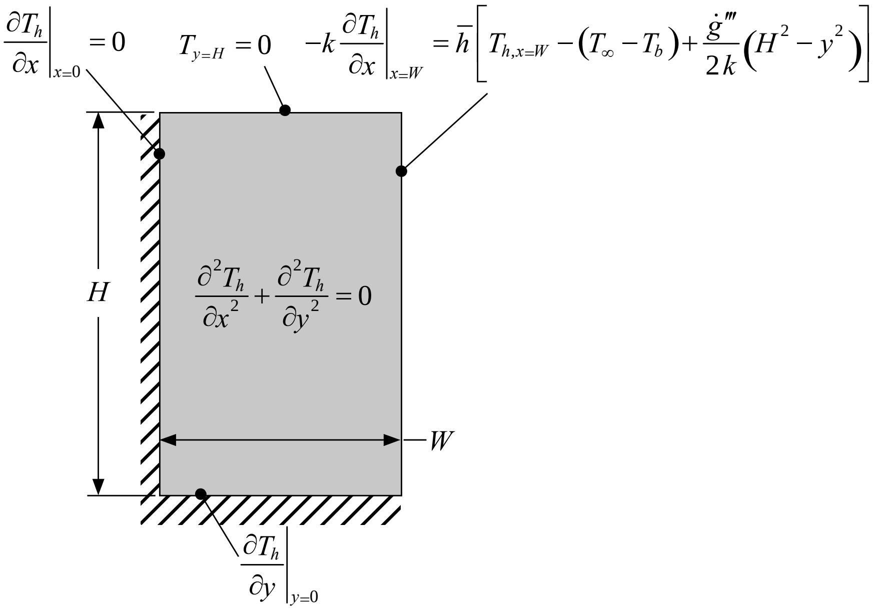


Figure 2-12: Heating element.

The heating element is symmetric so that a quarter-symmetry model can be used. The computational domain is shown in Figure 2-12 and the problem is mathematically specified in Figure 2-13(a).



(a)



(b)

Figure 2-13: Problem specified in terms of (a) temperature *T*, and (b) the homogeneous component of the solution *T*$_h$.

The governing differential equation may be obtained from an energy balance on the differential control volume shown in Figure 2-13(a):

$$\dot{q}_x + \dot{q}_y + \dot{g} = \dot{q}_{x+dx} + \dot{q}_{y+dy} \tag{2-77}$$

or

$$\dot{g} = \frac{\partial \dot{q}_x}{\partial x}dx + \frac{\partial \dot{q}_y}{\partial y}dy \tag{2-78}$$

Substituting the rate equations into the differential energy balance leads to:

$$L\,dx\,dy\,\dot{g}''' = \frac{\partial}{\partial x}\left[-k\,L\,dy\frac{\partial T}{\partial x}\right]dx + \frac{\partial}{\partial y}\left[-k\,L\,dx\frac{\partial T}{\partial y}\right]dy \tag{2-79}$$

where L is the length of the heater (into the page). Assuming that conductivity is constant, the governing differential equation can be simplified:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{g}'''}{k} = 0 \tag{2-80}$$

The boundary conditions are:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0}=0 \tag{2-81}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=W}=\bar{h}\left(T_{x=W}-T_{\infty}\right) \tag{2-82}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0}=0 \tag{2-83}$$

$$T_{y=H}=T_b \tag{2-84}$$

The introduction of volumetric generation in Eq. (2-80) leads to a non-homogeneous partial differential equation; any multiple CT of a solution T will not satisfy the equation. Also, neither the *x*- or *y*-directions are homogeneous; therefore, it is not possible to solve this problem directly using the separation of variables technique described in Section 2.2.2.

Split Solution into Homogeneous and Particular Components

To proceed, it is necessary to split the solution T into a homogeneous (T_h) and one or more non-homogeneous components (X and Y); this process is analogous to splitting the solution to an ordinary differential equation into a homogeneous and non-homogeneous (or particular) component. The homogeneous problem for T_h can be solved using separation of variables. The particular problems for X and Y must be solved separately.

$$T(x,y)=T_h(x,y)+X(x)+Y(y) \tag{2-85}$$

The assumed, split solution, Eq. (2-85), is substituted into the partial differential equation, Eq.(2-80):

$$\underbrace{\frac{\partial^2 T_h}{\partial x^2}+\frac{\partial^2 T_h}{\partial y^2}}_{\substack{=0 \text{ for homogeneous} \\ \text{differential equation}}}+\frac{d^2 X}{dx^2}+\frac{d^2 Y}{dy^2}+\frac{\dot{g}'''}{k}=0 \tag{2-86}$$

and the boundary conditions, Eqs. (2-81) through (2-84):

$$\left.\frac{\partial T_h}{\partial x}\right|_{x=0}+\left.\frac{dX}{dx}\right|_{x=0}=0 \tag{2-87}$$

$$-k\left.\frac{\partial T_h}{\partial x}\right|_{x=W}-k\left.\frac{dX}{dx}\right|_{x=W}=\bar{h}\left(T_{h,x=W}+X_{x=W}+Y-T_{\infty}\right) \tag{2-88}$$

$$\underbrace{\left.\frac{\partial T_h}{\partial y}\right|_{y=0}}_{\substack{\text{=0 for homogeneous}\\ \text{boundary condition}}} + \underbrace{\left.\frac{dY}{dy}\right|_{y=0} = 0}_{\text{boundary condition for } Y} \tag{2-89}$$

$$\underbrace{T_{h,y=H}}_{\substack{\text{=0 for homogeneous}\\ \text{boundary condition}}} + \underbrace{X + Y_{y=H} = T_b}_{\text{boundary conditions for } X \text{ and } Y} \tag{2-90}$$

Enforce a Homogeneous PDE to obtain ODEs for the Particular Solutions

The process must result in a homogeneous partial differential equation for T_h:

$$\frac{\partial^2 T_h}{\partial x^2} + \frac{\partial^2 T_h}{\partial y^2} = 0 \tag{2-91}$$

If Eq. (2-91) is valid, then Eq. (2-86) is reduced to:

$$\underbrace{\frac{d^2 X}{dx^2}}_{\substack{\text{=0,}\\ \text{ODE for X}}} + \underbrace{\frac{d^2 Y}{dy^2} + \frac{\dot{g}'''}{k} = 0}_{\text{ODE for Y}} \tag{2-92}$$

We need to identify functions X and Y that satisfy Eq. (2-92); for example, Eq. (2-92) can be satisfied if:

$$\boxed{\frac{d^2 Y}{dy^2} + \frac{\dot{g}'''}{k} = 0} \tag{2-93}$$

and

$$\boxed{\frac{d^2 X}{dx^2} = 0} \tag{2-94}$$

Note that if the volumetric generation term could also have been included with the X function. The process of enforcing a homogeneous partial differential equation for the homogeneous component of the solution leads to the ordinary differential equations that characterize the particular components of the solution.

Solve the Ordinary Differential Equations for the Particular Solutions

Integrating Eqs. (2-93) and (2-94) one time leads to:

$$\frac{dY}{dy} = -\frac{\dot{g}'''}{k} y + C_1 \tag{2-95}$$

and

$$\frac{dX}{dx} = C_3 \tag{2-96}$$

where C_1 and C_3 are arbitrary constants. Integrating again leads to:

$$\boxed{Y = -\frac{\dot{g}'''}{k}\frac{y^2}{2} + C_1\,y + C_2} \tag{2-97}$$

and

$$\boxed{X = C_3\,x + C_4} \tag{2-98}$$

Equations (2-97) and (2-98) specify functions X and Y that successfully transform the partial differential equation for T into a homogeneous differential equation for T_h regardless of the constants C_1 through C_4. The next step is to select these constants so that the problem for T_h includes a homogeneous direction, the remaining requirement for the application of separation of variables.

Enforce a Homogeneous Direction

Both of the boundary conditions for T_h in one direction must be homogeneous in order to apply separation of variables. In this problem, the y-direction is selected as the homogeneous direction. This choice is enforced through proper selection of the boundary conditions for X and Y. The homogeneous boundary condition required at $y = 0$ is:

$$\boxed{\left.\frac{\partial T_h}{\partial y}\right|_{y=0} = 0} \tag{2-99}$$

Therefore, Eq. (2-89) is reduced to:

$$\left.\frac{dY}{dy}\right|_{y=0} = 0 \tag{2-100}$$

Substituting Eq. (2-95) into Eq. (2-100) leads to $C_1 = 0$; therefore:

$$Y = -\frac{\dot{g}'''}{k}\frac{y^2}{2} + C_2 \tag{2-101}$$

The homogeneous boundary condition required at $y = H$ is:

$$\boxed{T_{h,y=H} = 0} \tag{2-102}$$

Therefore, Eq. (2-90) is reduced to:

$$\underbrace{X}_{=0} + \underbrace{Y_{y=H} = T_b}_{\substack{\text{boundary condition} \\ \text{for Y}}} \tag{2-103}$$

Recall that Y is a function of y, but X is not. In order for Eq. (2-103) to be true over the entire boundary, it is necessary that:

$$\boxed{X = 0} \tag{2-104}$$

and

$$Y_{y=H} = T_b \tag{2-105}$$

Substituting Eq. (2-101) into Eq. (2-105) leads to:

$$Y_{y=H} = -\frac{\dot{g}'''}{k}\frac{H^2}{2} + C_2 = T_b \tag{2-106}$$

Solving for C_2 leads to:

$$C_2 = T_b + \frac{\dot{g}'''}{k}\frac{H^2}{2} \tag{2-107}$$

and therefore:

$$\boxed{Y = T_b + \frac{\dot{g}'''}{2k}\left(H^2 - y^2\right)} \tag{2-108}$$

The process of enforcing a homogeneous direction for the homogeneous component of the solution has led to the necessary boundary conditions for the particular solutions and, therefore, to the particular solutions themselves. We have identified functions X and Y that successfully transform our problem for T from one with a non-homogeneous partial differential equation and no homogeneous direction to a problem for T_h with a homogeneous partial differential equation and two homogeneous boundary conditions in the y-direction.

Determine the Non-homogeneous Direction Boundary Conditions

All that remains is to determine the boundary conditions for T_h in the non-homogeneous direction. There are no particular requirements on the boundary conditions in the non-homogeneous direction and so this is just a matter of substitution. Substituting Eq. (2-104) into the boundary condition at $x = 0$, Eq. (2-87) leads to:

$$\boxed{\left.\frac{\partial T_h}{\partial x}\right|_{x=0} = 0} \tag{2-109}$$

Substituting Eqs. (2-104) and (2-108) into the boundary condition at $x = W$, Eq. (2-88), leads to:

$$\boxed{-k\left.\frac{\partial T_h}{\partial x}\right|_{x=W} = \overline{h}\left[T_{h,x=W} - \left(T_\infty - T_b\right) + \frac{\dot{g}'''}{2\,k}\left(H^2 - y^2\right)\right]} \tag{2-110}$$

Solve the Homogeneous Problem using Separation of Variables

The problem for T_h is shown mathematically in Figure 2-13(b) and the solution follows the steps for separation of variables that are laid out in Section 2.2.2. The solution for T_h is separated into TX_h and TY_h which leads to the ordinary differential equations:

$$\frac{d^2TX_h}{dx^2} - \lambda^2\,TX_h = 0 \tag{2-111}$$

$$\frac{d^2TY_h}{dy^2} + \lambda^2\,TY_h = 0 \tag{2-112}$$

The eigenproblem is solved; the solution in the homogeneous direction (y) is:

$$TY_h = C_5\cos\left(\lambda\,y\right) + C_6\sin\left(\lambda\,y\right) \tag{2-113}$$

The boundary condition at $y = 0$, Eq. (2-99), leads to:

$$\left.\frac{dTY_h}{dy}\right|_{y=0} = -C_5\,\lambda\underbrace{\sin\left(\lambda\,0\right)}_{0} + C_6\,\lambda\underbrace{\cos\left(\lambda\,0\right)}_{1} = 0 \tag{2-114}$$

which requires that $C_6 = 0$:

$$TY_h = C_5\cos\left(\lambda\,y\right) \tag{2-115}$$

The boundary condition at $y = H$, Eq. (2-102), leads to:

$$TY_{h,y=H} = C_5\cos\left(\lambda\,H\right) = 0 \tag{2-116}$$

Therefore, the eigenfunctions and eigenvalues of the problem are:

$$TY_{h,i} = C_{5,i}\cos\left(\lambda_i\, y\right) \text{ where } \lambda_i = \frac{\left(2i-1\right)\pi}{2H} \text{ for } i = 1,2,..\infty \tag{2-117}$$

The solution to the ordinary differential equation in the non-homogeneous direction, Eq. (2-111), for each eigenvalue is:

$$TX_{h,i} = C_{7,i}\cosh\left(\lambda_i\, x\right) + C_{8,i}\sinh\left(\lambda_i\, x\right) \tag{2-118}$$

The general solution is therefore:

$$T_h = \sum_{i=1}^{\infty}\cos\left(\lambda_i\, y\right)\left[C_{7,i}\cosh\left(\lambda_i\, x\right) + C_{8,i}\sinh\left(\lambda_i\, x\right)\right] \tag{2-119}$$

The general solution is forced to satisfy the boundary conditions in the non-homogeneous (x) direction. Substituting Eq. (2-119) into the boundary condition at $x = 0$, Eq. (2-109), leads to:

$$\left.\frac{\partial T_h}{\partial x}\right|_{x=0} = \sum_{i=1}^{\infty}\cos\left(\lambda_i\, y\right)\left[C_{7,i}\,\lambda_i\underbrace{\sinh\left(\lambda_i\, 0\right)}_{0} + C_{8,i}\,\lambda_i\underbrace{\cosh\left(\lambda_i\, 0\right)}_{1}\right] = 0 \tag{2-120}$$

which can only be true if $C_{8,i} = 0$; therefore:

$$T_h = \sum_{i=1}^{\infty} C_i\cos\left(\lambda_i\, y\right)\cosh\left(\lambda_i\, x\right) \tag{2-121}$$

Substituting Eq. (2-121) into the boundary condition at $x = W$, Eq. (2-110), leads to:

$$-k\sum_{i=1}^{\infty} C_i\cos\left(\lambda_i\, y\right)\lambda_i\sinh\left(\lambda_i\, W\right) = \overline{h}\left[\sum_{i=1}^{\infty} C_i\cos\left(\lambda_i\, y\right)\cosh\left(\lambda_i\, W\right) - \left(T_\infty - T_b\right) + \frac{\dot{g}'''}{2k}\left(H^2 - y^2\right)\right] \tag{2-122}$$

The constants C_i are determined using the orthogonality property of the eigenfunctions, as explained in Section 2.2. Equation (2-122) is multiplied by $\cos(\lambda_j\, y)$ and integrated from 0 to H:

$$\begin{aligned} &-k\sum_{i=1}^{\infty} C_i\,\lambda_i\sinh\left(\lambda_i\, W\right)\int_0^H \cos\left(\lambda_i\, y\right)\cos\left(\lambda_j\, y\right)dy = \\ &= \overline{h}\sum_{i=1}^{\infty} C_i\cosh\left(\lambda_i\, W\right)\int_0^H \cos\left(\lambda_i\, y\right)\cos\left(\lambda_j\, y\right)dy + \\ &+\overline{h}\left[\frac{\dot{g}'''\, H^2}{2k} - \left(T_\infty - T_b\right)\right]\int_0^H \cos\left(\lambda_j\, y\right)dy - \overline{h}\,\frac{\dot{g}'''}{2k}\int_0^H y^2\cos\left(\lambda_j\, y\right)dy \end{aligned} \tag{2-123}$$

Since the eigenfunctions are orthogonal, Eq. (2-123) simplifies to:

$$
\begin{gathered}
-C_i\left[k\lambda_i \sinh(\lambda_i W)+\bar{h}\cosh(\lambda_i W)\right]\underbrace{\int_0^H \cos^2(\lambda_i y)\,dy}_{\text{Integral } 1_i}= \\
\bar{h}\left[\frac{\dot{g}''' H^2}{2k}-(T_\infty - T_b)\right]\underbrace{\int_0^H \cos(\lambda_i y)\,dy}_{\text{Integral } 2_i}-\bar{h}\,\frac{\dot{g}'''}{2k}\underbrace{\int_0^H y^2\cos(\lambda_i y)\,dy}_{\text{Integral } 3_i}
\end{gathered}
\tag{2-124}
$$

The integrals in Eq. (2-124) are evaluated using Maple:

```
> restart;
> assume(i,integer);
> lambda:=(2*i-1)*Pi/(2*H);
```

$$\lambda := \frac{(2\,i\sim - 1)\,\pi}{2\,H}$$

```
> Integral1:=int((cos(lambda*y))^2,y=0..H);
```

$$\mathit{Integral1} := \frac{H}{2}$$

```
> Integral2:=int(cos(lambda*y),y=0..H);
```

$$\mathit{Integral2} := \frac{2\,(-1)^{(1+i\sim)}\,H}{(2\,i\sim - 1)\,\pi}$$

```
> Integral3:=int(y^2*cos(lambda*y),y=0..H);
```

$$\mathit{Integral3} := \frac{2\,(-1)^{(1+i\sim)}\,H^3\,(-8+4\,i\sim^2\,\pi^2-4\,i\sim\,\pi^2+\pi^2)}{\pi^3\,(8\,i\sim^3-12\,i\sim^2+6\,i\sim-1)}$$

Equation (2-124) is solved for the constants:

$$
C_i=\frac{2\bar{h}\left[(T_\infty - T_b)-\dfrac{\dot{g}''' H^2}{2k}\right]\text{Integral2}+\bar{h}\,\dfrac{\dot{g}'''}{k}\,\text{Integral3}}{H\left[k\lambda_i\sinh(\lambda_i W)+\bar{h}\cosh(\lambda_i W)\right]}
\tag{2-125}
$$

The position to evaluate the solution is specified in terms of a dimensionless x and y position:

```
x_bar=0.5                                    "dimensionless position"
y_bar=0.5
x=x_bar*W                                    "position"
y=y_bar*H
```

The first N terms in the series solution are evaluated within a duplicate loop in EES; the eigenfunctions and the values of the integrals are evaluated for each term:

```
N=100                                                   "number of terms"
duplicate i=1,N
   lambda[i]=(2*i-1)*pi/(2*H)                           "eigenvalue"
   Integral2[i] = 2*(-1)^(1+i)/(2*i-1)/Pi*H             "value of the 2nd integral"
   Integral3[i] = 2*(-1)^(1+i)*H^3*(-8+4*i^2*Pi^2-4*i*Pi^2+Pi^2)/Pi^3/(8*i^3-12*i^2+6*i-1)
             "value of the 3rd integral"
```

Note that the integrals are copied directly into EES from Maple. The value of each constant is evaluated according to Eq. (2-125):

```
   C[i]=(2*h_bar*(T_infinity-T_b-gv_dot*H^2/(2*k))*Integral2[i]+h_bar*gv_dot*Integral3[i]/k)/&
             (H*(k*lambda[i]*sinh(lambda[i]*W)+h_bar*cosh(lambda[i]*W)))
```

and used to determine each term of the series in the homogeneous solution. The homogeneous solution is obtained by summing each term:

```
   T_h[i]=C[i]*cos(lambda[i]*y)*cosh(lambda[i]*x)       "each term in homogeneous solution"
end
T_h=sum(T_h[1..N])                                      "homogeneous solution"
```

The solution for temperature is obtained according to Eq. (2-85) with Eq. (2-108) used to evaluate Y (recall that X= 0):

```
T=T_h+T_b+gv_dot*(H^2-y^2)/(2*k)                        "total solution"
T_C=converttemp(K,C,T)                                   "in C"
```

A parametric table is used to generate results over a uniformly spaced grid and these results are used to generate the contour plot shown in Figure 2-14.

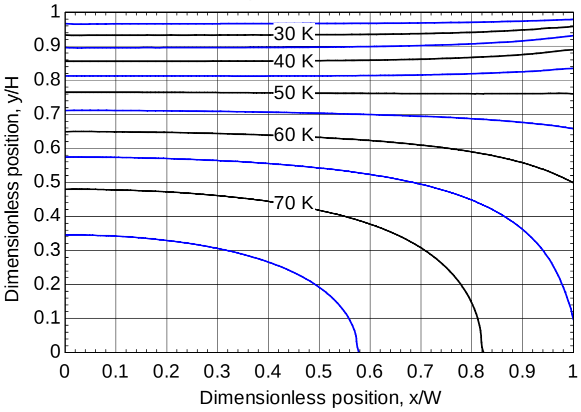


Figure 2-14: Contour plot of temperature in heater.

Summary of Steps

The steps required to apply separation of variables to a problem with non-homogeneous terms are summarized below. For a more complete description of this process, the reader is referred to Myers (1987).

1. Assume that the solution to the non-homogeneous problem can be written as the sum of a homogeneous solution (that can be solved with separation of variables) and particular

solutions that are functions only of x and y. Substitute this form of the solution into the partial differential equation and the boundary conditions for the non-homogeneous problem.

2. Enforce the requirement that the homogeneous solution must be characterized by a homogeneous partial differential equation. By inspection, this should lead to two ordinary differential equations for the particular solutions.
3. Solve the ordinary differential equations in order to obtain the particular solutions to within undetermined constants.
4. Enforce the requirement that both boundary conditions for the homogeneous problem in one direction must be homogeneous. This requirement should lead to sufficient boundary conditions for the particular solutions to determine the undetermined constants. Note that there may not be a unique solution for these particular solutions; any particular solutions that satisfy steps 2 and 4 will be sufficient.
5. Determine the boundary conditions for the homogeneous solution in the non-homogeneous direction.
6. Obtain the homogeneous solution using separation of variables. Assemble the solution from the homogeneous and particular solutions.

2.3.3 Cylindrical Coordinate System

The separation of variables technique can be extended to cylindrical coordinate systems. The steps used to obtain the solution remain the same as those introduced in Section 2.2 for Cartesian coordinate problems. The solution to the ordinary differential equation that results in the radial direction will be in the form of Bessel functions or modified Bessel functions. If the two homogeneous boundary conditions are in the radial direction then the solution to the ordinary differential equation in the radial direction will be the eigenfunctions of the problem. The eigenfunctions should be expressed in terms of Bessel functions. Recall from Section 1.8 that Bessel functions oscillate in a manner that is similar to sines and cosines. It can be shown that, when used with an appropriately-selected weighting function, the Bessel functions are orthogonal and this property is exploited in the usual way to obtain the coefficients of the series solution. If the two homogeneous boundary conditions lie in the axial direction, then the eigenfunctions are sine/cosines and the solution to the ordinary differential equation in the radial direction takes the form of modified Bessel functions. These concepts are illustrated in the following example.

EXAMPLE 2.3-1: Laser Machining

Figure 1 illustrates a ceramic disk that is th = 1 cm thick with an outer radius r_{out} = 10 cm. Ceramic can be machined more easily if it is pre-heated using a laser. Therefore, a laser beam with radius r_{laser} = 0.5 cm applies a flux of $\dot{q}''_{laser}$ = 1x10^5 W/m^2 to the center of the disk. The conductivity of the ceramic is k = 1.4 W/m-K. The disk is clamped to a machine bed at T_c = 20ºC, the contact resistance between the machine bed and the disk is R''_c = 2.8x10^{-4} m^2-K/W. The outer edge is cooled by exposure to coolant and can be considered to be at T_c = 20º. The top surface is exposed to ambient air; however, the heat transfer coefficient is so small that the surface may be modeled as being adiabatic except where it is experiencing laser heating.

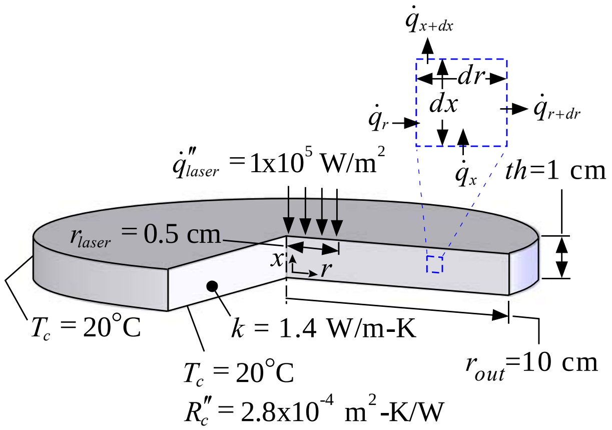


Figure 1: Ceramic disk exposed to laser pre-heating.

a.) Develop an analytical model capable of predicting the temperature in the disk.

A differential control volume is shown in Figure 1 and leads to the energy balance:

$$\dot{q}_x + \dot{q}_r = \dot{q}_{x+dx} + \dot{q}_{r+dr}$$

or

$$0 = \frac{\partial \dot{q}_x}{\partial x} dx + \frac{\partial \dot{q}_r}{\partial r} dr$$

Substituting the rate equations:

$$\dot{q}_x = -k\, 2\,\pi\, r\, dr \frac{\partial T}{\partial x}$$

and

$$\dot{q}_r = -k\, 2\,\pi\, r\, dx \frac{\partial T}{\partial r}$$

into the differential energy balance leads to:

$$0 = \frac{\partial}{\partial x}\left[-k\, 2\,\pi\, r\, dr \frac{\partial T}{\partial x}\right] dx + \frac{\partial}{\partial r}\left[-k\, 2\,\pi\, r\, dx \frac{\partial T}{\partial r}\right] dr$$

or

$$r \frac{\partial^2 T}{\partial x^2} + \frac{\partial}{\partial r}\left[r \frac{\partial T}{\partial r}\right] = 0$$

and the boundary conditions are:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$$

$$T_{r=r_{out}} = T_c$$

$$k\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{1}{R''_c}\left(T_{x=0} - T_c\right)$$

$$k\left.\frac{\partial T}{\partial x}\right|_{x=th} = \begin{cases} \dot{q}''_{laser} & \text{for } 0 < r < r_{laser} \\ 0 & \text{for } r_{laser} < r < r_{out} \end{cases}$$

As stated above, neither the *x* or the *r*-directions are homogeneous; however, the boundary conditions in the *x*-direction can be made homogeneous by defining the transformation variable:

$$\theta = T - T_c$$

so that the governing equation becomes:

$$r\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial}{\partial r}\left[r\frac{\partial \theta}{\partial r}\right] = 0 \tag{1}$$

and the boundary conditions become:

$$\left.\frac{\partial \theta}{\partial r}\right|_{r=0} = 0 \tag{2}$$

$$\theta_{r=r_{out}} = 0 \tag{3}$$

$$R''_c\, k\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = \theta_{x=0} \tag{4}$$

$$k\left.\frac{\partial \theta}{\partial x}\right|_{x=th} = \begin{cases} \dot{q}''_{laser} \;\; \text{for } 0 < r < r_{laser} \\ 0 \text{ for } r_{laser} < r < r_{out} \end{cases} \tag{5}$$

Notice that the two homogeneous boundary conditions are in the *r*-direction and therefore we will need eigenfunctions in this dimension. The solution is expressed as the product of *θX* and *θR* which are functions only of *x* and *r*, respectively.

$$\theta(x,y) = \theta X(x)\,\theta R(r)$$

Substituting the product solution into the governing partial differential equation, Eq. (1), leads to:

$$r\,\theta R\frac{d^2\theta X}{dx^2} + \theta X\frac{d}{dr}\left[r\frac{d\theta R}{dr}\right] = 0$$

Dividing by the product $r\ \theta X\ \theta R$ leads to:

$$\underbrace{\frac{\dfrac{d^2\theta X}{dx^2}}{\theta X}}_{\pm\lambda^2} + \underbrace{\frac{\dfrac{d}{dr}\left[r\dfrac{d\theta R}{dr}\right]}{r\,\theta R}}_{\mp\lambda^2} = 0$$

The choice of the sign for the λ^2 term is important. In this problem, the eigenfunctions must be in the radial direction so we choose the sign that leads to Bessel function solutions for θR:

$$\frac{d^2\theta X}{dx^2} - \lambda^2\,\theta X = 0 \qquad (6)$$

$$\frac{d}{dr}\left[r\frac{d\theta R}{dr}\right] + \lambda^2\,r\,\theta R = 0 \qquad (7)$$

We will focus on solving the eigenproblem first. The solution to Eq. (7) can be determined by returning to Section 1.8.4 and following the flow chart in Figure 1-54. Maple can be used as well:

```
> restart;
> ODEr:=diff(r*diff(thetar(r),r),r)+lambda^2*r*thetar(r)=0;
```

$$ODEr := \left(\frac{d}{dr}\,\mathrm{thetar}(r)\right) + r\left(\frac{d^2}{dr^2}\,\mathrm{thetar}(r)\right) + \lambda^2\,r\,\mathrm{thetar}(r) = 0$$

```
> thetars:=dsolve(ODEr);
```

$$thetars := \mathrm{thetar}(r) = \_C1\ \mathrm{BesselJ}(0, \lambda\, r) + \_C2\ \mathrm{BesselY}(0, \lambda\, r)$$

Therefore:

$$\theta R = C_1\,\mathrm{BesselJ}(0,\lambda r) + C_2\,\mathrm{BesselY}(0,\lambda r) \qquad (8)$$

Substituting Eq. (8) into the boundary condition at $r = 0$, Eq. (2), leads to:

$$C_1\frac{d}{dr}\left[\text{BesselJ}(0,\lambda\,r)\right]_{r=0}+C_2\frac{d}{dr}\left[\text{BesselY}(0,\lambda\,r)\right]_{r=0}=0$$

The derivatives are evaluated using Maple:

```
> diff(BesselJ(0,lambda*r),r);
                    −BesselJ(1, λ r) λ
> diff(BesselY(0,lambda*r),r);
                    −BesselY(1, λ r) λ
```

so that:

$$-C_1\,\lambda\,\underbrace{\text{BesselJ}(1,0)}_{0}-C_2\,\lambda\,\underbrace{\text{BesselY}(1,0)}_{\to-\infty}=0$$

Return to Section 1.8 and examine Figure 1-56, notice that BesselJ(1,0) approaches 0 while BesselY(1,0) approaches negative infinity; this information can also be obtained from Maple:

```
> limit(BesselJ(1,x),x=0);
                           0
> limit(BesselY(1,x),x=0);
                       undefined
```

Therefore, C_2 must be zero and θR is given by:

$$\theta R=C_1\,\text{BesselJ}(0,\lambda\,r)$$

Substituting Eq. (8) with $C_2 = 0$ into the boundary condition at $r = r_{out}$, Eq. (3), leads to:

$$C_1\,\text{BesselJ}(0,\lambda\,r_{out})=0 \tag{9}$$

Equation (9) is satisfied if $C_1 = 0$, but this would provide the solution $\theta R = 0$ which is not useful. Figure 2 shows the 0th order Bessel function of the 1st kind (i.e., Bessel_J(0,x)) as well as one of the more familiar eigenfunctions, cosine. Notice that BesselJ(0,x) oscillates about zero every time the argument changes by 2π in the same way that sine and cosine do; therefore, there are an infinite number of eigenvalues λ_i that will satisfy Eq. (9) associated with an infinite number of eigenfunctions. However, the eigencondition for this problem cannot be used to explicitly solve for the eigenvalues; rather, an implicit equation for the eigenvalues results from Eq. (9):

$$\text{BesselJ}(0,\lambda_i\,r_{out})=0 \text{ where } i=1,2,\ldots\infty \tag{10}$$

and the eigenfunctions for this problem are:

$$\theta R_i = C_{1,i}\,\text{BesselJ}\left(0,\lambda_i\, r\right) \text{ where } \text{BesselJ}\left(0,\lambda_i\, r_{out}\right)=0 \text{ for } i=1,2,\ldots\infty \tag{11}$$

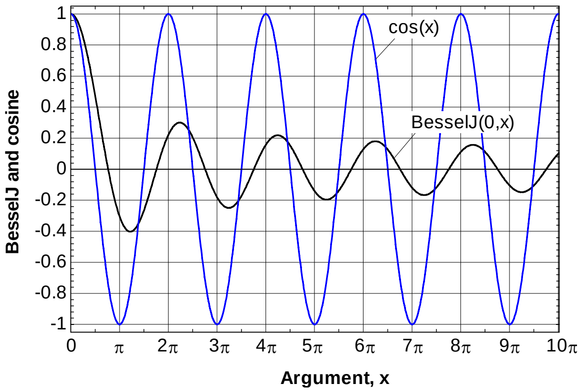


Figure 2: 0th order Bessel function of the 1st kind as well as the cosine function.

The solution to the ordinary differential equation for θX, Eq. (6), for each eigenvalue is:

$$\theta X_i = C_{3,i}\sinh\left(\lambda_i\, x\right)+C_{4,i}\cosh\left(\lambda_i\, x\right)$$

and so the general solution for each eigenvalue is:

$$\theta_i = \theta R_i\,\theta X_i = \text{BesselJ}\left(0,\lambda_i\, r\right)\left[C_{3,i}\sinh\left(\lambda_i\, x\right)+C_{4,i}\cosh\left(\lambda_i\, x\right)\right]$$

It is always a good idea to use Maple to check your work as you move through a problem like this one; enter the eigencondition and general solution into Maple:

```
> restart;
> BesselJ(0,lambda*r_out)=0;
```

$$\text{BesselJ}(0,\,\lambda\, r\_out) = 0$$

```
> theta:=(x,r)->BesselJ(0,lambda*r)*(C3*sinh(lambda*x)+C4*cosh(lambda*x));
```

$$\theta := (x,\,r) \rightarrow \text{BesselJ}(0,\,\lambda\, r)\,(C3\,\sinh(\lambda\, x) + C4\,\cosh(\lambda\, x))$$

and verify that the solution satisfies the governing differential equation, Eq. (1):

```
> r*diff(diff(theta(x,r),x),x)+diff(r*diff(theta(x,r),r),r);
```

$$r\,\text{BesselJ}(0,\,\lambda\, r)\,(C3\,\sinh(\lambda\, x)\,\lambda^2 + C4\,\cosh(\lambda\, x)\,\lambda^2)$$
$$-\text{BesselJ}(1,\,\lambda\, r)\,\lambda\,(C3\,\sinh(\lambda\, x) + C4\,\cosh(\lambda\, x))$$
$$-r\left(\text{BesselJ}(0,\,\lambda\, r) - \frac{\text{BesselJ}(1,\,\lambda\, r)}{\lambda\, r}\right)\lambda^2\,(C3\,\sinh(\lambda\, x) + C4\,\cosh(\lambda\, x))$$

```
> simplify(%);
```

$$0$$

and the boundary conditions in the r-direction:

```
> eval(diff(theta(x,r),r),r=0);
                                0

> theta(x,r_out);
             BesselJ( 0, λ r_out ) ( C3 sinh( λ x ) + C4 cosh( λ x ) )

> simplify(%);
                                0
```

The general solution is the sum of the solutions for each eigenvalue:

$$\theta = \sum_{i=1}^{\infty}\theta_i = \sum_{i=1}^{\infty}\text{BesselJ}\left(0,\lambda_i\, r\right)\left[C_{3,i}\sinh\left(\lambda_i\, x\right)+C_{4,i}\cosh\left(\lambda_i\, x\right)\right] \tag{12}$$

Equation (12) is substituted into the non-homogeneous boundary condition at $x = 0$, Eq. (4):

$$R_c''\, k\sum_{i=1}^{\infty}\text{BesselJ}\left(0,\lambda_i\, r\right)\lambda_i\left[C_{3,i}\underbrace{\cosh\left(\lambda_i\, 0\right)}_{1}+C_{4,i}\underbrace{\sinh\left(\lambda_i\, 0\right)}_{0}\right]=$$
$$\sum_{i=1}^{\infty}\text{BesselJ}\left(0,\lambda_i\, r\right)\left[C_{3,i}\underbrace{\sinh\left(\lambda_i\, 0\right)}_{0}+C_{4,i}\underbrace{\cosh\left(\lambda_i\, 0\right)}_{1}\right]$$

which can be simplified to:

$$\sum_{i=1}^{\infty}\text{BesselJ}\left(0,\lambda_i\, r\right)\left(R_c''\, k\,\lambda_i\, C_{3,i}-C_{4,i}\right)=0 \tag{13}$$

Equation (13) can only be true if:

$$R_c''\, k\,\lambda_i\, C_{3,i}=C_{4,i} \tag{14}$$

Substituting Eq. (14) into Eq. (12) leads to:

$$\theta = \sum_{i=1}^{\infty}C_i\,\text{BesselJ}\left(0,\lambda_i\, r\right)\left[\sinh\left(\lambda_i\, x\right)+R_c''\, k\,\lambda_i\cosh\left(\lambda_i\, x\right)\right] \tag{15}$$

Equation (15) is substituted into the final non-homogeneous boundary condition at $x = th$, Eq. (5):

$$k\left.\frac{\partial\theta}{\partial x}\right|_{x=th}=k\sum_{i=1}^{\infty}C_i\,\text{BesselJ}\left(0,\lambda_i\, r\right)\lambda_i\left[\cosh\left(\lambda_i\, th\right)+R_c''\, k\,\lambda_i\,\sinh\left(\lambda_i\, th\right)\right]=\begin{cases}\dot{q}_{laser}'' & \text{for } 0<r<r_{laser}\\ 0 \text{ for } r_{laser}<r<r_{out}\end{cases} \tag{16}$$

When we reached this point for our previous problems, the eigenfunctions were either sines or cosines and we took advantage of their orthogonality to reduce the summation to an equation for each constant C_i. Bessel functions are also orthogonal provided that they are multiplied by a weighting function, r, before integrating. That is, Eq. (16) is multiplied by r Bessel_J($0,\lambda_j\, r$) and then integrated from $r = 0$ to $r = r_{out}$:

$$k\sum_{i=1}^{\infty} C_i\, \lambda_i \left[\cosh\left(\lambda_i\, th\right) + R''_c\, k\, \lambda_i\, \sinh\left(\lambda_i\, th\right)\right] \int_0^{r_{out}} r\, \mathrm{BesselJ}\left(0,\lambda_j\, r\right) \mathrm{BesselJ}\left(0,\lambda_i\, r\right) dr =$$

$$\dot{q}''_{laser} \int_0^{r_{laser}} r\, \mathrm{BesselJ}\left(0,\lambda_j\, r\right) dr$$

Because the weighted eigenfunctions are orthogonal, all of the terms in the summation integrate to zero except for the one in which $i = j$; therefore, the summation is again reduced to an equation for each constant:

$$k\, C_i\, \lambda_i \left[\cosh\left(\lambda_i\, th\right) + R''_c\, k\, \lambda_i\, \sinh\left(\lambda_i\, th\right)\right] \int_0^{r_{out}} r\, \mathrm{BesselJ}^2\left(0,\lambda_i\, r\right) dr = \dot{q}''_{laser} \int_0^{r_{laser}} r\, \mathrm{BesselJ}\left(0,\lambda_i\, r\right) dr \tag{17}$$

The integrals in Eq. (17) may be evaluated using tables of Bessel function relations or, more conveniently, using Maple. The first integral is:

```
> restart;
> int(r*(BesselJ(0,lambda*r))^2,r=0..r_out);
```

$$\frac{1}{2}\,\frac{r\_out\,\left(\sqrt{\pi}\,\lambda\, r\_out\, \mathrm{BesselJ}(0,\lambda\, r\_out)^2 + \sqrt{\pi}\,\lambda\, r\_out\, \mathrm{BesselJ}(1,\lambda\, r\_out)^2\right)}{\sqrt{\pi}\,\lambda}$$

```
> simplify(%);
```

$$\frac{1}{2}\, r\_out^2\, \left(\mathrm{BesselJ}(0,\lambda\, r\_out)^2 + \mathrm{BesselJ}(1,\lambda\, r\_out)^2\right)$$

and the second integral is:

```
> int(r*BesselJ(0,lambda*r),r=0..r_laser);
```

$$\frac{r\_laser\, \mathrm{BesselJ}(1, r\_laser\, \lambda)}{\lambda}$$

Substituting these results into Eq. (17) leads to:

$$\frac{k\,C_i\,\lambda_i\,r_{out}^2}{2}\left[\cosh\left(\lambda_i\,th\right)+R''_c\,k\,\lambda_i\,\sinh\left(\lambda_i\,th\right)\right]\left[\text{BesselJ}^2\left(0,\lambda_i\,r_{out}\right)+\text{BesselJ}^2\left(1,\lambda_i\,r_{out}\right)\right]=$$
$$\frac{\dot{q}''_{laser}\,r_{laser}\,\text{BesselJ}\left(1,\lambda_i\,r_{laser}\right)}{\lambda_i}$$

Solving for the constants:

$$C_i=\frac{2\,\dot{q}''_{laser}\,r_{laser}\,\text{BesselJ}\left(1,\lambda_i\,r_{laser}\right)}{k\,\lambda_i^2\,r_{out}^2\left[\cosh\left(\lambda_i\,th\right)+R''_c\,k\,\lambda_i\,\sinh\left(\lambda_i\,th\right)\right]\left[\text{BesselJ}^2\left(0,\lambda_i\,r_{out}\right)+\text{BesselJ}^2\left(1,\lambda_i\,r_{out}\right)\right]}\tag{18}$$

The solution is programmed in EES. The inputs are entered:

```
"EXAMPLE 2.3-2: Laser Machining"
$UnitSystem SI MASS  RAD PA K J
$Tabstops 0.2 0.4 0.6 0.8 3.5

"Inputs"
th=1 [cm]*convert(cm,m)                        "thickness of disk"
r_out=10 [cm]*convert(cm,m)                    "outer radius of disk"
r_laser= 0.5 [cm]*convert(cm,m)                "radius of laser"
q``_dot_laser=1e5 [W/m^2]                      "laser flux"
k=1.4 [W/m-K]                                  "conductivity "
T_c=ConvertTemp(C,K,20 [C])                    "temperature of coolant and bed"
R``_c=2.8e-4 [m^2-K/W]                         "contact resistance"
```

The eigenvalues are computed using Eq. (10), which is repeated below:

$$\text{BesselJ}\left(0,\lambda_i\,r_{out}\right)=0\text{ where } i=1,2,\ldots\infty\tag{10}$$

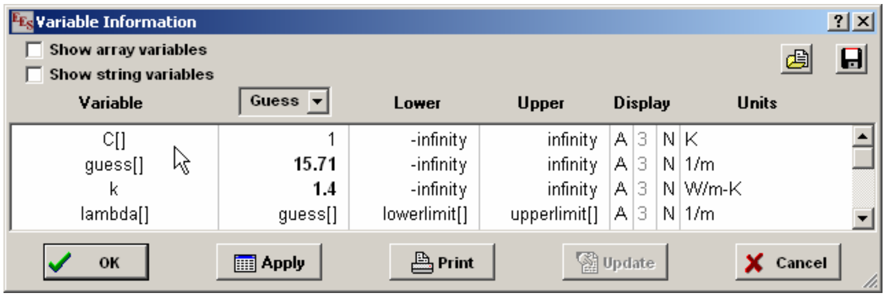
The eigenvalues cannot be computed explicitly. Instead, the implicit equation must be solved over different ranges of the independent variable in order to obtain subsequent eigenvalues. Without a computer, we would be reduced to using only a few of these eigenvalues which would be manually entered from a table. However, it is possible to automate the process of finding these eigenvalues using EES. First, specify how many terms will be used in the series and identify the lower and upper limits that characterize the range of each of these eigenvalues. Figure 2 shows that the zeros of the function BesselJ(0,x) are located in successive multiples of π, much like the cosine function. An array is created that defines the lower and upper limits as well as suitable guess values for each of the eigenvalues:

```
N=100                                          "number of terms in solution"
duplicate i=1,N
    lowerlimit[i]=(i-1)*pi/r_out               "lower limit of eigenvalue"
    upperlimit[i]=i*pi/r_out                   "upper limit of eigenvalue"
    guess[i]=lowerlimit[i]+pi/(2*r_out)        "guess value for eigenvalue"
end
```

The implicit equation that defines the eigenvalue is entered into EES:

```
duplicate i=1,N
    BesselJ(0,lambda[i]*r_out)=0                            "solve for eigenvalues"
end
```

Solving the EES program at this point will lead to the same eigenvalue for each term in the series, which can be observed by opening the Array table. EES determines the solution to the set of equations that falls within the upper and lower limits specified in the Variable Information window. It is possible to specify an array of limits and guess values that are defined by the arrays lowerlimit, upperlimit, and guess. Navigate to the Variable Information Window and de-select the show arrays option. Enter the arrays (specified by the [ ] indicator) into the appropriate column of the lambda entry, as shown in Figure 3.



☐ Show array variables
☐ Show string variables

Variable	Guess	Lower	Upper	Display	Units
C[]	1	-infinity	infinity	A 3 N	K
guess[]	**15.71**	-infinity	infinity	A 3 N	1/m
k	**1.4**	-infinity	infinity	A 3 N	W/m-K
lambda[]	guess[]	lowerlimit[]	upperlimit[]	A 3 N	1/m

Figure 3: Variable Information window with limits and guess values set for lambda

Solving the EES problem with these limits and guess values will result in the identification of the correct eigenvalues for each term in the series.

A position to evaluate the solution is specified in terms of a dimensionless axial and radial coordinate:

```
x_bar=0.5 [-]                                               "dimensionless axial position"
r_bar=0.5 [-]                                               "dimensionless radial position"
x=x_bar*th                                                  "x-position"
r=r_bar*r_out                                               "r-position"
```

Equation (18) is used to evaluate each of the constants and Eq. (15) is used to evaluate the associated term of the series:

```
duplicate i=1,N
    C[i]=2*q``_dot_laser*r_laser*BesselJ(1,lambda[i]*r_laser)/(k*lambda[i]^2*r_out^2*&
        (cosh(lambda[i]*th)+R``_c*k*lambda[i]*sinh(lambda[i]*th)*((BesselJ(0,lambda[i]*r_out))^2+&
        (BesselJ(1,lambda[i]*r_out))^2)))
    theta[i]=C[i]*BesselJ(0,lambda[i]*r)*(sinh(lambda[i]*x)+R``_c*k*lambda[i]*cosh(lambda[i]*x))
end
```

The solution for the temperature is obtained by summing the terms and adding T_c:

```
T=sum(theta[1..N])+T_c                                      "temperature"
T_Celsius=converttemp(K,C,T)                                "in C"
```

A parametric table is generated that contains the temperature evaluated over a grid that includes the entire computational domain. The contour plot generated in EES is shown in Figure 4. Note that the highest temperatures occur at *r* = 0 and *x* = *th* where the laser strikes the disk, as expected.

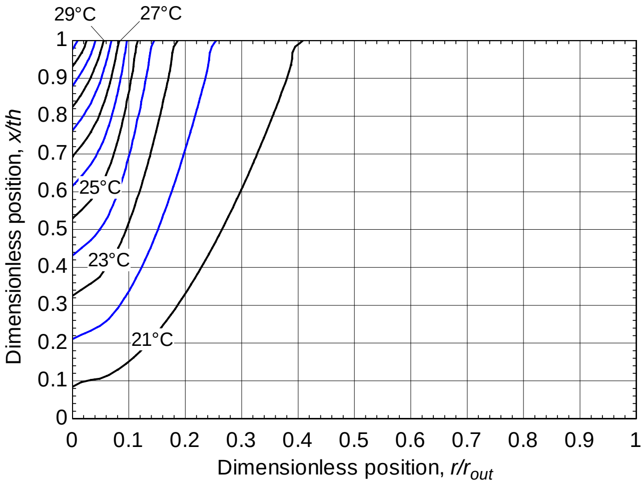


Figure 4: Contour plot of the temperature distribution in the work piece.

This problem is a good example of a situation where an analytical solution might be useful beyond simply providing verification for a numerical solution. The analytical solution developed here will not break down, even at very small length scales. However, a numerical solution might have problems simultaneously resolving both the large spatial scales associated with the entire disk and the very small spatial scales associated with the irradiated spot (particularly if the laser is focused more tightly). Consequently, a hybrid modeling approach is sometimes used. A numerical solution of the work piece would deal with the overall, possibly complex shape of the material as well as the details of the cooling provided to its surfaces. The numerical model would not consider in detail the region struck by the laser; rather the heat must be applied uniformly over a region that is large enough to be sufficiently resolved. The numerical model can be used to provide boundary conditions (i.e., the “far-field” conditions”) for the analytical solution which is used to analyze the very local effects of the laser heating. There are many situations where it is necessary to consider effects that occur over a wide range of length or time scales. It is often the case that multiple models, each tailored to resolve effects of a certain type, must be “stitched together” through suitably defined boundary conditions. We will not deal with these problems substantially in this textbook, but you should be alert for opportunities where two models of various types can be used together to provide more insight and accuracy than either one alone.