3.5.5 Non-homogeneous Boundary Conditions

The method of separation of variables can only be applied to 1-D transient problems where both spatial boundary conditions are homogeneous. In Sections 3.5.2 and 3.5.4, a single and obvious transformation was sufficient to make both spatial boundary conditions homogeneous. In many problems this will not be the case and therefore more advanced techniques will be required. Section 2.3.2 discusses methods for breaking 2-D steady problems with non-homogeneous terms into sub-problems that can be solved either by separation of variables or by the solution of an ordinary differential equation. In Section 2.4, superposition for 2-D steady-state problems is discussed. These techniques for solving problems with non-homogeneous boundary conditions using separation of variables remain valid for 1-D transient problems.

For example, consider the problem illustrated in Figure 3-31. A plane wall is initially at temperature T_{ini} = 20°C when at time t = 0 the right side of the wall (at x = L) is subjected to a heat flux $\dot{q}''_s$ = 5000 W/m^2. The left side of the wall is maintained at T_s = 20°C. The wall is L= 0.1 m thick and made of material with k = 10 W/m-K and α = 1x10^{-4} m^2/s.

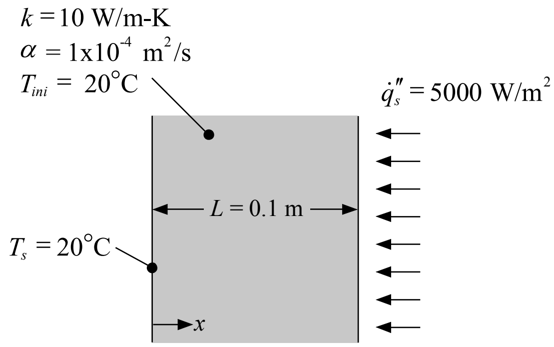

Figure 3-31: Plane wall subjected to a heat flux.

The inputs are entered in EES:

```
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

"Inputs"
T_ini=converttemp(C,K,20 [C])          "initial temperature"
T_s=converttemp(C,K,20 [C])            "surface temperature"
q``_dot=5000 [W/m^2]                   "heat flux"
k=10 [W/m-K]                           "conductivity"
alpha=1e-4 [m^2/s]                     "thermal diffusivity"
L=0.1 [m]                              "thickness of the wall"
```

The governing partial differential equation for the problem is:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3-376}$$

The boundary conditions for the problem are:

$$T_{x=0} = T_s \tag{3-377}$$

$$k \left. \frac{\partial T}{\partial x} \right|_{x=L} = \dot{q}_s'' \tag{3-378}$$

$$T_{t=0} = T_{ini} \tag{3-379}$$

Split Solutions into Homogeneous and Particular Components

Both of the spatial boundary conditions for this problem are non-homogeneous and there is no simple transformation (e.g., subtracting T_s) that can make them both homogeneous. Therefore, it is not possible to apply separation of variables to the problem as it is stated. However, by following the steps outlined in Section 2.3, it is possible to divide the problem into a homogeneous problem, $T_h(x,t)$ that can be solved using separation of variables and a particular solution that is only a function of x, $X(x)$:

$$T(x,t) = T_h(x,t) + X(x) \tag{3-380}$$

Equation (3-380) is substituted into the partial differential equation, Eq. (3-376), and each of the boundary conditions, Eqs. (3-377) through (3-379):

$$\underbrace{\frac{d^2 X}{dx^2}}_{\text{ODE for } X} + \underbrace{\frac{\partial^2 T_h}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_h}{\partial t}}_{\text{homogeneous PDE}} \tag{3-381}$$

$$\underbrace{T_{h,x=0}}_{\substack{=0 \text{ for homogeneous} \\ \text{boundary condition}}} + \underbrace{X_{x=0} = T_s}_{\text{boundary condition for X}} \tag{3-382}$$

$$\underbrace{k \left. \frac{\partial T_h}{\partial x} \right|_{x=L}}_{\substack{=0 \text{ for homogeneous} \\ \text{boundary condition}}} + \underbrace{k \left. \frac{dX}{dx} \right|_{x=L} = \dot{q}_s''}_{\text{boundary condition for X}} \tag{3-383}$$

$$T_{h,t=0} + X = T_{ini} \tag{3-384}$$

Enforce a Homogeneous Partial Differential Equation

It is necessary that the partial differential equation for T_h be homogeneous:

$$\frac{\partial^2 T_h}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_h}{\partial t} \tag{3-385}$$

The ordinary differential equation for the particular solution results from whatever is left in Eq. (3-381) once Eq. (3-385) is enforced:

$$\frac{d^2 X}{dx^2} = 0 \tag{3-386}$$

Solve the Ordinary Differential Equation for the Particular Solution

Integrating Eq. (3-386) one time leads to:

$$\frac{dX}{dx} = C_1 \tag{3-387}$$

where C_1 is a constant of integration. Integrating again leads to:

$$X = C_1 x + C_2 \tag{3-388}$$

where C_2 is another constant of integration.

Enforce Spatial Homogeneous Boundary Conditions

Both boundary conditions for T_h must be homogeneous in order to apply separation of variables. The homogeneous boundary condition required at $x = 0$ is:

$$T_{h,x=0} = 0 \tag{3-389}$$

Therefore, Eq. (3-382) is reduced to:

$$X_{x=0} = T_s \tag{3-390}$$

The homogeneous boundary condition required at $x = L$ is:

$$k \left. \frac{\partial T_h}{\partial x} \right|_{x=L} = 0 \tag{3-391}$$

Therefore, Eq. (3-383) is reduced to:

$$k \left. \frac{dX}{dx} \right|_{x=L} = \dot{q}_s'' \tag{3-392}$$

Substituting Eq. (3-388) into Eqs. (3-392) and (3-390) leads to the particular solution:

$$X = T_s + \frac{\dot{q}_s''}{k} x \tag{3-393}$$

Notice that the particular solution is equivalent to the steady-state temperature distribution in the material; this is the typical result.

Determine the Initial Condition for the Homogeneous Solution

Substituting Eq. (3-393) into Eq. (3-384) leads to:

$$T_{h,t=0} = T_{ini} - T_s - \frac{\dot{q}_s''}{k} x \tag{3-394}$$

Solve the Homogeneous Problem using Separation of Variables

It is straightforward to apply separation of variables to solve for T_h. The general solution that satisfies the partial differential equation, Eq. (3-385), and both spatial boundary conditions, Eqs. (3-389) and (3-391), is:

$$T_h = \sum_{i=1}^{\infty} C_i \sin(\lambda_i x) \exp(-\lambda_i^2 \alpha t) \tag{3-395}$$

where the eigenvalues are given by:

$$\lambda_i = \frac{(2i-1)\pi}{2L} \tag{3-396}$$

The constants are selected so that the general solution satisfies the initial condition, Eq. (3-394):

$$T_{h,t=0} = \sum_{i=1}^{\infty} C_i \sin(\lambda_i x) = T_{ini} - T_s - \frac{\dot{q}_s''}{k} x \tag{3-397}$$

Applying the orthogonality of the eigenfunctions leads to:

$$C_i \int_0^L \sin^2(\lambda_i x)\, dx = (T_{ini} - T_s) \int_0^L \sin(\lambda_i x)\, dx - \frac{\dot{q}_s''}{k} \int_0^L x \sin(\lambda_i x)\, dx \tag{3-398}$$

The integrals are evaluated in Maple:

```
> restart;
> assume(i,integer);
> lambda:=(2*i-1)*Pi/(2*L);
```

$$\lambda := \frac{(2\,i\sim - 1)\,\pi}{2\,L}$$

```
> int(sin(lambda*x)*sin(lambda*x),x=0..L);
```

$$\frac{L}{2}$$

```
> int(sin(lambda*x),x=0..L);
```

$$\frac{2\,L}{(2\,i\sim-1)\,\pi}$$

```
> int(x*sin(lambda*x),x=0..L);
```

$$\frac{4\,(-1)^{(1+i\sim)}\,L^2}{\pi^2\,(4\,i\sim^2-4\,i\sim+1)}$$

Substituting these results into Eq. (3-398) leads to:

$$C_i = \left(T_{ini} - T_s\right)\frac{4}{\left(2\,i-1\right)\pi} - \frac{\dot{q}_s''}{k}\,\frac{8\left(-1\right)^{(1+i)}\,L}{\pi^2\left(4\,i^2-4\,i+1\right)} \tag{3-399}$$

The solution is obtained by substituting Eq. (3-395) and (3-393) into Eq. (3-380):

$$T\left(x,t\right) = \sum_{i=1}^{\infty} C_i \sin\left(\lambda_i\,x\right)\exp\left(-\lambda_i^2\,\alpha\,t\right) + T_s + \frac{\dot{q}_s''}{k}\,x \tag{3-400}$$

where the constants are given by Eq. (3-399) and the eigenvalues are given by Eq. (3-396). The solution is implemented in EES:

```
N_term=10 [-]                                              "number of terms"
duplicate i=1,N_term
    lambda[i]=(2*i-1)*pi/(2*L)                             "eigenvalue"
    C[i]=(T_ini-T_s)*(4/(2*i-1)/Pi)-q``_dot*(8*(-1)^(1+i)*L/Pi^2/(4*i^2-4*i+1))/k
        "constants"
end
```

The solution is evaluated at a particular value of dimensionless position and Fourier number:

```
x_hat=0.1 [-]                                              "dimensionless position"
Fo=1    [-]                                                "Fourier number"
x=x_hat*L                                                  "position"
time=Fo*L^2/alpha                                          "time"
duplicate i=1,N_term
    T_h[i]=C[i]*sin(lambda[i]*x)*exp(-lambda[i]^2*alpha*time)
end
T=sum(T_h[1..N_term])+T_s+q``_dot*x/k                      "temperature"
T_C=converttemp(K,C,T)                                     "in C"
```

Figure 3-32 illustrates temperature distribution in the wall at various values of Fourier number. Notice that a thermal wave emanates from the right side of the wall (where the flux is applied) and reaches the left side at $Fo \approx 0.2$. The wall reaches steady state at $Fo \approx 1$.

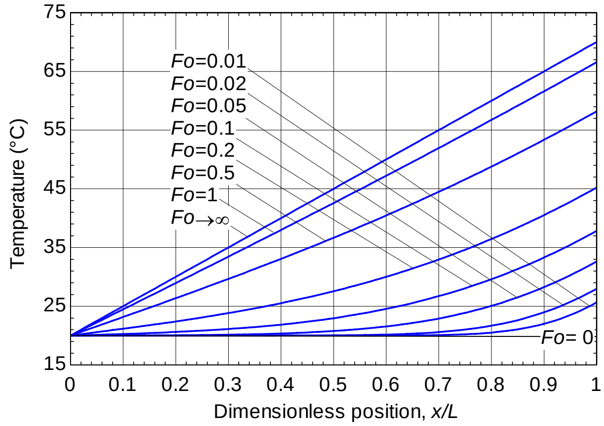

Figure 3-32: Temperature as a function of dimensionless position for various values of Fourier number.