3.9: Reduction of Multi-Dimensional Transient Problems

3.9.1 Introduction

Section 3.5 discusses the solution to 1-D transient problems using separation of variables. In some cases, it is possible to solve a multidimensional transient problem using the product of 1-D transient solutions. This is not always possible and there are some fairly substantial restrictions on the usefulness of this technique. The problem must be linear and completely homogeneous for this process to work. Completely homogeneous indicates that: (1) the governing differential equation is homogeneous (e.g., there is no generation term), and (2) all of the spatial boundary conditions are homogeneous. The initial condition does not have to be homogeneous but it must be relatively simple, as we will see. If the problem satisfies these conditions, then the steps outlined below will indicate whether the multidimensional problem can be re-cast as several 1-D problems. It is advisable to follow the steps rigorously in order to ensure that the product solution satisfies the initial condition, boundary conditions, and governing partial differential equation. However, if the 1-D solutions that are required are readily available (for example, within the Transient Conduction EES library) or are relatively easy to generate (using, for example, separation of variables or a numerical technique) then the technique presented in this section provides an efficient and straightforward method for solving what would otherwise be complicated problems.

3.9.2 The Dimensional Reduction Process

The dimensional reduction problem proceeds according to the following steps:

1. Derive the governing partial differential equation, spatial boundary conditions, and initial condition for the multi-dimensional problem.
2. Make sure that the governing partial differential equation and the spatial boundary conditions are homogeneous; if necessary and possible, transform them so that they are.
3. Express the multidimensional solution as the product of two or more 1-D transient solutions.
4. Substitute the expression from (3) into the governing partial differential equation from (2) in order to break it into several 1-D partial differential equations.
5. Substitute the expression from (3) into the boundary conditions from (2) in order to determine the boundary conditions for the 1-D partial differential equations identified in (4).
6. Substitute the expression from (3) into the initial conditions from (2) in order to obtain a suitable set of initial conditions for the 1-D partial differential equations identified in (4).
7. Obtain 1-D transient solutions and assemble the multidimensional solution.

The process of reducing the dimensionality of a transient problem is best illustrated by example and learned by practice. The process is demonstrated in the context of the problem illustrated in Figure 3-48. A rectangular steel bar is removed from a furnace and rapidly cooled using a water spray system. Through dimensional reduction, this 3-D transient problem in cartesian coordinates can be divided into three separate 1-D transient problems.

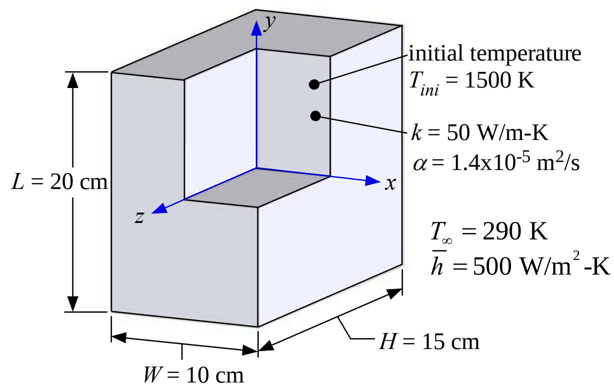

Figure 3-48: **A rectangular bar exposed to a sudden change in surface temperature.**

The bar has dimension W = 10 cm in the x-direction, L = 20 cm in the y-direction, and H = 15 cm in the z-direction. The bar material has conductivity k = 50 W/m-K and thermal diffusivity α = 1.4x10^{-5} m^2/s. The bar is initially at a uniform temperature T_{ini} = 1500 K. At time t = 0, the surfaces of the bar are sprayed with water at T_∞ = 290 K. The heat transfer coefficient between the water and the surface is $\overline{h}$ = 500 W/m^2-K.

The known information is entered in EES:

```
$UnitSystem SI MASS  RAD PA K J
$Tabstops 0.2 0.4 0.6 0.8 3.5

"Inputs"
W=10 [cm]*convert(cm,m)                          "dimension of bar in x-direction"
L=20 [cm]*convert(cm,m)                          "dimension of bar in y-direction"
H=15 [cm]*convert(cm,m)                          "dimension of bar in z-direction"
k=50 [W/m-K]                                     "conductivity"
alpha=1.4e-5 [m^2/s]                             "thermal diffusivity"
T_ini=1500 [K]                                   "initial temperature"
T_infinity=290 [K]                               "surface temperature"
h_bar=500 [W/m^2-K]                              "convection coefficient"
```

Derive the Mathematical Description of the Multi-Dimensional Problem

The governing partial differential equation is derived by considering a control volume that is differential in x, y, and z. The energy balance suggested by such a control volume is:

$$\dot{q}_x + \dot{q}_y + \dot{q}_z = \dot{q}_{x+dx} + \dot{q}_{y+dy} + \dot{q}_{z+dz} + \frac{\partial U}{\partial t} \tag{3-598}$$

Expanding the terms in Eq. (3-598) leads to:

$$0 = \frac{\partial \dot{q}_x}{\partial x} dx + \frac{\partial \dot{q}_y}{\partial y} dy + \frac{\partial \dot{q}_z}{\partial z} dz + \frac{\partial U}{\partial t} \tag{3-599}$$

The rate equations for conduction are:

$$\dot{q}_x = -k\, dy\, dz \frac{\partial T}{\partial x} \tag{3-600}$$

$$\dot{q}_y = -k\, dx\, dz \frac{\partial T}{\partial y} \tag{3-601}$$

$$\dot{q}_z = -k\, dx\, dy \frac{\partial T}{\partial z} \tag{3-602}$$

and the energy stored in the control volume is:

$$U = dx\, dy\, dz\, \rho\, c\, T \tag{3-603}$$

Substituting Eqs. (3-600) through (3-603) into Eq. (3-599) leads to:

$$0 = \frac{\partial}{\partial x}\left[-k\, dy\, dz \frac{\partial T}{\partial x}\right] dx + \frac{\partial}{\partial y}\left[-k\, dx\, dz \frac{\partial T}{\partial y}\right] dy + \frac{\partial}{\partial z}\left[-k\, dx\, dy \frac{\partial T}{\partial z}\right] dz + \frac{\partial}{\partial t}\left[dx\, dy\, dz\, \rho\, c\, T\right] \tag{3-604}$$

or:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3-605}$$

where $\alpha = k/\rho\, c$. The problem is symmetric around the center of the block in each coordinate direction and therefore a one-eighth symmetry model will be generated. With the coordinate system origins placed at the center of the block (see Figure 3-48), the boundary conditions are:

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=W/2} = \overline{h}\left(T_{x=W/2} - T_\infty\right) \tag{3-606}$$

$$-k \left.\frac{\partial T}{\partial y}\right|_{y=L/2} = \overline{h}\left(T_{y=L/2} - T_\infty\right) \tag{3-607}$$

$$-k \left.\frac{\partial T}{\partial z}\right|_{z=H/2} = \overline{h}\left(T_{z=H/2} - T_\infty\right) \tag{3-608}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \tag{3-609}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0 \tag{3-610}$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = 0 \tag{3-611}$$

The initial condition is:

$$T_{t=0} = T_{ini} \tag{3-612}$$

Ensure that Problem is Homogeneous

The spatial boundary conditions at the surfaces, Eqs. (3-606) through (3-608), are not homogeneous. The problem can be transformed by defining the temperature difference relative to the fluid temperature:

$$\theta = T - T_{\infty} \tag{3-613}$$

Substituting Eq. (3-613) into Eqs. (3-605) through (3-612) leads to a completely homogeneous problem:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{3-614}$$

$$-k\left.\frac{\partial \theta}{\partial x}\right|_{x=W/2} = \overline{h}\,\theta_{x=W/2} \tag{3-615}$$

$$-k\left.\frac{\partial \theta}{\partial y}\right|_{y=L/2} = \overline{h}\,\theta_{y=L/2} \tag{3-616}$$

$$-k\left.\frac{\partial \theta}{\partial z}\right|_{z=H/2} = \overline{h}\,\theta_{z=H/2} \tag{3-617}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0 \tag{3-618}$$

$$\left.\frac{\partial \theta}{\partial y}\right|_{y=0} = 0 \tag{3-619}$$

$$\left.\frac{\partial \theta}{\partial z}\right|_{z=0} = 0 \tag{3-620}$$

$$\theta_{t=0} = T_{ini} - T_{\infty} \tag{3-621}$$

Express the Solution as the Product of 1-D Transient Solutions

The multidimensional, transient solution is written as the product of 1-D transient solutions (θXt, θYt, and θZt) according to:

$$\theta\left(x,y,z,t\right) = \theta Xt\left(x,t\right)\theta Yt\left(y,t\right)\theta Zt\left(z,t\right) \tag{3-622}$$

Substitute the Product Solution into the Partial Differential Equation

Equation (3-622) is substituted into Eq. (3-614) with the goal of splitting it into several, 1-D transient partial differential equations:

$$\frac{\partial^2}{\partial x^2}\left(\theta Xt\,\theta Yt\,\theta Zt\right) + \frac{\partial^2}{\partial y^2}\left(\theta Xt\,\theta Yt\,\theta Zt\right) + \frac{\partial^2}{\partial z^2}\left(\theta Xt\,\theta Yt\,\theta Zt\right) = \frac{1}{\alpha}\frac{\partial}{\partial t}\left(\theta Xt\,\theta Yt\,\theta Zt\right) \tag{3-623}$$

or

$$\begin{aligned}&\theta Yt\,\theta Zt\frac{\partial^2\theta Xt}{\partial x^2} + \theta Xt\,\theta Zt\frac{\partial^2\theta Yt}{\partial y^2} + \theta Xt\,\theta Yt\frac{\partial^2\theta Zt}{\partial z^2} = \\ &\frac{1}{\alpha}\left(\theta Yt\,\theta Zt\frac{\partial\theta Xt}{\partial t} + \theta Xt\,\theta Zt\frac{\partial\theta Yt}{\partial t} + \theta Xt\,\theta Yt\frac{\partial\theta Zt}{\partial t}\right)\end{aligned} \tag{3-624}$$

Dividing Eq. (3-624) through by the product $\theta Xt\ \theta Yt\ \theta Zt$ leads to:

$$\underbrace{\frac{\dfrac{\partial^2\theta Xt}{\partial x^2}}{\theta Xt} - \frac{\dfrac{1}{\alpha}\dfrac{\partial\theta Xt}{\partial t}}{\theta Xt}}_{=0,\ \text{PDE in } x \text{ and } t} + \underbrace{\frac{\dfrac{\partial^2\theta Yt}{\partial y^2}}{\theta Yt} - \frac{\dfrac{1}{\alpha}\dfrac{\partial\theta Yt}{\partial t}}{\theta Yt}}_{=0,\ \text{PDE in } y \text{ and } t} + \underbrace{\frac{\dfrac{\partial^2\theta Zt}{\partial z^2}}{\theta Zt} - \frac{\dfrac{1}{\alpha}\dfrac{\partial\theta Zt}{\partial t}}{\theta Zt}}_{=0,\ \text{PDE in } z \text{ and } t} = 0 \tag{3-625}$$

Equation (3-625) can be grouped into terms that depend only on x and t, only on y and t, and only on z and t. The sum of these terms must be zero for any values of the independent variables x, y, z, and t. This is possible only if each of the equations for each spatial direction is zero. Therefore, Eq. (3-625) identifies three separate, 1-D transient partial differential equations:

$$\frac{\partial^2\theta Xt}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta Xt}{\partial t} \tag{3-626}$$

$$\frac{\partial^2\theta Yt}{\partial y^2} = \frac{1}{\alpha}\frac{\partial\theta Yt}{\partial t} \tag{3-627}$$

$$\frac{\partial^2 \theta Zt}{\partial z^2} = \frac{1}{\alpha}\frac{\partial \theta Zt}{\partial t} \tag{3-628}$$

Notice that Eqs. (3-626) through (3-628) are identical to the governing equations for a transient plane wall that were solved in Section 3.5.3 using separation of variables. For example, compare Eqs. (3-626) through (3-628) with Eq. (3-293).

Substitute the Product Solution into the Boundary Conditions

The product solution, Eq. (3-622), must be substituted into the boundary conditions, Eqs. (3-615) through (3-620):

$$-k\left.\frac{\partial \theta Xt}{\partial x}\right|_{x=W/2} \theta Yt\,\theta Zt = \overline{h}\,\theta Xt_{x=W/2}\,\theta Yt\,\theta Zt \tag{3-629}$$

$$-k\left.\frac{\partial \theta Yt}{\partial y}\right|_{y=L/2} \theta Xt\,\theta Zt = \overline{h}\,\theta Yt_{y=L/2}\,\theta Xt\,\theta Zt \tag{3-630}$$

$$-k\left.\frac{\partial \theta Zt}{\partial z}\right|_{z=H/2} \theta Xt\,\theta Yt = \overline{h}\,\theta Zt_{z=H/2}\,\theta Xt\,\theta Yt \tag{3-631}$$

$$\left.\frac{\partial \theta Xt}{\partial x}\right|_{x=0} \theta Yt\,\theta Zt = 0 \tag{3-632}$$

$$\left.\frac{\partial \theta Yt}{\partial y}\right|_{y=0} \theta Xt\,\theta Zt = 0 \tag{3-633}$$

$$\left.\frac{\partial \theta Zt}{\partial z}\right|_{z=0} \theta Xt\,\theta Yt = 0 \tag{3-634}$$

Equations (3-629) through (3-634) can be simplified:

$$-k\left.\frac{\partial \theta Xt}{\partial x}\right|_{x=W/2} = \overline{h}\,\theta Xt_{x=W/2} \tag{3-635}$$

$$-k\left.\frac{\partial \theta Yt}{\partial y}\right|_{y=L/2} = \overline{h}\,\theta Yt_{y=L/2} \tag{3-636}$$

$$-k\left.\frac{\partial \theta Zt}{\partial z}\right|_{z=H/2} = \overline{h}\,\theta Zt_{z=H/2} \tag{3-637}$$

$$\left.\frac{\partial \theta Xt}{\partial x}\right|_{x=0} = 0 \tag{3-638}$$

$$\left.\frac{\partial \theta Yt}{\partial y}\right|_{y=0} = 0 \tag{3-639}$$

$$\left.\frac{\partial \theta Zt}{\partial z}\right|_{z=0} = 0 \tag{3-640}$$

Equations (3-635) and (3-638) are the boundary conditions for θXt, Eqs. (3-636) and (3-639) are the boundary conditions for θYt, and Eqs. (3-637) and (3-640) are the boundary conditions for θZt.

Substitute the Product Solution into the Initial Condition

Equation (3-622) is substituted into the initial condition, Eq. (3-621):

$$\theta Xt_{x,t=0}\,\theta Yt_{y,t=0}\,\theta Zt_{z,t=0} = T_{ini} - T_{\infty} \tag{3-641}$$

The initial condition is satisfied if:

$$\theta Xt_{x,t=0} = T_{ini} - T_{\infty} \tag{3-642}$$

and

$$\theta Yt_{y,t=0} = 1 \tag{3-643}$$

$$\theta Zt_{z,t=0} = 1 \tag{3-644}$$

Obtain 1-D Transient Solutions and Assemble the Solution

The original multidimensional transient problem and the three, 1-D transient sub-problems are summarized in Table 3-1.

Table 3-1: **Multidimensional transient problem shown in Figure 3-48 and the three 1-D transient sub-problems that are associated with reducing the dimensionality of the problem**

Multidimensional problem for $T(x, y, z, t)$		
partial differential equation (PDE): $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$		
spatial boundary condition (SBC):		
$-k\left.\frac{\partial T}{\partial x}\right\|_{x=W/2}=\bar{h}\left(T_{x=W/2}-T_\infty\right)$,	$-k\left.\frac{\partial T}{\partial y}\right\|_{y=L/2}=\bar{h}\left(T_{y=L/2}-T_\infty\right)$,	$-k\left.\frac{\partial T}{\partial z}\right\|_{z=H/2}=\bar{h}\left(T_{z=H/2}-T_\infty\right)$,
$-\left.\frac{\partial T}{\partial x}\right\|_{x=0}=0$,	$\left.\frac{\partial T}{\partial x}\right\|_{y=0}=0$,	$\left.\frac{\partial T}{\partial x}\right\|_{z=0}=0$
initial condition (IC): $T_{t=0}=T_{ini}$		
Corresponding 1-D transient sub-problems		
$T=\theta+T_\infty$ where $\theta=\theta Xt\ \theta Yt\ \theta Zt$		
Sub-problem for $\theta Xt(x, t)$	Sub-problem for $\theta Yt(y, t)$	Sub-problem for $\theta Zt(z, t)$
PDE: $\frac{\partial^2 \theta Xt}{\partial x^2}=\frac{1}{\alpha}\frac{\partial \theta Xt}{\partial t}$	PDE: $\frac{\partial^2 \theta Yt}{\partial y^2}=\frac{1}{\alpha}\frac{\partial \theta Yt}{\partial t}$	PDE: $\frac{\partial^2 \theta Zt}{\partial z^2}=\frac{1}{\alpha}\frac{\partial \theta Zt}{\partial t}$
SBC: $\left.\frac{\partial \theta Xt}{\partial x}\right\|_{x=0}=0$,	SBC: $\left.\frac{\partial \theta Yt}{\partial y}\right\|_{y=0}=0$,	SBC: $\left.\frac{\partial \theta Zt}{\partial z}\right\|_{z=0}=0$,
$\left.\frac{\partial \theta Xt}{\partial x}\right\|_{x=W/2}=-\frac{\bar{h}}{k}\theta Xt_{x=W/2}$	$\left.\frac{\partial \theta Yt}{\partial y}\right\|_{y=L/2}=-\frac{\bar{h}}{k}\theta Yt_{y=L/2}$	$\left.\frac{\partial \theta Zt}{\partial z}\right\|_{z=H/2}=-\frac{\bar{h}}{k}\theta Zt_{z=H/2}$
IC: $\theta Xt_{t=0}=T_{ini}-T_\infty$	IC: $\theta Yt_{t=0}=1$	IC: $\theta Zt_{t=0}=1$

The position and time at which the temperature will be evaluated is specified:

```
x=0       [m]                                    "x-position"
y=0       [m]                                    "y-position"
z=0       [m]                                    "z-position"
time=500 [s]                                     "time"
```

The solution to the three sub-problems can be obtained using separation of variables. These sub-problems correspond to a plane wall subjected to a step change in convection and the solution was derived in Section 3.5.3. The solution is programmed in EES as the function planewall_T. The temperature within the rectangular bar can therefore be assembled from three separate calls to the function planewall_T to evaluate θXt (with fluid temperature 0 and initial temperature T_{ini} – T_∞), θYt (with fluid temperature 0 and initial temperature 1), and θZt (with fluid temperature 0 and initial temperature 1).

```
thetaXt=planewall_T(x, time, (T_ini-T_infinity), 0 [K], alpha, k, h_bar, W/2)    "solution for thetaXt"
thetaYt=planewall_T(y, time, 1 [K], 0 [K], alpha, k, h_bar, L/2)                  "solution for thetaYt"
thetaZt=planewall_T(z, time, 1 [K], 0 [K], alpha, k, h_bar, H/2)                  "solution for thetaZt"
```

The transformed multidimensional solution, θ, is the product of the 1-D solutions, Eq. (3-622):

```
theta=thetaXt*thetaYt*thetaZt     "solution for theta"
```

and the temperature is calculated using Eq. (3-613):

```
T=theta+T_infinity          "solution for T"
```

Figure 3-49 illustrates the temperature at the center of the block ($x = y = z = 0$) and the corner of the block ($x = W/2$, $y = L/2$, and $z = H/2$) as a function of time.

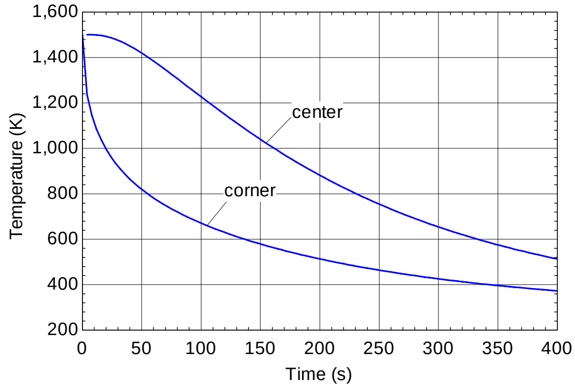

Figure 3-49: **Temperature at the center and corner of the rectangular block as a function of time.**