4.4.4 The Falkner-Skan Transformation

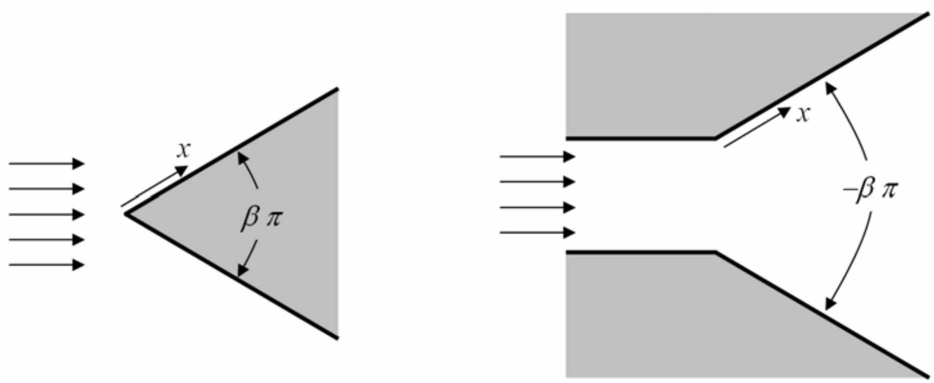
The self similar solutions provided in Sections 4.4.2 and 4.4.3 do not allow the consideration of any variation in the free-stream velocity or surface temperature. The Falkner-Skan transformation can be used to develop self-similar solutions to a class of problems where the velocity varies according to:

$$u_\infty = C_1\, x^m \tag{E12-1}$$

and the wall temperature varies according to:

$$T_s = T_\infty + C_2\, x^n \tag{E12-2}$$

where C_1, C_2, m, and n are constants (Cebeci (2002)). Potential flow theory shows that the free stream velocity over a wedge, shown in Figure E12-1, will have a power-law functional form; therefore the solutions to the Falkner-Skan transformation are sometimes referred to as wedge-flow solutions.



(a) **(b)**

Figure E12-1: Flows that can be solved using the Falkner-Skan transformation include (a) flow over a wedge of with angle $\beta\pi$ and (b) flow through an expansion with angle -$\beta\pi$.

The free stream velocity in the x-direction (along the surface, see Figure E12-1) for either of the cases shown in Figure E12-1 is given by Eq. (E12-1), where the exponent m is related to the angle β according to:

$$m = \frac{\beta}{2-\beta} \tag{E12-3}$$

Note that if $\beta > 0$, as shown in Figure E12-1(a), then the flow will be accelerating along the surface whereas if $\beta < 0$, as shown in Figure E12-1(b), then the flow is decelerating. If $\beta = 0$ then $m = 0$ and the free-stream velocity is constant; in this limit, the Falkner-Skan solution becomes identical to the Blasius solution for flow over a flat plate. If $\beta = 1$, then the Falkner-Skan transformation yields the solution to a stagnation flow.

4.4.4.1 Transformation of the Momentum Equation

The Falkner-Skan transformation can be accomplished using the similarity variable, η, originally defined in Eq. (4-197) and discussed in Section 4.4.2:

$$\eta = \frac{y}{2}\sqrt{\frac{u_\infty}{\upsilon\, x}} \tag{E12-4}$$

where, u_∞ is a function of x, according to Eq. (E12-1). Substituting Eq. (E12-1) into Eq. (E12-4) leads to:

$$\eta = \frac{y}{2}\sqrt{\frac{C_1}{\upsilon}}\, x^{\frac{m-1}{2}} \tag{E12-5}$$

The stream function, Ψ, is defined in terms of a dimensionless stream function, f, according to:

$$\Psi = 2\sqrt{u_\infty\, \upsilon\, x}\, f\left(\eta\right) \tag{E12-6}$$

Substituting Eq. (E12-1) into Eq. (E12-6) leads to:

$$\Psi = 2\sqrt{C_1\, \upsilon}\, x^{\frac{m+1}{2}}\, f\left(\eta\right) \tag{E12-7}$$

The transformation from the x, y plane to the x, η plane proceeds using the same chain-rule relations discussed in Sections 4.4.2 and 4.4.3. We know that the quantities Ψ or u or T are functions of x and η and we wish to transform them to functions of x and y. The chain rule for the partial derivative of any of these quantities, for example Ψ, with respect to x at constant y is:

$$\frac{\partial}{\partial x}\left[\Psi\left(x,\eta\left(x,y\right)\right)\right]_y = \left(\frac{\partial \Psi}{\partial x}\right)_\eta \underbrace{\left(\frac{\partial x}{\partial x}\right)_y}_{=1} + \left(\frac{\partial \Psi}{\partial \eta}\right)_x \left(\frac{\partial \eta}{\partial x}\right)_y \tag{E12-8}$$

or

$$\left(\frac{\partial \Psi}{\partial x}\right)_y = \left(\frac{\partial \Psi}{\partial x}\right)_\eta + \left(\frac{\partial \Psi}{\partial \eta}\right)_x \left(\frac{\partial \eta}{\partial x}\right)_y \tag{E12-9}$$

The chain rule for the partial derivative with respect to y at constant x is:

$$\frac{\partial}{\partial y}\left[\Psi\left(x,\eta\left(x,y\right)\right)\right]_x = \left(\frac{\partial \Psi}{\partial x}\right)_\eta \underbrace{\left(\frac{\partial x}{\partial y}\right)_x}_{=0} + \left(\frac{\partial \Psi}{\partial \eta}\right)_x \left(\frac{\partial \eta}{\partial y}\right)_x \tag{E12-10}$$

or

$$\left(\frac{\partial\Psi}{\partial y}\right)_x = \left(\frac{\partial\Psi}{\partial\eta}\right)_x \left(\frac{\partial\eta}{\partial y}\right)_x \tag{E12-11}$$

Therefore, the x-velocity component becomes:

$$u = \left(\frac{\partial\Psi}{\partial y}\right)_x = \frac{\partial}{\partial\eta}\left[2\sqrt{C_1 x^{m+1}\upsilon}\, f(\eta)\right]_x \left(\frac{\partial\eta}{\partial y}\right)_x \tag{E12-12}$$

or

$$u = 2\sqrt{C_1 x^{m+1}\upsilon}\,\frac{df}{d\eta}\frac{1}{2}\sqrt{\frac{C_1 x^{m-1}}{\upsilon}} \tag{E12-13}$$

which can be simplified to:

$$u = C_1 x^m \frac{df}{d\eta} \tag{E12-14}$$

or

$$\boxed{u = u_\infty \frac{df}{d\eta}} \tag{E12-15}$$

The y-velocity component becomes:

$$v = -\left(\frac{\partial\Psi}{\partial x}\right)_y = -\left(\frac{\partial\Psi}{\partial x}\right)_\eta - \left(\frac{\partial\Psi}{\partial\eta}\right)_x \left(\frac{\partial\eta}{\partial x}\right)_y \tag{E12-16}$$

Substituting Eq. (E12-6) into Eq. (E12-16) leads to:

$$v = -\frac{\partial}{\partial x}\left[2\sqrt{C_1\upsilon}\, x^{\frac{m+1}{2}} f(\eta)\right]_\eta - \frac{\partial}{\partial\eta}\left[2\sqrt{C_1\upsilon}\, x^{\frac{m+1}{2}} f\right]_x \left(\frac{\partial\eta}{\partial x}\right)_y \tag{E12-17}$$

or

$$v = -2\sqrt{C_1\upsilon}\left(\frac{m+1}{2}\right)x^{\frac{m-1}{2}} f - 2\sqrt{C_1\upsilon}\, x^{\frac{m+1}{2}}\frac{df}{d\eta}\frac{y}{2}\sqrt{\frac{C_1}{\upsilon}}\left(\frac{m-1}{2}\right)x^{\frac{m-3}{2}} \tag{E12-18}$$

which can be rearranged:

$$v=-2\sqrt{\frac{C_1\,x^m\,\upsilon}{x}}\left(\frac{m+1}{2}\right)f-2\sqrt{C_1\,x^m\,\upsilon\,x}\,\frac{df}{d\eta}\,\frac{y}{2}\left(\frac{m-1}{2}\right)\sqrt{\frac{C_1\,x^m}{\upsilon\,x^3}} \tag{E12-19}$$

and simplified:

$$v=-2\sqrt{\frac{u_\infty\,\upsilon}{x}}\left(\frac{m+1}{2}\right)f-2\frac{u_\infty}{x}\frac{df}{d\eta}\frac{y}{2}\left(\frac{m-1}{2}\right) \tag{E12-20}$$

or

$$\boxed{v=2\sqrt{\frac{u_\infty\,\upsilon}{x}}\left(\frac{m+1}{2}\right)\left[-f+\frac{(1-m)}{(1+m)}\eta\frac{df}{d\eta}\right]} \tag{E12-21}$$

The partial derivatives of u are required; the first derivative of u with respect to x is:

$$\left(\frac{\partial u}{\partial x}\right)_y=\left(\frac{\partial u}{\partial x}\right)_\eta+\left(\frac{\partial u}{\partial \eta}\right)_x\left(\frac{\partial \eta}{\partial x}\right)_y \tag{E12-22}$$

Substituting Eq. (E12-13) into Eq. (E12-22) leads to:

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left[C_1\,x^m\,\frac{df}{d\eta}\right]_\eta+\frac{\partial}{\partial \eta}\left[C_1\,x^m\,\frac{df}{d\eta}\right]_x\left(\frac{\partial \eta}{\partial x}\right)_y \tag{E12-23}$$

or

$$\frac{\partial u}{\partial x}=C_1\,m\,x^{m-1}\,\frac{df}{d\eta}+C_1\,x^m\,\frac{d^2 f}{d\eta^2}\,\frac{y}{2}\,\frac{(m-1)}{2}\sqrt{\frac{C_1\,x^{m-3}}{\upsilon}} \tag{E12-24}$$

which can be simplified to:

$$\boxed{\frac{\partial u}{\partial x}=m\frac{df}{d\eta}\frac{u_\infty}{x}+\frac{u_\infty}{x}\frac{d^2 f}{d\eta^2}\eta\frac{(m-1)}{2}} \tag{E12-25}$$

The first derivative of u with respect to y is:

$$\left(\frac{\partial u}{\partial y}\right)_x=\left(\frac{\partial u}{\partial \eta}\right)_x\left(\frac{\partial \eta}{\partial y}\right)_x \tag{E12-26}$$

Substituting Eq. (E12-14) into Eq. (E12-26) leads to:

$$\left(\frac{\partial u}{\partial y}\right)_x = C_1\, x^m \frac{d^2 f}{d\eta^2}\frac{1}{2}\sqrt{\frac{C_1\, x^{m-1}}{\upsilon}} \tag{E12-27}$$

or

$$\boxed{\frac{\partial u}{\partial y} = u_\infty \frac{d^2 f}{d\eta^2}\frac{1}{2}\sqrt{\frac{u_\infty}{x\,\upsilon}}} \tag{E12-28}$$

The second derivative of u with respect to y is:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \eta}\left[\frac{\partial u}{\partial y}\right]_x \left(\frac{\partial \eta}{\partial y}\right)_x \tag{E12-29}$$

Substituting Eq. (E12-27) into Eq. (E12-29) leads to:

$$\frac{\partial^2 u}{\partial y^2} = C_1\, x^m \frac{d^3 f}{d\eta^3}\frac{1}{2}\sqrt{\frac{C_1\, x^{m-1}}{\upsilon}}\,\frac{1}{2}\sqrt{\frac{C_1\, x^{m-1}}{\upsilon}} \tag{E12-30}$$

or

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{d^3 f}{d\eta^3}\frac{u_\infty^2}{4\upsilon\, x}} \tag{E12-31}$$

The momentum equation, including the pressure gradient (which is not zero in the case of a wedge flow), is:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \upsilon\frac{\partial^2 u}{\partial y^2} \tag{E12-32}$$

The flow outside of the boundary layer (i.e., the free stream) can be considered inviscid and therefore the free-stream pressure can be related to the free stream velocity distribution according to Bernoulli's equation:

$$p_\infty + \frac{\rho\, u_\infty^2}{2} = \text{constant} \tag{E12-33}$$

Taking the derivative of Eq. (E12-33) leads to:

$$\frac{dp_\infty}{dx} + \rho u_\infty \frac{du_\infty}{dx} = 0 \tag{E12-34}$$

Substituting Eq. (E12-34) into Eq. (E12-32) leads to:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_\infty \frac{du_\infty}{dx} + \upsilon \frac{\partial^2 u}{\partial y^2} \tag{E12-35}$$

Substituting Eqs. (E12-1), (E12-15), (E12-25), (E12-21), (E12-28), and (E12-31) into Eq. (E12-35) leads to:

$$\begin{aligned} & u_\infty \frac{df}{d\eta}\left[m\frac{df}{d\eta}\frac{u_\infty}{x} + \frac{u_\infty}{x}\frac{d^2 f}{d\eta^2}\eta\frac{(m-1)}{2}\right] + \\ & 2\sqrt{\frac{u_\infty \upsilon}{x}}\left(\frac{m+1}{2}\right)\left[-f + \frac{(1-m)}{(1+m)}\eta\frac{df}{d\eta}\right]u_\infty \frac{d^2 f}{d\eta^2}\frac{1}{2}\sqrt{\frac{u_\infty}{x\upsilon}} = \\ & u_\infty \frac{du_\infty}{dx} + \upsilon\frac{d^3 f}{d\eta^3}\frac{u_\infty^2}{4\upsilon x} \end{aligned} \tag{E12-36}$$

Dividing Eq. (E12-36) through by u_∞^2 / x leads to:

$$\frac{df}{d\eta}\left[m\frac{df}{d\eta} + \frac{d^2 f}{d\eta^2}\eta\frac{(m-1)}{2}\right] + \left(\frac{m+1}{2}\right)\left[-f + \frac{(1-m)}{(1+m)}\eta\frac{df}{d\eta}\right]\frac{d^2 f}{d\eta^2} = \frac{x}{u_\infty}\frac{du_\infty}{dx} + \frac{1}{4}\frac{d^3 f}{d\eta^3} \tag{E12-37}$$

or

$$m\left(\frac{df}{d\eta}\right)^2 - f\left(\frac{m+1}{2}\right)\frac{d^2 f}{d\eta^2} = \frac{x}{u_\infty}\frac{du_\infty}{dx} + \frac{1}{4}\frac{d^3 f}{d\eta^3} \tag{E12-38}$$

Substituting Eq. (E12-1) into Eq. (E12-38) leads to:

$$m\left(\frac{df}{d\eta}\right)^2 - f\left(\frac{m+1}{2}\right)\frac{d^2 f}{d\eta^2} = \frac{x}{C_1 x^m} m C_1 x^{m-1} + \frac{1}{4}\frac{d^3 f}{d\eta^3} \tag{E12-39}$$

which can be simplified to:

$$m\left(\frac{df}{d\eta}\right)^2 - f\left(\frac{m+1}{2}\right)\frac{d^2 f}{d\eta^2} = m + \frac{1}{4}\frac{d^3 f}{d\eta^3} \tag{E12-40}$$

or, finally:

$$\boxed{\frac{d^3 f}{d\eta^3} = 4m\left[\left(\frac{df}{d\eta}\right)^2 - 1\right] - 4f\left(\frac{m+1}{2}\right)\frac{d^2 f}{d\eta^2}} \tag{E12-41}$$

Notice that if $m = 0$ then the free stream velocity is constant and Eq. (E12-41) reduces to:

$$\frac{d^3 f}{d\eta^3} = -2f\frac{d^2 f}{d\eta^2} \tag{E12-42}$$

which is the same ordinary differential equation derived in Section 4.4.2 for flow over a flat plate. The boundary conditions for Eq. (E12-41) are:

$$u_{y=0} = 0 \rightarrow \boxed{\left.\frac{df}{d\eta}\right|_{\eta=0} = 0} \tag{E12-43}$$

$$v_{y=0} = 0 \rightarrow \boxed{f_{\eta=0} = 0} \tag{E12-44}$$

$$u_{y\rightarrow\infty} = u_\infty \rightarrow \boxed{\left.\frac{df}{d\eta}\right|_{\eta\rightarrow\infty} = 1} \tag{E12-45}$$

4.4.4.2 Solution of the Momentum Equation

Here we will use EES' internal integration functions in order to solve Eq. (E12-41) subject to the boundary conditions provided by Eqs. (E12-43) through (E12-45). The solution consists of three integrals:

$$f = f_{\eta=0} + \int_{\eta=0}^{\eta} \frac{df}{d\eta} d\eta \tag{E12-46}$$

$$\frac{df}{d\eta} = \left.\frac{df}{d\eta}\right|_{\eta=0} + \int_{\eta=0}^{\eta} \frac{d^2 f}{d\eta^2} d\eta \tag{E12-47}$$

$$\frac{d^2 f}{d\eta^2} = \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} + \int_{\eta=0}^{\eta} \frac{d^3 f}{d\eta^3} d\eta \tag{E12-48}$$

where the third derivative of f with respect to η is provided by the ordinary differential equation, Eq. (E12-41).

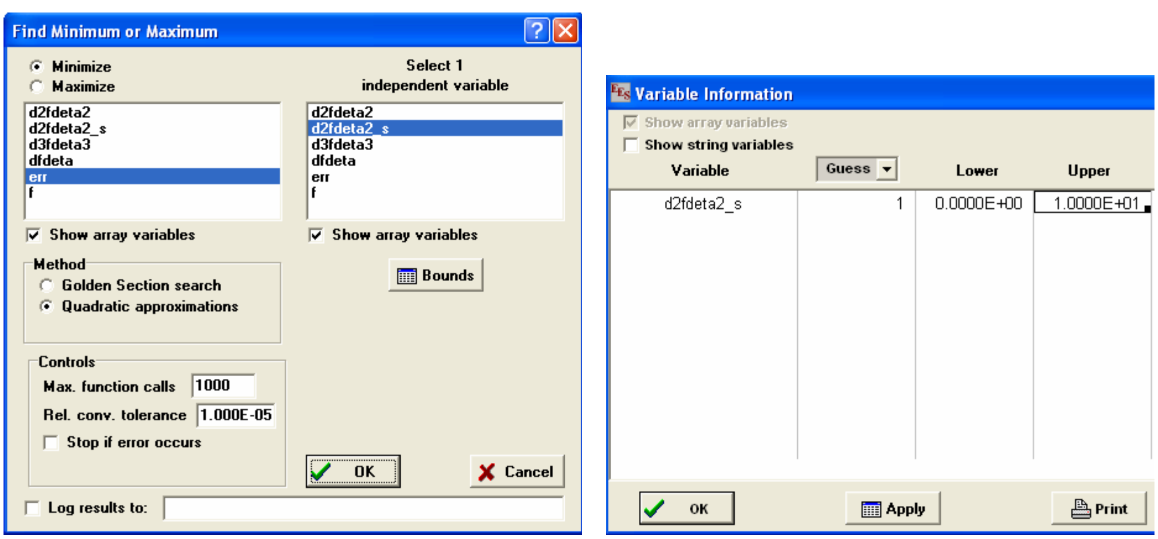
```
$UnitSystem SI MASS RAD PA K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in
```

```
eta_infinity=10 [-]              "outer edge of the computational domain"
beta=0 [-]                       "shape of flow"
m=beta/(2-beta)                  "exponent on free stream velocity variation"

"initial conditions for integration"
d2fdeta2_s=0.664 [-]         "assumed"
dfdeta_s=0 [-]
f_s=0 [-]

f=f_s+integral(dfdeta,eta,0,eta_infinity)
dfdeta=dfdeta_s+integral(d2fdeta2,eta,0,eta_infinity)
d2fdeta2=d2fdeta2_s+integral(d3fdeta3,eta,0,eta_infinity)
d3fdeta3=4*m*(dfdeta^2-1)-4*f*(m+1)*d2fdeta2/2
```

Notice that the initial conditions for the integrals in Eqs. (E12-46) and (E12-47) are provided by Eqs. (E12-43) and (E12-44); however, the initial condition for the integral in Eq. (E12-48) is not known and must be adjusted in order to achieve the third boundary condition, Eq. (E12-45). Here we will set this problem up as an optimization problem; the error between the desired and calculated values of the variable dfdeta at the outer edge of the computational domain is defined and used as the target of EES' built-in optimization algorithms:

```
err=abs(dfdeta-1)                                          "error"
```

The assumed value of the variable d2fdeta2_s is commented out and Min/Max is selected from the Calculate menu (Figure E12-2(a)). The problem is setup to minimize the value of the variable err by adjusting the independent variable d2fdeta2_s. It is necessary to set up bounds for the independent variable by selecting Bounds (Figure E12-2(b)). This problem is very sensitive to the value of d2fdeta2_s and therefore reasonable bounds are important (but not critical, as we shall see). In Section 4.4.2, we found that the value of d2fdeta2_s is proportional to the shear stress; therefore, d2fdeta2_s can feasibly range from 0 (no shear stress, indicating that the flow is separated and our boundary layer assumptions invalid) to a very high value (e.g., 10).



(a) **(b)**

Figure E12-2: (a) Find Minimum or Maximum Window and (b) Bounds Window.

Select OK twice to carry out the optimization. The program will likely work as expected for β = 0 (i.e., a flat plate) and accurately reproduce the self-similar solution discussed in Section 4.4.2. However, if β is not zero then you are likely to obtain a convergence error because the problem is inherently very sensitive to the value of d2fdeta2_s; when the optimization evaluates the integral at the limits of d2fdeta2_s (set in the Figure E12-2(b)), the integration quickly diverges. Fortunately, such a solution is not physical and should be discarded. This problem can be overcome by de-selecting the Stop if error occurs button in the Find Minimum or Maximum Window (Figure S4.4.4-2(a)); values of d2fdeta2_s that provide a convergence error are discarded and the optimizer continues to work until an optimal solution is identified.

The intermediate values of the integration variables are recorded using an Integral Table:

```
$IntegralTable eta:0.1, f,dfdeta,d2fdeta2
```

The dimensionless velocity $u/u_{\infty} = \frac{df}{d\eta}$ is shown in Figure E12-3 as a function of dimensionless position, η, for various values of β. Notice that when β = 0, the solution is identical to Figure 4-11 with $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$ =0.664. A positive value of β indicates a favorable pressure gradient; as β increases, the flow acceleration increases so that the boundary layer thins and the shear stress at the surface (indicated by the gradient of the velocity distribution) increases. A negative value of β indicates an adverse pressure gradient; as β decreases, the flow deceleration increases so that the boundary layer becomes thicker and the shear stress at the surface is reduced. At β = -0.198, the shear stress reaches zero and the flow will separate.

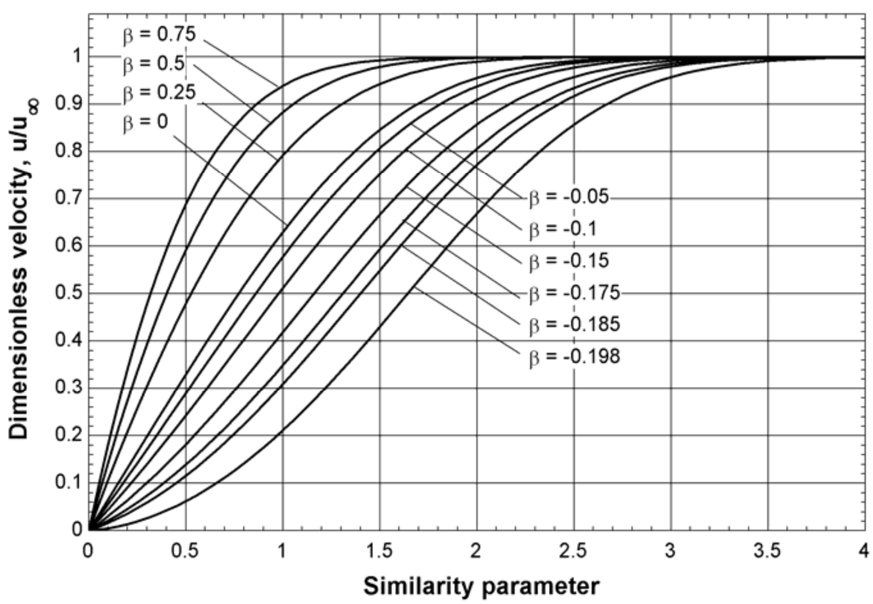


Figure E12-3: Dimensionless velocity (u/u_{∞}) as a function of the similarity variable η for various values of β.

The shear stress at the surface of the plate is given by:

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \tag{E12-49}$$

Substituting Eq. (E12-28) into Eq. (E12-49) leads to:

$$\tau_s = \mu u_\infty \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} \frac{1}{2}\sqrt{\frac{u_\infty}{x\upsilon}} \tag{E12-50}$$

The friction coefficient is:

$$C_f = \frac{2\tau_s}{\rho u_\infty^2} \tag{E12-51}$$

Substituting Eq. (E12-50) into Eq. (E12-51) leads to:

$$C_f = \frac{2}{\rho u_\infty^2} \mu u_\infty \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} \frac{1}{2}\sqrt{\frac{u_\infty}{x\upsilon}} \tag{E12-52}$$

which can be simplified to:

$$C_f = \frac{\left.\dfrac{d^2 f}{d\eta^2}\right|_{\eta=0}}{\sqrt{Re_x}} \tag{E12-53}$$

where

$$Re_x = \frac{\rho u_\infty x}{\mu} \tag{E12-54}$$

Figure E12-4 shows $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$ as a function of β and indicates that the friction coefficient is reduced as the shape parameter β is reduced until separation occurs at β = -0.198.

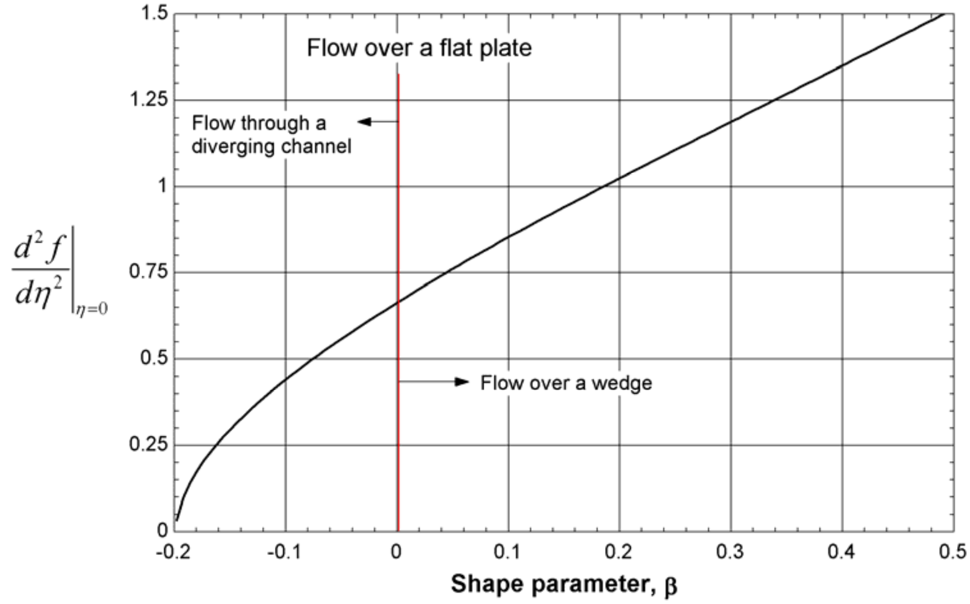


Figure E12-4: $C_f/\sqrt{Re_x}$ as a function of the shape parameter.

Transformation of the Energy Equation

The dimensionless temperature, $\tilde{\theta}$, is assumed to be a function of the similarity parameter η. The dimensionless temperature is defined according to:

$$\tilde{\theta} = \frac{T - T_s}{T_\infty - T_s} \tag{E12-55}$$

and so the temperature may be written as:

$$T = T_s - \left(T_s - T_\infty\right)\tilde{\theta} \tag{E12-56}$$

or

$$T = T_\infty + \left(T_s - T_\infty\right)\left(1 - \tilde{\theta}\right) \tag{E12-57}$$

Substituting the function form of the surface-to-free stream temperature difference given by Eq. (E12-2) into Eq. (E12-57) leads to:

$$T = T_\infty + C_2\, x^n \left(1 - \hat{\theta}\right) \tag{E12-58}$$

The transformation of the problem from the x, y plane to the x, η plane progresses in the same manner. The partial derivative of temperature with respect to x is:

$$\left(\frac{\partial T}{\partial x}\right)_y = \left(\frac{\partial T}{\partial x}\right)_\eta + \left(\frac{\partial T}{\partial \eta}\right)_x \frac{\partial \eta}{\partial x} \tag{E12-59}$$

Substituting Eqs. (E12-58) and (E12-5) into Eq. (E12-59) leads to:

$$\frac{\partial T}{\partial x} = C_2\, n\, x^{n-1}\left(1-\tilde{\theta}\right) - C_2\, x^n \frac{d\tilde{\theta}}{dx} \frac{y}{2}\sqrt{\frac{C_1}{\upsilon}}\left(\frac{m-1}{2}\right) x^{\frac{m-3}{2}} \tag{E12-60}$$

which can be rewritten as:

$$\boxed{\frac{\partial T}{\partial x} = C_2\, x^{n-1}\left[n\left(1-\tilde{\theta}\right) - \frac{d\tilde{\theta}}{dx}\eta\left(\frac{m-1}{2}\right)\right]} \tag{E12-62}$$

The partial derivative of temperature with respect to *y* is:

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{E12-63}$$

or

$$\boxed{\frac{\partial T}{\partial y} = -C_2\, x^n \frac{d\tilde{\theta}}{d\eta}\frac{1}{2}\sqrt{\frac{u_\infty}{\upsilon\, x}}} \tag{E12-64}$$

The second derivative of temperature with respect to *y* is:

$$\frac{\partial}{\partial y}\left[\frac{\partial T}{\partial y}\right] = \frac{\partial}{\partial \eta}\left[\frac{\partial T}{\partial y}\right]\frac{\partial \eta}{\partial y} \tag{E12-65}$$

Substituting Eq. (E12-64) into Eq. (E12-65) leads to:

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial \eta}\left[-C_2\, x^n \frac{d\tilde{\theta}}{d\eta}\frac{1}{2}\sqrt{\frac{u_\infty}{\upsilon\, x}}\right]\frac{\partial \eta}{\partial y} \tag{E12-66}$$

or

$$\boxed{\frac{\partial^2 T}{\partial y^2} = -C_2\, x^n \frac{d^2\tilde{\theta}}{d\eta^2}\frac{1}{4}\frac{u_\infty}{\upsilon\, x}} \tag{E12-67}$$

Substituting Eqs. (E12-62), (E12-64), (E12-67), (E12-15), and (E12-21) into the energy equation simplified for a boundary layer and neglecting viscous dissipation, Eq. (4-264), leads to:

$$u_\infty \frac{df}{d\eta} C_2\, x^{n-1}\left[n\left(1-\tilde{\theta}\right)-\frac{d\tilde{\theta}}{dx}\eta\left(\frac{m-1}{2}\right)\right]$$
$$-2\sqrt{\frac{u_\infty\,\upsilon}{x}}\left(\frac{m+1}{2}\right)\left[-f+\frac{\left(1-m\right)}{\left(1+m\right)}\eta\frac{df}{d\eta}\right]C_2\, x^n \frac{d\tilde{\theta}}{d\eta}\frac{1}{2}\sqrt{\frac{u_\infty}{\upsilon\, x}}= \qquad \text{(E12-68)}$$
$$-\alpha\, C_2\, x^n \frac{d^2\tilde{\theta}}{d\eta^2}\frac{1}{4}\frac{u_\infty}{\upsilon\, x}$$

Dividing Eq. (E12-68) through by $u_\infty\, C_2\, x^{n-1}$ leads to:

$$\boxed{\frac{df}{d\eta}\left[n\left(1-\tilde{\theta}\right)-\frac{d\tilde{\theta}}{dx}\eta\left(\frac{m-1}{2}\right)\right]-\left(\frac{m+1}{2}\right)\left[-f+\frac{\left(1-m\right)}{\left(1+m\right)}\eta\frac{df}{d\eta}\right]\frac{d\tilde{\theta}}{d\eta}=-\frac{d^2\tilde{\theta}}{d\eta^2}\frac{1}{4\,Pr}} \qquad \text{(E12-69)}$$

which is an ordinary differential equation for the dimensionless temperature. Note that in the limit that $m = n = 0$ (i.e., constant free stream velocity and surface temperature), Eq. (E12-69) becomes:

$$\frac{d^2\tilde{\theta}}{d\eta^2}+2\, f\, Pr\frac{d\tilde{\theta}}{d\eta}=0 \qquad \text{(E12-70)}$$

which is identical to Eq. (4-282), derived in Section 4.4.3. The boundary conditions for Eq. (E12-69) are related to the surface temperature:

$$\tilde{\theta}_{\eta=0}=0 \qquad \text{(E12-71)}$$

and recovering the free stream temperature:

$$\tilde{\theta}_{\eta\to\infty}=1 \qquad \text{(E12-72)}$$

Solution of the Energy Equation

EES' internal integration functions are used to solve Eq. (E12-70) subject to the boundary conditions provided by (E12-71) and (E12-72). The momentum equation must be solved prior to solving the energy equation using the technique discussed in the previous section, this is possible because the thermal energy equation is decoupled from the momentum equation by the assumption of temperature-independent properties.

The solution technique remains the same. The solution consists of two integrals:

$$\tilde{\theta}=\tilde{\theta}_{\eta=0}+\int_{\eta=0}^{\eta}\frac{d\tilde{\theta}}{d\eta}d\eta \qquad \text{(E12-73)}$$

$$\frac{d\tilde{\theta}}{d\eta} = \left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0} + \int_{\eta=0}^{\eta} \frac{d^2\tilde{\theta}}{d\eta^2} d\eta \tag{E12-74}$$

where the second derivative of $\tilde{\theta}$ with respect to η is provided by the ordinary differential equation, Eq. (E12-69). The following code is appended to the previous EES program used to solve the momentum equation:

```
Pr=1.0                                                        "Prandtl number"
theta_s=0 [-]
n=0.1 [-]
dthetadeta_s=0.7 [-]                                          "guess"

theta=theta_s+integral(dthetadeta,eta,0,eta_infinity)
dthetadeta=dthetadeta_s+integral(d2thetadeta2,eta,0,eta_infinity)
d2thetadeta2=-4*Pr*(dfdeta*(n*(1-theta)-eta*dthetadeta*((m-1)/2))-&
     (m+1)/2*(-f+(1-m)*eta*dfdeta/(1+m))*dthetadeta)

err2=abs(theta-1)
```

The value of the variable dthetadeta_s is adjusted to minimize the value of err2 using EES' internal minimization function. Note that the outer limit of the integration should be adjusted as discussed in Section 4.4.3:

```
eta_infinity=MAX(10,10/sqrt(Pr))          "outer edge of the computational domain"
```

and the additional integration variables are included in the Integral Table:

```
$IntegralTable eta:0.05, f,dfdeta,d2fdeta2,theta,dthetadeta,d2thetadeta2
```

The heat flux at the surface of the plate is given by:

$$\dot{q}_s'' = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{E12-75}$$

Substituting Eq. (E12-64) into Eq. (E12-75) leads to:

$$\dot{q}_s'' = k\, C_2\, x^{n-0.5} \frac{d\tilde{\theta}}{d\eta} \frac{1}{2} \sqrt{\frac{u_\infty}{\upsilon}} \tag{E12-76}$$

and therefore $n = 0.5$ with $\beta = 0$ corresponds to the case of a flat plate with a constant heat flux. The Nusselt number is given by:

$$Nu_x = \frac{\dot{q}_s''\, x}{k\left(T_s - T_\infty\right)} \tag{E12-77}$$

or, substituting Eqs. (E12-76) and (E12-2) into Eq. (E12-77):

$$Nu_x = k\,C_2\,x^{n-0.5}\,\frac{d\tilde{\theta}}{d\eta}\frac{1}{2}\sqrt{\frac{u_\infty}{\upsilon}}\frac{x}{k\,C_2\,x^n} \tag{E12-78}$$

which can be simplified to:

$$Nu_x = \left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}\frac{1}{2}\sqrt{\frac{u_\infty\,x}{\upsilon}} \tag{E12-79}$$

or

$$Nu_x = \left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}\frac{\sqrt{Re_x}}{2} \tag{E12-80}$$

Figure E12-5 illustrates the quantity $\left.\frac{1}{2}\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$ which, according to Eq. (E12-80) is equal to the ratio of Nu_x to $\sqrt{Re_x}$, as a function of Pr for the case where $\beta = 0$ (i.e., flow over a flat plate) and $n = 0.5$ (i.e., a constant heat flux).

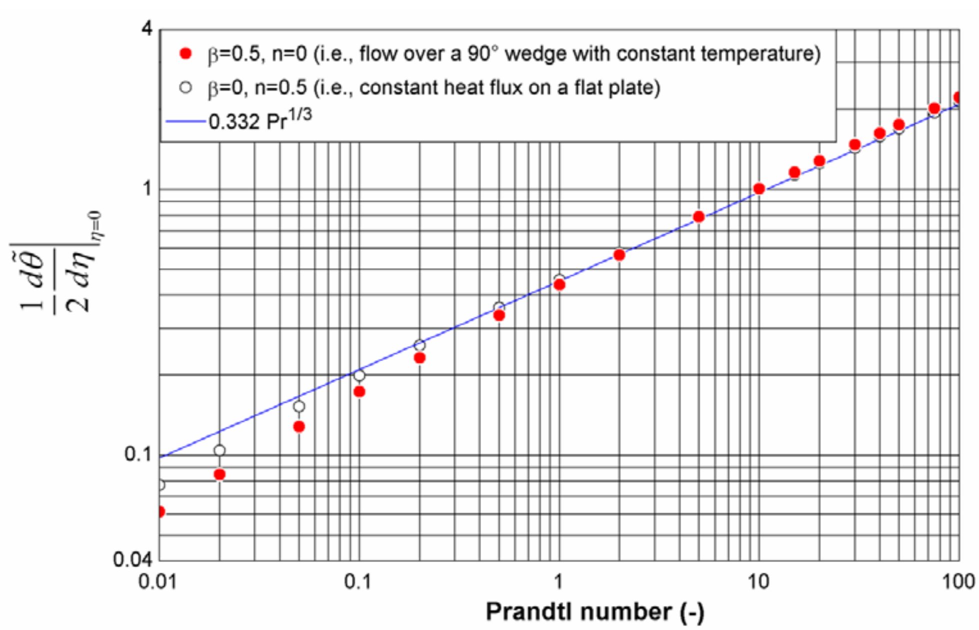


Figure E12-5: The quantity $\left.\frac{1}{2}\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$ as a function of Pr provided by the numerical solution for $\beta = 0$ and $n = 0.5$ (constant heat flux over a flat plate) and the curve fit to the solution provided by Eq. (E12-81). The solution for $n = 0$ and $\beta = 0.5$, which corresponds to flow over a 90° wedge with constant temperature is also shown.

Also shown in Figure E12-5 is the curve fit for $\left.\frac{1}{2}\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$ for the flat plate with a constant heat flux:

$$\left.\frac{1}{2}\frac{d\hat{\theta}}{d\eta}\right|_{\eta=0} = 0.453\,Pr^{1/3} \tag{E12-81}$$

which is reasonably accurate for $Pr > 0.7$.

We can use the solution to investigate the effect of varying free stream velocity and/or plate temperature on the heat flux and the temperature distribution. For example, Figure 4 also shows the quantity $\left.\frac{1}{2}\frac{d\hat{\theta}}{d\eta}\right|_{\eta=0}$ as a function of Pr for $\beta = 0.5$ and $n = 0$, which corresponds to flow over a 90° wedge with constant temperature.