The Reynolds Equation

The *x*-momentum equation for inertia-free flow, Eq. (5-116), provides the basis for the Reynolds equation. The Reynolds equation is used to analyze the flow of viscous fluids through small gaps (i.e., lubrication problems). For example, Figure 5-25 illustrates an inertia-free flow through the gap formed between a stationary surface (at $y = H$, where H may be a function of x and t) and a surface moving with velocity u_p (at $y = 0$).

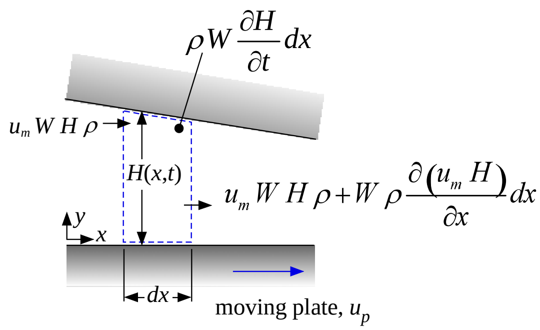

Figure 5-25: Inertia-free flow through the gap between a stationary and moving surface.

The height of the gap may be a function of position, *x*, and time, *t*, and therefore the pressure is a function of position and time as well. However, provided that the modified Reynolds number defined by Eq. (5-115) is small and the flow is quasi-steady, the inertia-free *x*-momentum equation, Eq. (5-116) can be applied. Quasi-steady implies that the characteristic time associated with any changes in the flow (e.g., oscillations of the gap or variations in the flow boundary conditions) is much larger than the characteristic time required for momentum to diffuse across the channel:

$$\tau_{diff} \approx \frac{H^2}{\upsilon} \tag{5-134}$$

Equation (5-116) is restated below:

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \tag{5-135}$$

and integrated twice in *y* in order to obtain:

$$u = \frac{\partial p}{\partial x} \frac{y^2}{2\,\mu} + C_1\, y + C_2 \tag{5-136}$$

The constants of integration for Eq. (5-136) are obtained by applying the no-slip condition at $y = 0$ and $y = H$. At $y = 0$, the fluid must reach the velocity of the moving surface:

$$u_{y=0} = u_p \rightarrow C_2 = u_p \tag{5-137}$$

At $y = H$, the fluid must be stationary:

$$u_{y=H} = 0 \rightarrow \frac{\partial p}{\partial x}\frac{H^2}{2\,\mu} + C_1\,H + u_p = 0 \tag{5-138}$$

Substituting Eqs. (5-137) and (5-138) into Eq. (5-122) leads to:

$$\boxed{u = \frac{1}{2\,\mu}\frac{\partial p}{\partial x}\left(y^2 - H\,y\right) + u_p\left(1 - \frac{y}{H}\right)} \tag{5-139}$$

The mean velocity is obtained by integrating the velocity distribution, Eq. (5-139), across the gap:

$$u_m = \frac{1}{H}\int_0^H u\,dy = \frac{1}{2\,\mu\,H}\frac{\partial p}{\partial x}\int_0^H \left(y^2 - H\,y\right)dy + \frac{u_p}{H}\int_0^H \left(1 - \frac{y}{H}\right)dy \tag{5-140}$$

which leads to:

$$\boxed{u_m = -\frac{\partial p}{\partial x}\frac{H^2}{12\,\mu} + \frac{u_p}{2}} \tag{5-141}$$

Equation (5-141) indicates that the mean velocity in the gap will increase with either the plate velocity or the pressure gradient.

A mass balance on a differential control volume (in *x*) is shown in Figure 5-25:

$$u_m\,W\,H\,\rho = u_m\,W\,H\,\rho + W\,\rho\frac{\partial}{\partial x}\left(u_m\,H\right)dx + \rho W\frac{\partial H}{\partial t}dx \tag{5-142}$$

where *W* is the width of the passage (into the page). Equation (5-142) is simplified to:

$$0 = \frac{\partial}{\partial x}\left(u_m\,H\right) + \frac{\partial H}{\partial t} \tag{5-143}$$

Equation (5-141) is substituted into Eq. (5-143) in order to obtain a partial differential equation for the pressure in terms of the gap height.

$$0 = \frac{\partial}{\partial x}\left(-\frac{\partial p}{\partial x}\frac{H^3}{12\,\mu} + \frac{u_p}{2}H\right) + \frac{\partial H}{\partial t} \tag{5-144}$$

or

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}H^{3}\right)=6\,\mu\,u_{p}\,\frac{\partial H}{\partial x}+12\,\mu\frac{\partial H}{\partial t}} \qquad (5\text{-}145)$$

Equation (5-145) is one form of the Reynolds equation for lubrication problems. Given an arbitrary description of the gap height as a function of position and time and boundary conditions for the pressure, it is possible to solve Eq. (5-145) either analytically or numerically in order to obtain the pressure variation. The pressure distribution coupled with Eq. (5-139) provides the velocity distribution.