6.3: Self-Similar Solution

6.3.1 Introduction

Section 6.1 discussed the basic concepts that underlie natural convection problems and, most importantly, lead to the appropriate scaling velocity and therefore dimensionless parameters for natural convection problems. In this section, the development of the self-similar solution for natural convection from a vertical, isothermal heated plate is presented using the methodology provided by Ostrach (1953) and discussed in Kays and Crawford (1993). The steps leading to a solution are similar to those presented in Section 4.4.2 in order to derive the self-similar solution for laminar forced convection flow over a flat plate. However, the definitions of the similarity variable and stream function are slightly different and the inclusion of the buoyancy force in the *x*-momentum equation complicates the solution.

6.3.2 Self-Similar Solution

Figure 6-20 illustrates, qualitatively, the velocity and temperature distributions that are expected adjacent to a vertical, isothermal, heated flat plate under laminar conditions.

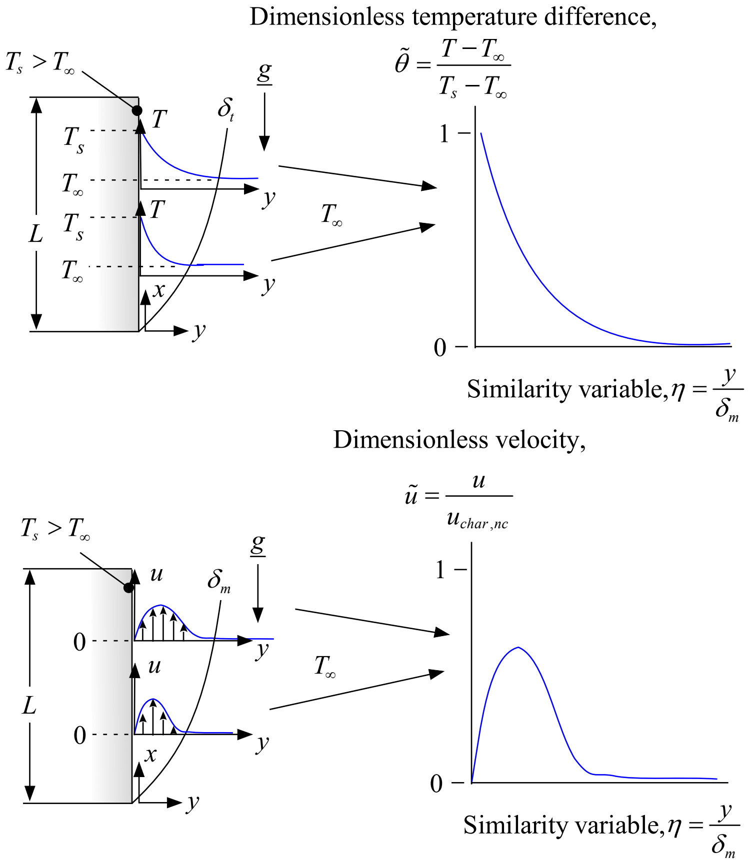

Figure 6-20: Laminar natural convection from a heated vertical plate.

The similarity solution is based on the idea that the temperature and velocity distributions at any position along the plate surface, *x*, will collapse if they are plotted in dimensionless form as a function of an appropriately defined similarity variable, $\eta = y/\delta_m$ (see Figure 6-20). Therefore, the partial differential equations that describe the problem in terms of x and y will collapse to ordinary differential equations in η for dimensionless velocity and dimensionless temperature.

The Problem Statement

The governing equations for the situation shown in Figure 6-20 are presented in Section 6.1.2. The continuity equation is:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{6-105}$$

The x-momentum equation in the boundary layer, including the buoyancy term represented using the Boussinesq approximation, is:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\,\beta\left(T-T_{\infty}\right)+\upsilon\frac{\partial^2 u}{\partial y^2} \tag{6-106}$$

Note that x is in the vertical direction, as shown in Figure 6-20. The thermal energy equation is:

$$\rho\, c\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=k\frac{\partial^2 T}{\partial y^2} \tag{6-107}$$

The boundary conditions include no-slip at the wall:

$$u_{y=0}=0 \tag{6-108}$$

$$v_{y=0}=0 \tag{6-109}$$

The plate temperature is specified:

$$T_{y=0}=T_s \tag{6-110}$$

As y becomes large, the fluid becomes stagnant:

$$u_{y\to\infty}=0 \tag{6-111}$$

and the ambient temperature is recovered:

$$T_{y\to\infty}=T_{\infty} \tag{6-112}$$

The Similarity Variables

The growth of the velocity and thermal boundary layers in a laminar flow occur primarily due to the molecular diffusion of momentum and energy. Therefore, the momentum boundary layer (δ_m) will grow approximately according to:

$$\delta_m \approx 2\sqrt{\upsilon\, t} \tag{6-113}$$

where υ is the kinematic viscosity and t is time, which is related to the distance from the leading edge (x) and the characteristic velocity for natural convection ($u_{char,nc}$) according to:

$$t = \frac{x}{u_{char,nc}} \tag{6-114}$$

Substituting Eq. (6-114) into Eq. (6-113) leads to:

$$\delta_m \approx 2\sqrt{\frac{\upsilon\, x}{u_{char,nc}}} \tag{6-115}$$

The local characteristic velocity follows from the discussion in Section 6.1:

$$u_{char,nc} = \sqrt{g\, x\, \beta\left(T_s - T_\infty\right)} \tag{6-116}$$

Substituting Eq. (6-116) into Eq. (6-115) leads to:

$$\delta_m \approx 2\sqrt{\frac{\upsilon\, x}{\sqrt{g\, x\, \beta\left(T_s - T_\infty\right)}}} = 2\left[\frac{\upsilon^2\, x}{g\, \beta\left(T_s - T_\infty\right)}\right]^{1/4} \tag{6-117}$$

The similarity variable is defined as the ratio of the distance from the plate surface (y) to the approximate thickness of the momentum boundary layer:

$$\eta = \frac{y}{\delta_m} \tag{6-118}$$

Substituting Eq. (6-117) into Eq. (6-118) leads to:

$$\eta = \frac{y}{2}\left[\frac{g\, \beta\left(T_s - T_\infty\right)}{\upsilon^2\, x}\right]^{1/4} = \frac{y}{2\, x}\left[\underbrace{\frac{g\, \beta\, x^3\left(T_s - T_\infty\right)}{\upsilon^2}}_{Gr_x}\right]^{1/4} \tag{6-119}$$

The second term in brackets in Eq. (6-119) is the local Grashof number, discussed in Section 6.1:

$$Gr_x = \frac{g\,\beta\,x^3\left(T_s - T_\infty\right)}{\upsilon^2} \tag{6-120}$$

Therefore, Eq. (6-119) can be written as:

$$\eta = \frac{y}{2\,x} Gr_x^{1/4} \tag{6-121}$$

Following the presentation of Ostrach (1953), the constant used to define the similarity parameter is adjusted slightly:

$$\boxed{\eta = \frac{y}{x}\left[\frac{Gr_x}{4}\right]^{1/4}} \tag{6-122}$$

It is anticipated that the dimensionless velocity and temperature difference:

$$\tilde{u} = \frac{u}{u_{char,nc}} \tag{6-123}$$

$$\tilde{\theta} = \frac{T - T_\infty}{T_s - T_\infty} \tag{6-124}$$

at any position x will collapse when expressed in terms of η, as shown in Figure 6-20:

$$\tilde{u} = \tilde{u}\left(x, y\right) = \tilde{u}\left(\eta\right) \tag{6-125}$$

$$\tilde{\theta} = \tilde{\theta}\left(x, y\right) = \tilde{\theta}\left(\eta\right) \tag{6-126}$$

The stream function (Ψ) was introduced in Section 4.4 as part of the development of the self-similar solution for forced convection laminar flow over a flat plate. The stream function is defined so that the continuity equation, Eq. (6-105), is automatically satisfied:

$$u = \left(\frac{\partial\,\Psi}{\partial\,y}\right)_x \tag{6-127}$$

$$v = -\left(\frac{\partial\,\Psi}{\partial\,x}\right)_y \tag{6-128}$$

The stream function is related to the volumetric flow between the surface of the plate and any position y according to:

$$\dot{V} = W\,\Psi \tag{6-129}$$

where W is the width of the plate (into the page). The volumetric flow rate is calculated from the velocity according to:

$$\dot{V} = W \int_0^y u\,dy \tag{6-130}$$

Equation (6-130) can be expressed in terms of the dimensionless variables ($\tilde{u}$ and η) using Eqs. (6-118) and (6-123):

$$\dot{V} = W\,u_{char,nc}\,\delta_m \int_0^\eta \tilde{u}\,d\eta \tag{6-131}$$

Substituting Eq. (6-129) into Eq. (6-131) leads to:

$$\Psi = u_{char,nc}\,\delta_m \underbrace{\int_0^\eta \tilde{u}(\eta)\,d\eta}_{f(\eta)} \tag{6-132}$$

The integral labeled $f(\eta)$ in Eq. (6-132) can be thought of as the dimensionless form of the stream function it and must be a function only of the similarity variable, η. Substituting Eqs. (6-116) and (6-117) into Eq. (6-132) leads to:

$$\Psi = \underbrace{\sqrt{g\,x\,\beta\left(T_s - T_\infty\right)}}_{u_{char,nc}}\;\underbrace{2\left[\frac{\upsilon^2\,x}{g\,\beta\left(T_s - T_\infty\right)}\right]^{1/4}}_{\delta_m} f(\eta) = 2\left[\upsilon^2\,g\,\beta\left(T_s - T_\infty\right)x\right]^{1/4} f(\eta) \tag{6-133}$$

Equation (6-133) can be simplified to:

$$\Psi = 2\left[\upsilon^4\,\underbrace{\frac{g\,\beta\left(T_s - T_\infty\right)x^3}{\upsilon^2}}_{Gr_x}\right]^{1/4} f(\eta) = 2\,\upsilon\,Gr_x^{1/4} f(\eta) \tag{6-134}$$

The constant in the definition of the stream function, Eq. (6-134), is also adjusted slightly in order to be consistent with the solution of Ostrach:

$$\boxed{\Psi = 4\upsilon\left(\frac{Gr_x}{4}\right)^{1/4} f\left(\eta\right)} \tag{6-135}$$

The Problem Transformation

The similarity variables must be substituted into both the governing x-momentum and thermal energy conservation equations as well as the boundary conditions for velocity and temperature in order to transform the two coupled partial differential equations into two coupled ordinary differential equations that can be solved more easily. (The continuity equation is automatically satisfied due to the use of a stream function.) Note that it is not possible to sequentially solve the x-momentum and then the thermal energy equations as was the case in Section 4.4, because of the fundamentally coupled nature of a natural convection problem.

The transformation process takes a problem that is stated in terms of x and y and re-states it in terms of x and η; the x-dependence drops out of the transformed problem (as it must if the similarity variable is well-defined). The mathematical manipulations are facilitated by expressing the parameter $Gr_x^{1/4}/4$, which appears in the equations for both η and Ψ, in terms of x rather than Gr_x:

$$\left(\frac{Gr_x}{4}\right)^{1/4} = \left(\frac{g\,\beta\left(T_s - T_\infty\right)x^3}{4\upsilon^2}\right)^{1/4} = B^{1/4}x^{3/4} \tag{6-136}$$

where

$$B = \frac{g\,\beta\left(T_s - T_\infty\right)}{4\upsilon^2} \tag{6-137}$$

Substituting Eq. (6-136) into Eqs. (6-122) and (6-135) allows the similarity variables to be written as:

$$\eta = y\,B^{1/4}\,x^{-1/4} \tag{6-138}$$

$$\Psi = 4\upsilon\,B^{1/4}\,x^{3/4}\,f\left(\eta\right) \tag{6-139}$$

The x-velocity is expressed in terms of the similarity variables by substituting Eq. (6-139) into Eq. (6-127):

$$u = \left(\frac{\partial\Psi}{\partial y}\right)_x = \frac{\partial}{\partial y}\left[4\upsilon\,B^{1/4}\,x^{3/4}\,f\left(\eta\right)\right]_x = 4\upsilon\,B^{1/4}\,x^{3/4}\underbrace{\frac{\partial}{\partial y}\left[f\left(\eta\right)\right]_x}_{\frac{df}{d\eta}\left(\frac{\partial\eta}{\partial y}\right)_x} \tag{6-140}$$

The partial derivative of η, Eq. (6-138), with respect to y at constant x is:

$$\left(\frac{\partial \eta}{\partial y}\right)_x = B^{1/4}\, x^{-1/4} \tag{6-141}$$

Substituting Eq. (6-141) into Eq. (6-140) leads to:

$$u = 4\upsilon\, B^{1/4}\, x^{3/4}\, \frac{df}{d\eta}\, B^{1/4}\, x^{-1/4} \tag{6-142}$$

or:

$$\boxed{u = 4\upsilon\, B^{1/2}\, x^{1/2}\, \frac{df}{d\eta}} \tag{6-143}$$

It is worth noting that the first derivative of f with respect to η is related to the dimensionless velocity, $\tilde{u}$, defined in Eq. (6-123). The definition of B, Eq. (6-137), is substituted into Eq. (6-143):

$$u = 4\upsilon\left[\frac{g\,\beta\left(T_s - T_\infty\right)}{4\upsilon^2}\right]^{1/2} x^{1/2}\, \frac{df}{d\eta} = 2\underbrace{\sqrt{g\,\beta\, x\left(T_s - T_\infty\right)}}_{u_{char,nc}}\, \frac{df}{d\eta} \tag{6-144}$$

Solving for $\frac{df}{d\eta}$:

$$\frac{df}{d\eta} = \frac{u}{2\,u_{char,nc}} = \frac{\tilde{u}}{2} \tag{6-145}$$

This result is consistent with the Blasius solution, where $\frac{df}{d\eta}$ was also proportional to the dimensionless velocity.

The y-velocity is obtained by substituting Eq. (6-139) into Eq. (6-128):

$$v = -\frac{\partial}{\partial x}\left[4\upsilon\, B^{1/4}\, x^{3/4}\, f\left(\eta\left(x,y\right)\right)\right]_y \tag{6-146}$$

It is important to recognize that η is a function of x and y and, as a result, f is itself a function of x. Therefore:

$$v = -f \frac{\partial}{\partial x}\left[4\upsilon B^{1/4} x^{3/4}\right]_y - 4\upsilon B^{1/4} x^{3/4} \frac{df}{d\eta}\left(\frac{\partial \eta}{\partial x}\right)_y \tag{6-147}$$

Substituting Eq. (6-138) into Eq. (6-147) leads to:

$$v = -f \frac{\partial}{\partial x}\left[4\upsilon B^{1/4} x^{3/4}\right]_y - 4\upsilon B^{1/4} x^{3/4} \frac{df}{d\eta}\frac{\partial}{\partial x}\left[y B^{1/4} x^{-1/4}\right]_y \tag{6-148}$$

or

$$v = -f 3\upsilon B^{1/4} x^{-1/4} + \upsilon B^{1/4} x^{3/4} \frac{df}{d\eta} y B^{1/4} x^{-5/4} \tag{6-149}$$

which can be rewritten as:

$$\boxed{v = -f 3\upsilon B^{1/4} x^{-1/4} + \upsilon B^{1/2} x^{-1/2} \frac{df}{d\eta} y} \tag{6-150}$$

In addition to *u* and *v*, we will need the partial derivatives of *u* (see Eq. (6-106)). The partial derivative of *u* with respect to *x* is:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\underbrace{4\upsilon B^{1/2} x^{1/2}\frac{df}{d\eta}}_{u}\right]_y = \frac{df}{d\eta}\frac{\partial}{\partial x}\left[4\upsilon B^{1/2} x^{1/2}\right]_y + 4\upsilon B^{1/2} x^{1/2}\frac{d^2 f}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)_y \tag{6-151}$$

Substituting Eq. (6-138) into Eq. (6-151) leads to:

$$\frac{\partial u}{\partial x} = \frac{df}{d\eta}\frac{\partial}{\partial x}\left[4\upsilon B^{1/2} x^{1/2}\right]_y + 4\upsilon B^{1/2} x^{1/2}\frac{d^2 f}{d\eta^2}\frac{\partial}{\partial x}\left(y B^{1/4} x^{-1/4}\right)_y \tag{6-152}$$

or

$$\frac{\partial u}{\partial x} = \frac{df}{d\eta} 2\upsilon B^{1/2} x^{-1/2} - \upsilon B^{1/2} x^{1/2}\frac{d^2 f}{d\eta^2} y B^{1/4} x^{-5/4} \tag{6-153}$$

which can be rewritten as:

$$\boxed{\frac{\partial u}{\partial x} = \frac{df}{d\eta} 2\upsilon B^{1/2} x^{-1/2} - \upsilon B^{3/4} x^{-3/4}\frac{d^2 f}{d\eta^2} y} \tag{6-154}$$

The partial derivative of *u* with respect to *y* is:

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{\partial}{\partial y}\left[4\upsilon B^{1/2}x^{1/2}\frac{df}{d\eta}\right] = 4\upsilon B^{1/2}x^{1/2}\frac{d^2 f}{d\eta^2}\left(\frac{\partial \eta}{\partial y}\right)_x \tag{6-155}$$

Substituting Eq. (6-138) into Eq. (6-155) leads to:

$$\left(\frac{\partial u}{\partial y}\right)_x = 4\upsilon B^{1/2}x^{1/2}\frac{d^2 f}{d\eta^2}\frac{\partial}{\partial y}\left[y B^{1/4}x^{-1/4}\right]_x \tag{6-156}$$

or

$$\left(\frac{\partial u}{\partial y}\right)_x = 4\upsilon B^{1/2}x^{1/2}\frac{d^2 f}{d\eta^2}B^{1/4}x^{-1/4} \tag{6-157}$$

which can be rewritten as:

$$\boxed{\left(\frac{\partial u}{\partial y}\right)_x = 4\upsilon B^{3/4}x^{1/4}\frac{d^2 f}{d\eta^2}} \tag{6-158}$$

Finally, the second derivative of *u* with respect to *y* is:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(4\upsilon B^{3/4}x^{1/4}\frac{d^2 f}{d\eta^2}\right) = 4\upsilon B^{3/4}x^{1/4}\frac{d^3 f}{d\eta^3}\left(\frac{\partial \eta}{\partial y}\right)_x \tag{6-159}$$

Substituting Eq. (6-138) into Eq. (6-159) leads to:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(4\upsilon B^{3/4}x^{1/4}\frac{d^2 f}{d\eta^2}\right) = 4\upsilon B^{3/4}x^{1/4}\frac{d^3 f}{d\eta^3}\frac{\partial}{\partial y}\left(y B^{1/4}x^{-1/4}\right)_x \tag{6-160}$$

or

$$\frac{\partial^2 u}{\partial y^2} = 4\upsilon B^{3/4}x^{1/4}\frac{d^3 f}{d\eta^3}B^{1/4}x^{-1/4} \tag{6-161}$$

which can be rewritten as:

$$\boxed{\frac{\partial^2 u}{\partial y^2} = 4\upsilon B\frac{d^3 f}{d\eta^3}} \tag{6-162}$$

Substituting Eqs. (6-124), (6-143), (6-150), (6-154), (6-158), and (6-162) into the x-momentum equation, Eq. (6-106), leads to:

$$\begin{gathered}
\underbrace{4\upsilon B^{1/2} x^{1/2}\frac{df}{d\eta}}_{u}\underbrace{\left[\frac{df}{d\eta}2\upsilon B^{1/2}x^{-1/2}-\upsilon B^{3/4}x^{-3/4}\frac{d^2f}{d\eta^2}y\right]}_{\frac{\partial u}{\partial x}}+\\
\underbrace{\left[-f3\upsilon B^{1/4}x^{-1/4}+\upsilon B^{1/2}x^{-1/2}\frac{df}{d\eta}y\right]}_{v}\underbrace{4\upsilon B^{3/4}x^{1/4}\frac{d^2f}{d\eta^2}}_{\frac{\partial u}{\partial y}}=\\
g\,\beta\underbrace{\left(T_s-T_\infty\right)\tilde{\theta}}_{(T-T_\infty)}+\upsilon\underbrace{4\upsilon B\frac{d^3f}{d\eta^3}}_{\frac{\partial^2 u}{\partial y^2}}
\end{gathered}\tag{6-163}$$

Expanding the terms and substituting the definition of B, Eq. (6-137), into the buoyancy term leads to:

$$\begin{gathered}
4\upsilon B^{1/2}x^{1/2}\frac{df}{d\eta}\frac{df}{d\eta}2\upsilon B^{1/2}x^{-1/2}-4\upsilon B^{1/2}x^{1/2}\frac{df}{d\eta}\upsilon B^{3/4}x^{-3/4}\frac{d^2f}{d\eta^2}y\\
-f3\upsilon B^{1/4}x^{-1/4}4\upsilon B^{3/4}x^{1/4}\frac{d^2f}{d\eta^2}+\upsilon B^{1/2}x^{-1/2}\frac{df}{d\eta}y4\upsilon B^{3/4}x^{1/4}\frac{d^2f}{d\eta^2}=\\
4\upsilon^2B\tilde{\theta}+4\upsilon^2B\frac{d^3f}{d\eta^3}
\end{gathered}\tag{6-164}$$

Dividing Eq. (6-164) through by $4\upsilon^2 B$ leads to:

$$\boxed{\frac{d^3f}{d\eta^3}+3f\frac{d^2f}{d\eta^2}-2\left(\frac{df}{d\eta}\right)^2+\tilde{\theta}=0}\tag{6-165}$$

Equation (6-165) shows that the differential equation for f is coupled to the differential equation for $\tilde{\theta}$, which must be derived from the thermal energy equation, Eq. (6-107).

The first derivative of $\tilde{\theta}$ with respect to x is obtained using the chain rule, recognizing that $\tilde{\theta}$ is a function only of η, which is a function of x and y:

$$\frac{\partial}{\partial x}\left[\tilde{\theta}\left(\eta\left(x,y\right)\right)\right]=\frac{d\tilde{\theta}}{d\eta}\left(\frac{\partial\eta}{\partial x}\right)_y\tag{6-166}$$

Substituting Eq. (6-138) into Eq. (6-166) leads to:

$$\frac{\partial \tilde{\theta}}{\partial x} = \frac{d\tilde{\theta}}{d\eta}\frac{\partial}{\partial x}\left[y\,B^{1/4}\,x^{-1/4}\right]_y \tag{6-167}$$

which can be written as:

$$\boxed{\frac{\partial \tilde{\theta}}{\partial x} = -\frac{1}{4}\frac{d\tilde{\theta}}{d\eta}\,y\,B^{1/4}\,x^{-5/4}} \tag{6-168}$$

The first derivative of $\tilde{\theta}$ with respect to *y* is obtained in a similar manner:

$$\frac{\partial \tilde{\theta}}{\partial y} = \frac{d\tilde{\theta}}{d\eta}\left(\frac{\partial \eta}{\partial y}\right)_x \tag{6-169}$$

or

$$\boxed{\frac{\partial \tilde{\theta}}{\partial y} = \frac{d\tilde{\theta}}{d\eta}B^{1/4}\,x^{-1/4}} \tag{6-170}$$

The second derivative of $\tilde{\theta}$ with respect to *y* is:

$$\frac{\partial^2 \tilde{\theta}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \tilde{\theta}}{\partial y}\right)_x = \frac{\partial}{\partial \eta}\left[\frac{\partial \tilde{\theta}}{\partial y}\right]\left(\frac{\partial \eta}{\partial y}\right)_x \tag{6-171}$$

Substituting Eq. (6-170) into Eq. (6-171) leads to:

$$\frac{\partial^2 \tilde{\theta}}{\partial y^2} = \frac{\partial}{\partial \eta}\left[\frac{d\tilde{\theta}}{d\eta}B^{1/4}\,x^{-1/4}\right]\left(\frac{\partial \eta}{\partial y}\right)_x \tag{6-172}$$

or

$$\boxed{\frac{\partial^2 \tilde{\theta}}{\partial y^2} = \frac{d^2\tilde{\theta}}{d\eta^2}B^{1/2}\,x^{-1/2}} \tag{6-173}$$

Subsituting the definition of $\tilde{\theta}$, Eq. (6-124), into the thermal energy equation, Eq. (6-107), leads to:

$$u\frac{\partial \tilde{\theta}}{\partial x} + v\frac{\partial \tilde{\theta}}{\partial y} = \alpha\frac{\partial^2 \tilde{\theta}}{\partial y^2} \tag{6-174}$$

Substituting Eqs. (6-143), (6-168), (6-150), (6-170), and (6-173) into Eq. (6-174) leads to:

$$\begin{gathered}\underbrace{4\upsilon B^{1/2} x^{1/2} \frac{df}{d\eta}}_{u} \underbrace{\left[-\frac{1}{4}\frac{d\tilde{\theta}}{d\eta} y B^{1/4} x^{-5/4}\right]}_{\frac{\partial\tilde{\theta}}{\partial x}} + \\ \underbrace{\left[-f 3\upsilon B^{1/4} x^{-1/4} + \upsilon B^{1/2} x^{-1/2}\frac{df}{d\eta} y\right]}_{v} \underbrace{\frac{d\tilde{\theta}}{d\eta} B^{1/4} x^{-1/4}}_{\frac{\partial\tilde{\theta}}{\partial y}} = \alpha \underbrace{\frac{d^2\tilde{\theta}}{d\eta^2} B^{1/2} x^{-1/2}}_{\frac{\partial^2\tilde{\theta}}{\partial y^2}}\end{gathered} \tag{6-175}$$

Expanding the terms in Eq. (6-175) leads to:

$$\begin{gathered}-\upsilon B^{1/2} x^{1/2} \frac{df}{d\eta}\frac{d\tilde{\theta}}{d\eta} y B^{1/4} x^{-5/4} - f 3\upsilon B^{1/4} x^{-1/4}\frac{d\tilde{\theta}}{d\eta} B^{1/4} x^{-1/4} + \\ \upsilon B^{1/2} x^{-1/2}\frac{df}{d\eta} y \frac{d\tilde{\theta}}{d\eta} B^{1/4} x^{-1/4} = \alpha \frac{d^2\tilde{\theta}}{d\eta^2} B^{1/2} x^{-1/2}\end{gathered} \tag{6-176}$$

which leads to the ordinary differential equation for $\tilde{\theta}$:

$$\boxed{\frac{d^2\tilde{\theta}}{d\eta^2} + 3 f \, Pr \frac{d\tilde{\theta}}{d\eta} = 0} \tag{6-177}$$

The boundary conditions associated with the ordinary differential equations for *f* and $\tilde{\theta}$ must be obtained by transforming the boundary conditions. Equation (6-143) is substituted into Eq. (6-108):

$$u_{y=0} = 4\upsilon B^{1/2} x^{1/2} \left.\frac{df}{d\eta}\right|_{\eta=0} = 0 \tag{6-178}$$

or

$$\boxed{\left.\frac{df}{d\eta}\right|_{\eta=0} = 0} \tag{6-179}$$

Equation (6-143) is substituted into Eq. (6-111):

$$u_{y\to\infty} = 4\upsilon B^{1/2} x^{1/2} \left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 0 \tag{6-180}$$

or

$$\boxed{\left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 0} \tag{6-181}$$

Equation (6-150) is substituted into Eq. (6-109):

$$v_{y=0} = -f\,3\upsilon B^{1/4} x^{-1/4} + \underbrace{\upsilon B^{1/2} x^{-1/2} \frac{df}{d\eta} y}_{=0} = 0 \tag{6-182}$$

which can only be true if:

$$\boxed{f_{\eta=0} = 0} \tag{6-183}$$

Equation (6-124) is substituted into Eqs. (6-110) and (6-112):

$$\boxed{\tilde{\theta}_{\eta=0} = 1} \tag{6-184}$$

$$\boxed{\tilde{\theta}_{\eta\to\infty} = 0} \tag{6-185}$$

Equations (6-165) and (6-177) together with Eqs. (6-179), (6-181), (6-183), (6-184), and (6-185) represent coupled third order and second order ordinary differential equations with the five required boundary conditions.

Numerical Solution

The numerical solution of the problem is more complicated than it would appear and therefore also quite interesting. The numerical solution is implemented in MATLAB using the `fminsearch` command for multivariable optimization together with an explicit integration technique.

The five state variables are f, $\frac{df}{d\eta}$, $\frac{d^2 f}{d\eta^2}$, $\tilde{\theta}$, and $\frac{d\tilde{\theta}}{d\eta}$. The state equations provide the rate of change of these state variables:

$$\frac{d}{d\eta}\left[f\right] = \frac{df}{d\eta} \tag{6-186}$$

$$\frac{d}{d\eta}\left[\frac{df}{d\eta}\right] = \frac{d^2 f}{d\eta^2} \tag{6-187}$$

$$\frac{d^3 f}{d\eta^3} = -3 f \frac{d^2 f}{d\eta^2} + 2\left(\frac{df}{d\eta}\right)^2 - \tilde{\theta} \tag{6-188}$$

$$\frac{d}{d\eta}\left[\tilde{\theta}\right] = \frac{d\tilde{\theta}}{d\eta} \tag{6-189}$$

$$\frac{d^2\tilde{\theta}}{d\eta^2} = -3 f \, Pr \frac{d\tilde{\theta}}{d\eta} \tag{6-190}$$

Three of the boundary conditions for the state variables are specified at $\eta = 0$ while the remaining two are specified at $\eta \to \infty$; therefore, the shooting method is more complex than it was in Section 4.4 because two boundary conditions must be assumed and then adjusted at $\eta = 0$.

The Euler technique is implemented in MATLAB in order to integrate from $\eta = 0$ to a position far from the wall. Recall that η is defined as the ratio of the distance from the plate to the thickness of the boundary layer. Because the temperature rise in the fluid drives the velocity, it is reasonable to expect that the momentum and thermal boundary layers will have similar thickness unless the Prandtl number is very different than unity. Therefore, it should be reasonable to terminate the integration at a value of η that is much larger than unity, for example at $\eta_\infty = 10$, and enforce the boundary conditions given by Eqs. (6-181) and (6-185) at η_∞.

The computational domain ($0 < \eta < \eta_\infty$) is divided into steps of size $\Delta\eta$:

$$\Delta\eta = \frac{\eta_\infty}{(N-1)} \tag{6-191}$$

where N is the number of nodes. The location of each node is provided by:

$$\eta_i = \eta_\infty \frac{(i-1)}{(N-1)} \quad \text{for } i = 1..N \tag{6-192}$$

The solution is implemented in the MATLAB script `NC`.

```
clear all;
Pr=1;                                   % Prandtl number
n_infinity=10;                          % outer edge of computational domain
N=5001;                                 % number of steps in the numerical integration
Dn=n_infinity/(N-1);                    % size of the integration steps
for i=1:N
    n(i)=n_infinity*(i-1)/(N-1);        % location of integration steps
end
```

The initial conditions for the integration process are specified; note that the values of $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$ and $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$ (the variables `d2fdn2_0` and `dqdn_0`) are assumed and must be adjusted to complete the problem.

```
d2fdn2_0=0.5;                               % 2nd derivative of f at the wall
dqdn_0=-0.5;                                % 1st derivative of q at the wall

%boundary conditions at the wall
f(1)=0;
dfdn(1)=0;
d2fdn2(1)=d2fdn2_0;
q(1)=1;
dqdn(1)=dqdn_0;
```

Each integration step is taken using Euler's method; the rates of change of each state variable are evaluated at the beginning of the integration step using Eqs. (6-186) through (6-190):

$$f_{i+1} = f_i + \left.\frac{df}{d\eta}\right|_i \Delta\eta \tag{6-193}$$

$$\left.\frac{df}{d\eta}\right|_{i+1} = \left.\frac{df}{d\eta}\right|_i + \left.\frac{d^2 f}{d\eta^2}\right|_i \Delta\eta \tag{6-194}$$

$$\left.\frac{d^2 f}{d\eta^2}\right|_{i+1} = \left.\frac{d^2 f}{d\eta^2}\right|_i + \left[-3\, f_i \left.\frac{d^2 f}{d\eta^2}\right|_i + 2\left(\left.\frac{df}{d\eta}\right|_i\right)^2 - \tilde{\theta}_i\right]\Delta\eta \tag{6-195}$$

$$\tilde{\theta}_{i+1} = \tilde{\theta}_i + \left.\frac{d\tilde{\theta}}{d\eta}\right|_i \Delta\eta \tag{6-196}$$

$$\left.\frac{d\tilde{\theta}}{d\eta}\right|_{i+1} = \left.\frac{d\tilde{\theta}}{d\eta}\right|_i + \left[-3\, f_i\, Pr \left.\frac{d\tilde{\theta}}{d\eta}\right|_i\right]\Delta\eta \tag{6-197}$$

```
for i=1:(N-1)
    f(i+1)=f(i)+dfdn(i)*Dn;
    dfdn(i+1)=dfdn(i)+d2fdn2(i)*Dn;
    d2fdn2(i+1)=d2fdn2(i)+(-3*f(i)*d2fdn2(i)+2*(dfdn(i))^2-q(i))*Dn;
    q(i+1)=q(i)+dqdn(i)*Dn;
    dqdn(i+1)=dqdn(i)-3*Pr*f(i)*dqdn(i)*Dn;
end
```

Figure 6-21 illustrates $\tilde{\theta}$ and $\frac{df}{d\eta}$ as a function of η. Note that neither of the two boundary conditions at $\eta \to \infty$ are satisfied and, in fact, the solution diverges with η even for the reasonable assumed values of $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$ and $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$.

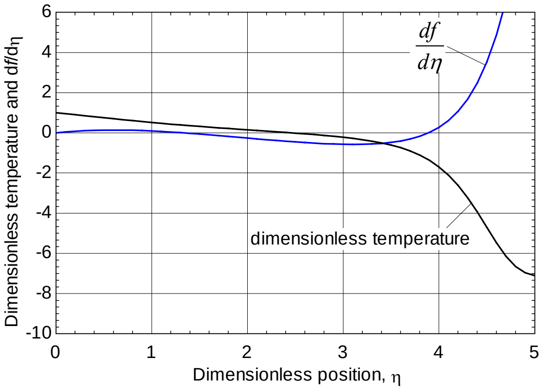

Figure 6-21: Dimensionless temperature and $\frac{df}{d\eta}$ as a function of η for *Pr* = 1.0 with $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} = 0.5$ and $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0} = -0.5$.

An error is calculated based on difference between the values of $\tilde{\theta}$ and $\frac{df}{d\eta}$ at η_∞ and their required values, based on Eqs. (6-181) and (6-185) (i.e., zero).

```
err=sqrt(dfdn(N)^2+q(N)^2);       % error in boundary conditions at n_infinity
```

The next step in the solution is to adjust $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$ and $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$ so that $\left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 0$ and $\tilde{\theta}_{\eta\to\infty} = 0$; unfortunately, the problem is sufficiently "stiff" (i.e., the values of the outputs are very sensitive to the values of the inputs) so that this is quite difficult to accomplish. Notice that the solution in Figure 6-21 grows rapidly as η becomes large. In fact, the solution for `dfdn` eventually becomes larger than MATLAB can represent and therefore the final entries in the vector `dfdn` are `NaN`, which is MATLAB shorthand for "not a number". One of the problems with this solution is that a lot of computational time is wasted finishing the numerical integration for a set of boundary conditions that are clearly not viable. Anytime the dimensionless temperature becomes less than 0 or greater than 1 or the dimensionless velocity becomes less than 0 or substantially greater than 1 then it is clear that the assumed values of $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$ and $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$ are not appropriate and therefore the integration should be terminated. For example, the integration shown in Figure 6-21 should have been terminated at about η = 1.3 when $\frac{df}{d\eta}$ became negative. The integration can be terminated if the velocity or temperature go out of reasonable bounds using the `break` command. The `break` command terminates the execution of a `for` loop in MATLAB. If the integration is terminated, then the value of $\tilde{\theta}$ and $\frac{df}{d\eta}$ at η_∞ are assigned to a large value that is inversely proportional to the point at which the integration went out of bounds.

```
    if((q(i+1)>1)|(q(i+1)<0)|(f(i+1)>2)|(f(i+1)<0))
        q(N)=1+1/i;
        dfdn(N)=1+1/i;
        break;
    end
```

Implementing the solution in MATLAB allows the use of the sophisticated multi-dimensional optimization functions that are native to MATLAB. In order to use these functions, it is necessary to convert the script `NC` into a function. The function takes as input arguments a vector `X` that includes the two unknown boundary conditions at η = 0 as well as the Prandtl number and returns as the first output the scalar argument `err`, which is related to how well the boundary conditions at η_∞ are met. The three additional outputs are the vectors `n`, `dfdn`, and `q`.

```
function[err,n,dfdn,q]=NC(X,Pr)

% Inputs:
% X - vector of unknown boundary conditions at n=0
%    X(1) - d2fdn2 at n=0 (-)
%    X(2) - dqdn at n=0 (-)
% Pr - Prandtl number (-)
%
% Outputs:
% err - error associated with mismatch in bc's at n_infinity (-)
% n - vector of nodal positions (-)
% dfdn - dfdn at each nodal position (-)
% q - q at each nodal position (-)

    Pr=1;                               % Prandtl number
    n_infinity=10;                      % outer edge of computational domain
    N=5001;                             % number of nodes in the integration
    Dn=n_infinity/(N-1);                % size of the integration steps
    for i=1:N
        n(i)=n_infinity*(i-1)/(N-1);    % location of integration steps
    end
    d2fdn2_0=X(1);                      % 2nd derivative of f at the wall
    dqdn_0=X(2);                        % 1st derivative of q at the wall

    % boundary conditions at the wall
    f(1)=0;
    dfdn(1)=0;
    d2fdn2(1)=d2fdn2_0;
    q(1)=1;
    dqdn(1)=dqdn_0;
    for i=1:(N-1)
        f(i+1)=f(i)+dfdn(i)*Dn;
        dfdn(i+1)=dfdn(i)+d2fdn2(i)*Dn;
```

```
        d2fdn2(i+1)=d2fdn2(i)+(-3*f(i)*d2fdn2(i)+2*(dfdn(i))^2-q(i))*Dn;
        q(i+1)=q(i)+dqdn(i)*Dn;
        dqdn(i+1)=dqdn(i)-3*Pr*f(i)*dqdn(i)*Dn;
        if((q(i+1)>1)|(q(i+1)<0)|(f(i+1)>2)|(f(i+1)<0))
            q(N)=1+1/i;
        dfdn(N)=1+1/i;
        break;
        end
    end
    err=sqrt(dfdn(N)^2+q(N)^2);
end
```

The function `fminsearch` in MATLAB allows multidimensional, unconstrained optimization using the Nelder-Mead algorithm. The calling protocol for `fminsearch` is:

```
X=fminsearch(fun,X0,OPTIONS)
```

where `fun` is a handle for the function to be minimized. This function must accept input `X` that can be a scalar or a vector of inputs. The vector `X0` contains the starting values of these inputs and is used to initiate the minimization and `OPTIONS` is a vector of options that can be used to control the optimization process.

A script called `SelfSimilar` is generated to control the optimization. The variable space is cleared and the Prandtl number set:

```
clear all;
Pr=1;
```

Reasonable values are used to setup `X0`:

```
dfdn_0=0.5;
dqdn_0=-0.5;
X0=[dfdn_0,dqdn_0];
```

The `OPTIONS` vector is set up using the `optimset` command; the `optimset` command is used in the same way that the `odeset` command was used to set up the `OPTIONS` vector for use with MATLAB's native ode solvers, as discussed in Section 3.2.2. Here, the `optimset` command is used to specify that the result of each iteration should be displayed and to set both termination tolerances to 1x10^{-6}.

```
OPTIONS=optimset('Display','iter','TolFun',1e-6,'TolX',1e-6);
```

The `fminsearch` command is called; notice that the function call is mapped onto a call of the function `NC`, as discussed in Section 3.2.2 and elsewhere.

```
X = fminsearch(@(X) NC(X,Pr),X0,OPTIONS)
```

The `NC` function is run again using the boundary conditions identified by the optimization:

```
[err,n,dfdn,q]=NC(X,Pr);
```

The script `SelfSimilar` is run from the command line:

```
>> Selfsimilar
 Iteration   Func-count     min f(x)         Procedure
     0            1          1.41554
     1            3          1.41542        initial simplex
     2            5          1.41537        expand
     3            7          1.41508        expand
     4            8          1.41508        reflect
     5           10          0.325811         expand
     6           11          0.325811         reflect
     7           13          0.325811         contract outside
...
    77          146     3.92452e-007         contract inside
    78          147     3.92452e-007         reflect
    79          148     3.92452e-007         reflect
    80          150     3.92452e-007         contract inside

Optimization terminated:
 the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-006
 and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-006


X =

    0.6421   -0.5668
```

The progress of the minimization algorithm is reported after each iteration and the optimization will identify the appropriate boundary conditions, $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} = 0.6421$ and $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0} \approx -0.5668$. Figure 6-22 illustrates the dimensionless temperature and $\frac{df}{d\eta}$ (which is proportional to the dimensionless velocity according to Eq. (6-145)) as a function of η for these boundary conditions.

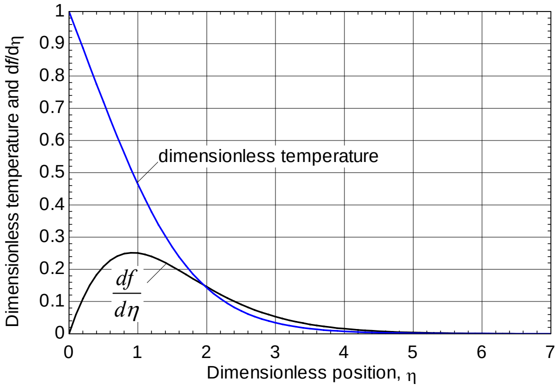

Figure 6-22: Dimensionless temperature and $\frac{df}{d\eta}$ as a function of η for Pr = 0.7 with $\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0}$ = 0.6419 and $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$ = -0.5668.

The Nusselt number is related to $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$. The local Nusselt number is defined as:

$$Nu_x = \frac{h\,x}{k} = \frac{\dot{q}_s''\,x}{\left(T_s - T_\infty\right)k} \tag{6-198}$$

The heat flux is given by:

$$\dot{q}_s'' = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{6-199}$$

Substituting the definition for dimensionless temperature into Eq. (6-199) leads to:

$$\dot{q}_s'' = -k\left(T_s - T_\infty\right)\left(\frac{\partial \tilde{\theta}}{\partial y}\right)_{y=0} \tag{6-200}$$

Substituting Eq. (6-170) into Eq. (6-200) leads to:

$$\dot{q}_s'' = -k\left(T_s - T_\infty\right)\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0} B^{1/4}\,x^{-1/4} \tag{6-201}$$

Substituting Eq. (6-137) into Eq. (6-201) leads to:

$$\dot{q}_s'' = -k\left(T_s - T_\infty\right)\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0}\left[\frac{g\,\beta\left(T_s - T_\infty\right)}{4\,\upsilon^2}\right]^{1/4} x^{-1/4} \tag{6-202}$$

Substituting Eq. (6-202) into Eq. (6-198) leads to:

$$Nu_x = -k\left(T_s - T_\infty\right)\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0}\left[\frac{g\,\beta\left(T_s - T_\infty\right)}{4\upsilon^2}\right]^{1/4} x^{-1/4}\frac{x}{\left(T_s - T_\infty\right)k} \tag{6-203}$$

which can be simplified to:

$$Nu_x = -\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0}\left(\frac{Gr_x}{4}\right)^{1/4} \tag{6-204}$$

The average Nusselt number is defined as:

$$\overline{h}_L = \frac{1}{L}\int_0^L h\,dx = \frac{1}{L}\int_0^L \frac{Nu_x\,k}{x}\,dx \tag{6-205}$$

Substituting Eq. (6-204) into Eq. (6-205) leads to:

$$\overline{h}_L = \frac{1}{L}\int_0^L -\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0}\left(\frac{Gr_x}{4}\right)^{1/4}\frac{k}{x}\,dx \tag{6-206}$$

Substituting the definition of the Grashof number into Eq. (6-206) leads to:

$$\overline{h}_L = -\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0}\frac{k}{L}\left(\frac{g\,\beta\left(T_s - T_\infty\right)}{4\upsilon^2}\right)^{1/4}\int_0^L x^{-1/4}\,dx \tag{6-207}$$

Carrying out the integration leads to:

$$\overline{h}_L = -\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0}\frac{k}{L}\left(\frac{g\,\beta\left(T_s - T_\infty\right)}{4\upsilon^2}\right)^{1/4}\frac{4}{3}L^{3/4} \tag{6-208}$$

Therefore, the average Nusselt number is given by:

$$\overline{Nu}_L = \frac{\overline{h}_L\,L}{k} = -\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0}\left(\underbrace{\frac{g\,\beta\left(T_s - T_\infty\right)L^3}{4\upsilon^2}}_{Gr_L/4}\right)^{1/4}\frac{4}{3} \tag{6-209}$$

or

$$\overline{Nu}_L = -\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0} \frac{4}{3}\left(\frac{Gr_L}{4}\right)^{1/4} \tag{6-210}$$

Equation (6-210) shows that the average Nusselt number is a function of the Grashof number and the Prandtl number through the parameter $\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$. The MATLAB script `SelfSimilar` is modified to identify the value of this parameter over a range of Prandtl number:

```
clear all;
Pr=[0.1,0.2,0.3,0.5,0.7,1,1.5,2,3,4,5,8,10,15,20,30,40,50,70,100,200,300,500,700,1000];
dfdn_0=0.7;
dqdn_0=-0.5;
for i=1:25
   X0=[dfdn_0,dqdn_0];
   OPTIONS=optimset('Display','iter','TolFun',1e-6,'TolX',1e-6);
   X = fminsearch(@(X) NC(X,Pr(i)),X0, OPTIONS)
   dfdn_0=X(1);
   dqdn_0=X(2);
   dqdn_0_v(i)=dqdn_0;
end
```

Notice that the results of each calculation are used as the starting point (i.e., the vector `X0`) for the next iteration in order to save computational time. Figure 6-23 illustrates the value of $\left(-\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}\right)$ as a function of *Pr*. Also shown in Figure 6-23 is the correlation provided by LeFevre (1956):

$$-\left(\frac{d\tilde{\theta}}{d\eta}\right)_{\eta=0} = \frac{0.75\sqrt{Pr}}{\left(0.609+1.221\sqrt{Pr}+1.238\,Pr\right)^{1/4}} \tag{6-211}$$

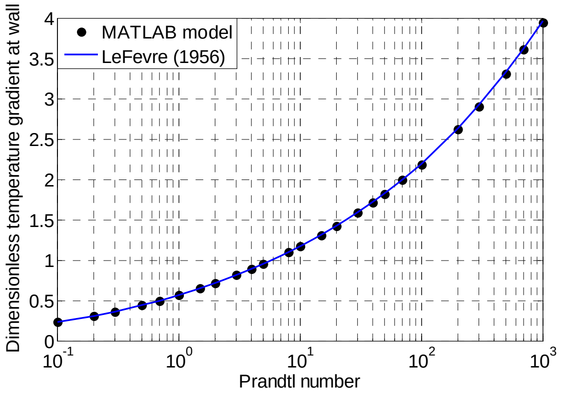

Figure 6-23: Value of $-\left.\frac{d\tilde{\theta}}{d\eta}\right|_{\eta=0}$ as a function of *Pr*; also shown is the correlation provided by LeFevre (1956).

The correlation for natural convection from a vertical plate that is presented in Section 6.2.2 and used for the EES function FC_Plate_vertical_ND is not the same as the self-similar solution presented in this section. However, the results provided by the correlation are consistent with the solution derived in this section. Figure 6-24 illustrates the average Nusselt number as a function of Grashof number for various values of the Prandtl number calculated using Eq. (6-210) together with the results obtained with the numerical model are shown in Figure 6-23. Also shown in Figure 6-23 is the average Nusselt number provided by the EES function FC_Plate_vertical_ND.

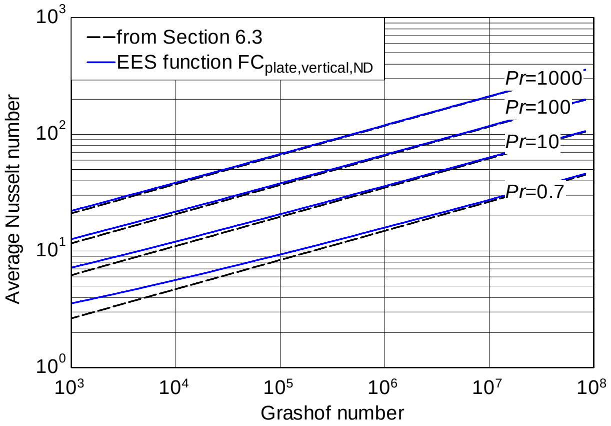

Figure 6-24: Average Nusselt number as a function of Grashof number for various values of the Prandtl number based on the self-similar solution presented in this section and the EES procedure FC_Plate_vertical_ND.