6.4: Integral Solution

6.4.1 Introduction

Section 4.8 presents integral techniques for laminar and turbulent external forced convection flows. These techniques can be extended to study problems where buoyancy induced flow is important by including the buoyancy force in the integral form of the momentum equation. If the steps discussed in Section 4.8.2 are carried starting with the *x*-momentum equation that includes the Boussinesq approximation for the buoyancy force then the result is:

$$\boxed{\begin{gathered}\underbrace{\frac{d}{dx}\left[\int_{y=0}^{y=\delta_m}\left(u^2-u\,u_\infty\right)dy\right]}_{\text{momentum change}}+\underbrace{\frac{du_\infty}{dx}\int_{y=0}^{y=\delta_m}\left(u-u_\infty\right)dy}_{\text{pressure force}}+\underbrace{v_{y=0}\left(u_\infty-u_{y=0}\right)}_{\text{momentum injected at surface}}=\\ \underbrace{\frac{1}{\rho}\left[\tau_{y=\delta_m}-\tau_s\right]}_{\text{shear force}}+\underbrace{g\,\beta\int_{y=0}^{\delta_m}\left(T-T_\infty\right)dy}_{\text{buoyancy force}}\end{gathered}} \qquad (6\text{-}212)$$

Equation (6-212) is generally useful and many of the terms that are included in Eq. (6-212) will be negligible for most problems. The integral form of the energy equation is the same as derived in Section 4.8.3:

$$\underbrace{\frac{d}{dx}\left[\int_0^{\delta_t}u\left(T-T_\infty\right)dy\right]}_{\text{enthalpy change}}=\underbrace{v_{y=0}\left(T_s-T_\infty\right)}_{\text{energy injected at surface}}+\underbrace{\frac{1}{\rho\,c}\left[\dot{q}_s''-\dot{q}_{y=\delta_t}''\right]}_{\text{conduction}}+\underbrace{\frac{\mu}{\rho\,c}\int_0^{\delta_t}\left(\frac{\partial u}{\partial y}\right)^2dy}_{\text{viscous dissipation}} \qquad (6\text{-}213)$$

In Section 6.4.2, an integral solution is presented for a vertical flat plate where the momentum and thermal boundary layer thicknesses are assumed to be identical (i.e., *Pr* = 1). Because the temperature elevation drives the velocity, the momentum and thermal boundary layers are of comparable size and therefore this solution is useful even if the Prandtl number is not exactly equal to unity. However, this technique can be extended to other Prandtl numbers and other flow situations.

6.4.2 Integral Solution

A simple integral solution for natural convection from an isothermal vertical plate is discussed in this section. The same problem is considered using a self-similar solution in Section 6.3 and correlations for a vertical heated plate are discussed in Section 6.2.2.

There is no forced convection flow and therefore u_∞ is zero; therefore, Eq. (6-212) can be simplified to:

$$\frac{d}{dx}\left[\int_{y=0}^{y=\delta_m}u^2\,dy\right]=-\frac{\tau_s}{\rho}+g\,\beta\int_{y=0}^{\delta_m}\left(T-T_\infty\right)dy \qquad (6\text{-}214)$$

The functional forms of the velocity distribution that are appropriate for the integral solution of laminar forced convection problems are discussed in Section 4.8.2 and presented in Table 4-4. These results are not valid for this problem because u_∞ is zero. Instead, the velocity distribution should look something like the curve for $\frac{df}{d\eta}$ shown in Section 6.3. However, we can use the steps identified in Section 4.8.2 to identify an appropriate velocity distribution for natural convection. A third order polynomial is assumed:

$$u = C_1 + C_2\, y + C_3\, y^2 + C_4\, y^3 \tag{6-215}$$

Equation (6-215) must satisfy the no-slip condition at the wall:

$$u_{y=0} = C_1 + C_2\, 0 + C_3\, 0^2 + C_4\, 0^3 = 0 \rightarrow C_1 = 0 \tag{6-216}$$

At the edge of the boundary layer the velocity must become zero:

$$u_{y=\delta_m} = C_1 + C_2\, \delta_m + C_3\, \delta_m^2 + C_4\, \delta_m^3 = 0 \tag{6-217}$$

The shear stress at the edge of the boundary layer should also go to zero:

$$\left.\frac{du}{dy}\right|_{y=\delta_m} = C_2 + 2\, C_3\, \delta_m + 3 C_4\, \delta_m^2 = 0 \tag{6-218}$$

Finally, the differential form of the momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\,\beta\left(T - T_\infty\right) + \upsilon\frac{\partial^2 u}{\partial y^2} \tag{6-219}$$

should be satisfied at the wall. Setting the velocities to zero in Eq. (6-219) leads to:

$$0 = g\,\beta\left(T_s - T_\infty\right) + \upsilon\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} \tag{6-220}$$

where T_s is the wall surface temperature. Substituting Eq. (6-215) into Eq. (6-220) leads to:

$$0 = g\,\beta\left(T_s - T_\infty\right) + \upsilon\, 2\, C_3 \tag{6-221}$$

Equations (6-216), (6-217), (6-218), and (6-221) provide four equations for the four unknowns, C_1 through C_4. These can be solved by hand or, more conveniently, using Maple:

```
> restart;
> EQ1:=C_1=0;
```

$$EQ1 := C\_1 = 0$$

```
> EQ2:=C_1+C_2*dm+C_3*dm^2+C_4*dm^3=0;
```

$$EQ2 := C\_1 + C\_2\ dm + C\_3\ dm^2 + C\_4\ dm^3 = 0$$

```
> EQ3:=C_2+2*C_3*dm+3*C_4*dm^2=0;
```

$$EQ3 := C\_2 + 2\ C\_3\ dm + 3\ C\_4\ dm^2 = 0$$

```
> EQ4:=g*beta*(T_s-T_infinity)+2*nu*C_3=0;
```

$$EQ4 := g\ \beta\ (T\_s - T\_infinity) + 2\ \nu\ C\_3 = 0$$

```
> constants:=solve({EQ1,EQ2,EQ3,EQ4},{C_1,C_2,C_3,C_4});
```

$$constants := \left\{ C\_1 = 0,\ C\_3 = \frac{g\ \beta\ (-T\_s + T\_infinity)}{2\ \nu},\ C\_4 = -\frac{g\ \beta\ (-T\_s + T\_infinity)}{4\ \nu\ dm},\right.$$
$$\left. C\_2 = -\frac{g\ \beta\ (-T\_s + T\_infinity)\ dm}{4\ \nu} \right\}$$

Substituting the constants identified by Maple into the velocity distribution, Eq. (6-215), leads to:

$$u = \frac{g\,\beta\left(T_s - T_\infty\right)\delta_m}{4\upsilon}\,y - \frac{g\,\beta\left(T_s - T_\infty\right)}{2\upsilon}\,y^2 + \frac{g\,\beta\left(T_s - T_\infty\right)}{4\upsilon\,\delta_m}\,y^3 \tag{6-222}$$

which can be rearranged:

$$u = \underbrace{\frac{g\,\beta\left(T_s - T_\infty\right)\delta_m^2}{\upsilon}}_{\text{velocity}}\left[\frac{1}{4}\frac{y}{\delta_m} - \frac{1}{2}\left(\frac{y}{\delta_m}\right)^2 + \frac{1}{4}\left(\frac{y}{\delta_m}\right)^3\right] \tag{6-223}$$

The term that is in front of the bracketed expression in Eq. (6-223) has units of velocity. Therefore, a dimensionless velocity can be defined:

$$\tilde{u} = \frac{u\,\upsilon}{g\,\beta\left(T_s - T_\infty\right)\delta_m^2} = \frac{1}{4}\frac{y}{\delta_m} - \frac{1}{2}\left(\frac{y}{\delta_m}\right)^2 + \frac{1}{4}\left(\frac{y}{\delta_m}\right)^3 \tag{6-224}$$

Figure 6-25 illustrates the dimensionless velocity calculated in Eq. (6-224) as a function of y/δ_m.

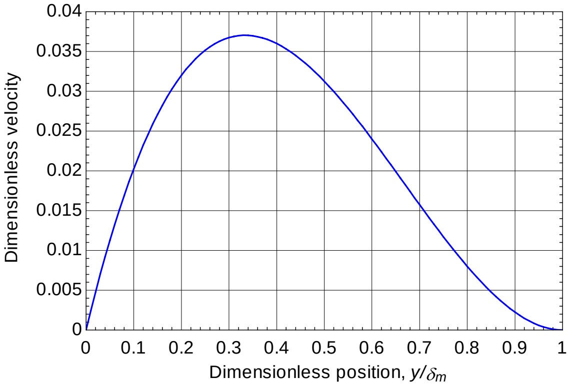

Figure 6-25: Dimensionless velocity as a function of y/δ_m.

The shear stress at the wall predicted by Eq. (6-223) is given by:

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu\, g\, \beta \left(T_s - T_\infty\right) \delta_m}{4\upsilon} \tag{6-225}$$

Both a velocity and temperature distribution must be substituted into Eq. (6-214) in order to carry out the integration because the momentum and energy equations are coupled. The temperature distributions identified in Section 4.8.3 for forced convection remain valid because the steps discussed in Section 4.8.3 do not change and the boundary conditions are the same. The third order polynomial presented in Table 4-5, simplified appropriately, is:

$$\frac{T - T_s}{T_\infty - T_s} = \left[\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right] \tag{6-226}$$

where δ_t is the thermal boundary layer thickness. Equation (6-226) can be rearranged:

$$T - T_\infty = \left(T_s - T_\infty\right)\left(1 - \frac{3}{2}\frac{y}{\delta_t} + \frac{1}{2}\frac{y^3}{\delta_t^3}\right) \tag{6-227}$$

The heat flux at the surface of the plate is therefore:

$$\dot{q}_s'' = \frac{3}{2} k \frac{\left(T_s - T_\infty\right)}{\delta_t} \tag{6-228}$$

Equations (6-223), (6-225), and (6-227) are substituted into the integral form of the momentum equation, Eq. (6-214):

$$\frac{d}{dx}\left[\int_{y=0}^{y=\delta_m}\left(\frac{g\,\beta\left(T_s-T_\infty\right)\delta_m^2}{\upsilon}\left[\frac{1}{4}\frac{y}{\delta_m}-\frac{1}{2}\left(\frac{y}{\delta_m}\right)^2+\frac{1}{4}\left(\frac{y}{\delta_m}\right)^3\right]\right)^2 dy\right]= -\frac{\mu\,g\,\beta\left(T_s-T_\infty\right)\delta_m}{4\upsilon\rho}+g\,\beta\left(T_s-T_\infty\right)\int_{y=0}^{\delta_m}\left(1-\frac{3}{2}\frac{y}{\delta_t}+\frac{1}{2}\frac{y^3}{\delta_t^3}\right)dy \qquad (6\text{-}229)$$

The problem is simplified by assuming that the momentum and thermal boundary layer thicknesses are equal ($\delta_m = \delta_t = \delta$):

$$\frac{d}{dx}\left[\int_{y=0}^{y=\delta}\left(\frac{g\,\beta\left(T_s-T_\infty\right)\delta^2}{\upsilon}\left[\frac{1}{4}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^2+\frac{1}{4}\left(\frac{y}{\delta}\right)^3\right]\right)^2 dy\right]= -\frac{\mu\,g\,\beta\left(T_s-T_\infty\right)\delta}{4\upsilon\rho}+g\,\beta\left(T_s-T_\infty\right)\int_{y=0}^{\delta}\left(1-\frac{3}{2}\frac{y}{\delta}+\frac{1}{2}\frac{y^3}{\delta^3}\right)dy \qquad (6\text{-}230)$$

Carrying out the integrations in Eq. (6-230) leads to an ordinary differential equation for δ that can be solved for δ as a function of position. Here, Eq. (6-230) will be solved using Maple. The velocity distribution, Eq. (6-223), is entered in Maple:

```
> restart;
> u:=(g*beta*(T_s-T_infinity)*d(x)^2/upsilon)*((y/d(x))/4-(y/d(x))^2/2+(y/d(x))^3/4);
```

$$u := \frac{g\,\beta\,(T\_s - T\_infinity)\,\mathrm{d}(x)^2\left(\frac{1}{4}\frac{y}{\mathrm{d}(x)}-\frac{1}{2}\frac{y^2}{\mathrm{d}(x)^2}+\frac{1}{4}\frac{y^3}{\mathrm{d}(x)^3}\right)}{\upsilon}$$

Note that the variable d(x) in the Maple code is the boundary layer thickness which is a function of x. The shear stress, Eq. (6-225), is entered in Maple:

```
> tau_s:=mu*g*beta*(T_s-T_infinity)*d(x)/(4*upsilon);
```

$$tau\_s := \frac{1}{4}\frac{\mu\,g\,\beta\,(T\_s - T\_infinity)\,\mathrm{d}(x)}{\upsilon}$$

The temperature distribution, Eq. (6-226), is entered in Maple:

```
> T:=T_infinity+(T_s-T_infinity)*(1-(3/2)*(y/d(x))+(y/d(x))^3/2);
```

$$T := T\_infinity + (T\_s - T\_infinity)\left(1-\frac{3}{2}\frac{y}{\mathrm{d}(x)}+\frac{1}{2}\frac{y^3}{\mathrm{d}(x)^3}\right)$$

These functions are substituted into the integral form of the momentum equation, Eq. (6-214):

```
> diff(int(u^2,y=0..d(x)),x)=-tau_s/rho+g*beta*int((T-T_infinity),y=0..d(x));
```

$$\frac{1}{336}\,\frac{g^2\,\beta^2\,(T\_s - T\_infinity)^2\,\mathrm{d}(x)^4\left(\frac{d}{dx}\,\mathrm{d}(x)\right)}{\upsilon^2} = -\frac{1}{4}\,\frac{\mu\,g\,\beta\,(T\_s - T\_infinity)\,\mathrm{d}(x)}{\upsilon\,\rho} + g\,\beta\left(-\frac{5}{8}\,(T\_s - T\_infinity)\,\mathrm{d}(x) + T\_s\,\mathrm{d}(x) - T\_infinity\,\mathrm{d}(x)\right)$$

Maple has identified the differential equation that describes the growth of the boundary layer:

$$\frac{g^2\,\beta^2\left(T_s - T_\infty\right)^2\delta^4}{336\,\upsilon^2}\,\frac{d\delta}{dx} = -\frac{\mu\,g\,\beta\left(T_s - T_\infty\right)\delta}{4\,\upsilon\,\rho} + \frac{3\,g\,\beta\left(T_s - T_\infty\right)\delta}{8} \tag{6-231}$$

Equation (6-231) can be simplified to:

$$\frac{g\,\beta\left(T_s - T_\infty\right)\delta^4}{336\,\upsilon^2}\,\frac{d\delta}{dx} = \frac{\delta}{8} \tag{6-232}$$

Separating variables leads to:

$$\frac{8\,g\,\beta\left(T_s - T_\infty\right)}{336\,\upsilon^2}\int_0^{\delta}\delta^3\,d\delta = \int_0^{x} dx \tag{6-233}$$

Carrying out the integration leads to:

$$\frac{2\,g\,\beta\left(T_s - T_\infty\right)}{336\,\upsilon^2}\,\delta^4 = x \tag{6-234}$$

Equation (6-234) is the usual result of an integral technique, an expression for the boundary layer as a function of position:

$$\delta = 3.60\left[\frac{x\,\upsilon^2}{g\,\beta\left(T_s - T_\infty\right)}\right]^{1/4} \tag{6-235}$$

The average Nusselt number is defined as:

$$\overline{Nu}_L = \frac{L}{k}\,\bar{h} = \frac{L}{k}\underbrace{\left[\frac{1}{L}\int_0^L h\,dx\right]}_{\bar{h}} = \frac{1}{k}\int_0^L \underbrace{\frac{\dot{q}_s''}{\left(T_s - T_\infty\right)}}_{h}\,dx \tag{6-236}$$

Substituting Eq. (6-228) into Eq. (6-236) leads to:

$$\overline{Nu}_L = \frac{1}{k}\int_0^L \underbrace{\frac{3}{2}k\frac{\left(T_s-T_\infty\right)}{\delta}}_{\dot{q}_s''}\frac{1}{\left(T_s-T_\infty\right)}dx = \frac{3}{2}\int_0^L \frac{dx}{\delta} \tag{6-237}$$

Substituting Eq. (6-235) into Eq. (6-237) leads to:

$$\overline{Nu}_L = 0.417\left[\frac{g\,\beta\left(T_s-T_\infty\right)}{\upsilon^2}\right]^{1/4}\int_0^L x^{-1/4}dx \tag{6-238}$$

Carrying out the integration leads to:

$$\overline{Nu}_L = 0.556\left[\frac{g\,\beta\left(T_s-T_\infty\right)}{\upsilon^2}\right]^{1/4} L^{3/4} \tag{6-239}$$

or

$$\overline{Nu}_L = 0.556\,Gr_L^{1/4} \tag{6-240}$$

Figure 6-26 illustrates the average Nusselt number predicted by Eq. (6-240) as well as the correlation presented in Section 6.2.2 and accessed by the EES function FC_plate_vertical_ND for various values of the Prandtl number. Notice that for Pr = 1 the agreement is good. However, the agreement is not as good when the Prandtl number is substantially different from unity. Equation (6-230) is strictly valid only if Pr = 1 and therefore the solution is most accurate in this limit. However, a more rigorous and broadly accurate solution can be obtained by substituting Eqs. (6-223), (6-227), and (6-228) into the integral form of the energy equation, Eq. (6-213), in order to obtain a set of coupled ordinary differential equations for δ_m and δ_t that could be solved numerically.

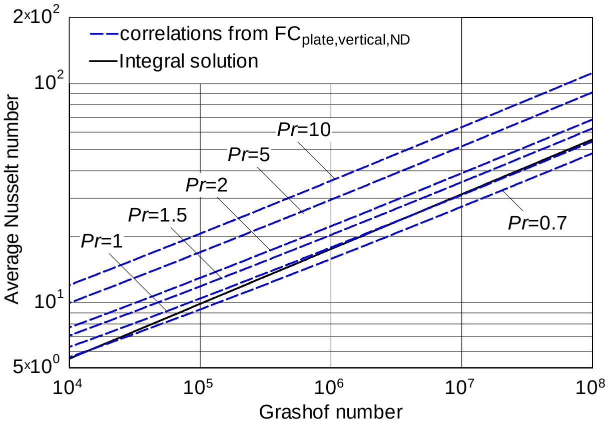

Figure 6-26: Average Nusselt number predicted by the integral solution, Eq. (6-240), and the EES function FC_plate_vertical_ND as a function of the Grashof number.

E16: Section 6.4 *Integral Solutions*

The integral technique discussed in this section provides a flexible but approximate technique for solving natural convection problems. The integral technique can be applied to a wide range of problems for which neither correlations or exact solutions exist. For example, it is possible to examine situations when there is some forced convection flow (i.e., u_∞ is not zero), the temperature of the plate is not constant (i.e., T_s is a function of x), there is transpiration, etc.