Planck blackbody function

Hale Bradt and Stanislaw Olbert

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What we learn in this Supplement

We derive the Planck blackbody function. We first present briefly the concepts of the **distribution function** (the density of **representative points (RP)** in a six-dimensional (6-D) ***x, p*** space), energy shells in **momentum space**, **thermodynamic equilibrium** and **indistinguishable particles**. The Bose-Einstein counting of arrangements of cells in 6-D space is worked out through the definitions of **microstates**, **macrostates**, and **global macrostates**. The **maximizing of the arrangements** for **massless bosons** (photons) yields the global macrostate at equilibrium, where constraints on the total (summed) energy and on the cell numbers are applied with **Lagrange multipliers**. This global macrostate provides the number of photons in each energy state at equilibrium, which is the desired spectrum. A **thermodynamic argument** relates a Lagrange constant to the **temperature *T*.** The final result is the distribution function f(m^{-3} (N s)$^{-3}$) for Bose-Einstein statistics of massless bosons. From this follows directly the **specific intensity** (W m^{-2} Hz^{-1} sr^{-1}), otherwise known as the **Planck function** of the **blackbody spectrum**.

The spectrum for **massless fermions** (e.g., neutrinos, neglecting their small mass) follows from a similar development. This is relevant to a predicted **cosmological background of neutrinos.** Similar calculations also produce the distribution functions for **bosons and fermions with mass**. Some of these cases are worked out in the problems.

Solutions to the problems are available to instructors on the password-protected portion of the CUP web site for this text (see URL above).

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1 Introduction

We derive here the *distribution function* $f(x, y, z, p_x, p_y, p_z)$ for a gas of photons in thermodynamic equilibrium. This quantity is the number density of points representing particles (or *representative points*, RP) in a 6-dimensional (6-D) phase space. The density has units m^{-3} $(\mathrm{N\ s})^{-3}$. The specific intensity I_ν, or Planck function (equation (6.6) in AP), follows directly from it.

The derivation here follows directly from the statistical ideas of Bose (Zietsch. f. Phys. 26, 178 (1924)) and Einstein (Berl. Ber. p. 261 (1924) and p. 18 (1925)). (Planck's blackbody law was published in 1901.) It follows the development and notation of Georg Joos (tr. by Freeman) in his classic book, *Theoretical Physics* (Hafner Publishing Co. New York, 2nd Edit. (1950), p. 617 ff. There are other approaches to these statistics; see textbooks on statistical mechanics.

Quantum statistics treats photons as discrete and *indistinguishable* quanta of energy $h\nu$, and it divides phase space into cells, each of volume of h^3. The quantization of photon energy was not known to Planck when he derived his law from a quantization of waves. The existence of the constant h followed from his work, and its numerical value followed from fits to measured energy spectra of thermal radiation.

The statistical approach differs from that of Boltzmann (outlined in Section 3.2 of Astrophysics Processes (AP)), which implicitly assumes the individual particles are *distinguishable*. Bose-Einstein (B-E) statistics are applicable to *bosons*, particles of integer spin (e.g., photons, pions), and there is no intrinsic limit to the number of bosons that may be placed in a given cell of $\boldsymbol{x}, \boldsymbol{p}$ phase space. In contrast, Fermi-Dirac (F-D) statistics apply to *fermions* which have half-integer spin (e.g., neutrinos, electrons), and each phase-space cell can contain only zero or one particle of a given type where spin state counts as a "type".

It turns out that the B-E or F-D statistics reduce to the exponential form of the Boltzmann distribution for sufficiently high temperatures, or equivalently, high particle energies ($E \approx kT$). For many practical situations, the quantization of energy states has negligible effect on the distributions, and one can use the Maxwell-Boltzmann distribution. It is only for very high densities and/or very low temperatures that the quantization becomes important.

Here we derive the B-E distribution for photons, which are massless bosons, when they are in thermal equilibrium. A simple modification to the derivation yields the F-D distribution for massless fermions in thermal equilibrium, which is applicable to a gas of *neutrinos* (if they may be considered massless). This is of great interest because a thermal distribution of neutrinos created in the early hot universe is believed to pervade the universe today according to the standard "big-bang" model. Originally of temperature ~10^{10} K, the neutrinos, if massless, would have cooled to ~2 K at the current epoch. It has not yet been possible to search for such very low-energy neutrinos.

This derivation may be further extended for particles of finite mass by adding one additional constraint for the conservation of particle number. Here we describe this and present the resulting distribution functions.

2 Basic concepts

Thermodynamic equilibrium

Consider the box of fixed volume V in Fig. 1. It is at *rest* in an *inertial frame* of reference and is filled with radiation. Treat the radiation as an ensemble of photons. The box may be a hollow cavity or it may contain a gas of ordinary particles. The photons collide with the walls and with the particles, exchanging energies and randomizing directions. As the energies and directions approach stable values, the distributions are said to become *stationary*. They become *time independent* and *isotropic*. In such a situation, the photons are said to be in *thermodynamic equilibrium.*

Distribution function

We operate every day in the usual physical x ,y, z space and are accustomed to the number density in such a space. In a Cartesian space with three directions, p_x, p_y, p_z, (Fig. 3.1c of AP), a point at the position indicated by a particle's momentum vector can represent the particle. One can therefore think of points representing particles distributed in momentum space. This may be generalized to a 6-D *phase space* with dimensions x, y, z, p_x, p_y, p_z wherein each particle is represented by a point. We call such points *representative points* (RP).

The density of RP for a gas in this 6-D space is called the *distribution function* $f(x, y, z, p_x, p_y, p_z, t)$; units m^{-3} (N s)$^{-3}$. The product $f(x, y, z, p_x, p_y, p_z, t)\ \boldsymbol{\Delta x}\ \boldsymbol{\Delta p}$ is the *number* of particles at time t with positions and momenta in the ranges $x \rightarrow x+\Delta x$, $y \rightarrow y+\Delta y$, $z \rightarrow z+\Delta z$, $p_x \rightarrow p_x+\Delta p_x$, $p_y \rightarrow p_y+\Delta p_y$, $p_z \rightarrow p_z+\Delta p_z$ where $\boldsymbol{\Delta x}\ \boldsymbol{\Delta p} \equiv \Delta x, \Delta y, \Delta z, \Delta p_x \Delta p_y \Delta p_z$ is the differential volume element in 6-D phase space occupied by these RP. Divide the number of RP by this volume to obtain the density in phase space, namely $f(x, y, z, p_x, p_y, p_z, t)$.

The distribution function f is a fundamental quantity from which all other characteristics of the particle flow can be derived, such as the specific intensity, spectral flux density, etc. See Section 3.3 of AP.

In general, the distribution function f depends on the variable, time t, because the distribution can change with time. However, since we consider here a photon gas in thermal equilibrium, the distribution function is independent of time. In this case, one can drop the t dependence; thus $f(x, y, z, p_x, p_y, p_z)$.

Photon gas

Here we present some background material regarding photon gases prior to deriving the Planck function.

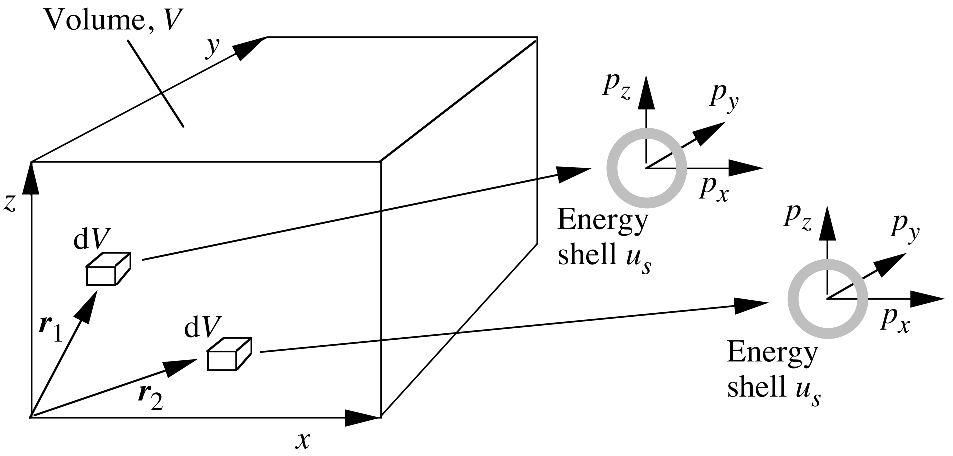

Figure 1. Energy shells (states) at two positions in physical (ordinary) space. The shells are in the momentum part of phase space where $p = h\nu/c$ is the momentum of a photon. All photons with representative points (RP) in the many shells of the same radius have the same energy.

Energy shells

Consider a physical (ordinary) volume V (Fig. 1), which may be subdivided into volume elements

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{1}$$

One can similarly define a volume element $\mathrm{d}V_\mathrm{p}$ in momentum space, in cartesian and also in spherical coordinates,

$$\mathrm{d}V_\mathrm{p} = \mathrm{d}^3p = \mathrm{d}p_\mathrm{x}\,\mathrm{d}p_\mathrm{y}\,\mathrm{d}p_\mathrm{z} = \mathrm{d}\Omega_\mathrm{p}\,p^2\,\mathrm{d}p \qquad \text{(Volume element in momentum space)} \tag{2}$$

Integration over all angles in momentum space yields a spherical shell of volume,

$$\mathrm{d}V_\mathrm{u} = \oiint_{\substack{\text{spherical}\\\text{surface}}} \mathrm{d}V_\mathrm{p} = 4\pi\,p^2\,\mathrm{d}p \qquad \text{(“Energy shell” in momentum space)} \tag{3}$$

This shell is illustrated in Fig. 2. All particles with momentum vectors terminating in this shell have the same magnitude p within dp. Since the photon energy is directly related to momentum, $u_s = p_s c$ (J), the energies of all such photons will also be the same, within the corresponding interval du_s. We thus call the spherical shell an *energy shell* and so label the volume element dV_u (“u” for energy), but keep in mind that it is a volume in momentum space. The subscript “s” in u_s indicates the energy shell (state) number.

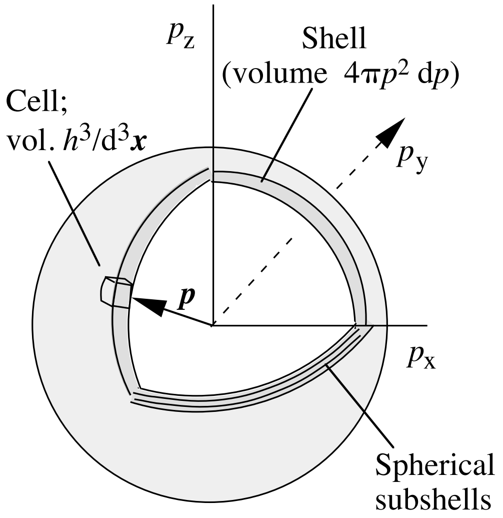

Figure 2. Combined shell in momentum space for all shells of same radius. Each such shell consists of many cells, each of which has volume h^3 in $\boldsymbol{x}, \boldsymbol{p}$ phase space.

If the particle has a rest mass m, the momentum–energy relation is $pc = (u^2 - (mc^2)^2)^{1/2}$ (AP, (7.20)). Again, there is a one to one correlation between the particle's momentum and energy. The intervals are related as $c^2\,p\,dp = u\,du$.

The radius of the spherical shell represents the momentum p of an *individual* photon. Other photons with the same energy will have momentum vectors that terminate in the same shell, regardless of their location in physical x, y, z space. Thus, the energy content of particles in a shell, $N_s h\nu_s$, will usually not equal the shell energy $u_s = p_s c = h\nu_s$, the exception being when there is just one photon in the shell.

Two such energy shells, of the same radius but at two different physical locations, are shown in Fig. 1. This figure illustrates that, at each point in physical space, one can have a complete set of energy shells each containing some number of particles, to be determined. In momentum space (Fig. 2), all such shells for a chosen volume are superimposed.

The distribution of photon number N_s (or energy $N_s h\nu_s$) with shell energy $h\nu_s$ (or frequency ν_s) is the spectrum we seek. It is the assumed isotropy in thermal equilibrium that permits us to describe the distribution solely in terms of energy, or equivalently momentum magnitude.

Indistinguishable particles

There are two kinds of photons, those with left-handed polarization and those with right-handed polarization (z spin states ±1). They are identical in all other respects. All photons with a given polarization state are indistinguishable except for their energy value $u_s = h\nu$.

The classical view is that particles in a box can be distinguished from one another, as in everyday life, because they are sufficiently separated to be tracked individually. This view is not generally correct in quantum mechanics. One might imagine a box of gas particles that are all totally at rest, with zero momentum. This is a plausible idea classically, although it would require a very low temperature. But, in quantum mechanics, the uncertainty principle tells us that a particle with zero momentum (with no uncertainty) has a huge uncertainty in position. Thus each of the particles in the box can occupy the *entire* box, like a wave. It is thus impossible to track individual particles where quantum mechanics is important. They are indistinguishable.

The statistics of Boltzmann yield the Maxwell-Boltzmann equation through a maximization of the number of ways the (distinguishable) particles can be arranged in cells in phase space. In quantum mechanics, the box of photons can be viewed as filled with waves (or indistinguishable particles). There are no distinguishable particles to arrange. A different kind of statistics is required to determine the energy distribution of a gas of photons.

Bose's idea was that, if one adopts cells of a fixed cell size in phase space (from Planck's quantization) and associates measurable quantities with the cells such as the number of photons of energy $h\nu$ in the cell, one can play statistics with the cells rather than the physical particles.

3 Bose-Einstein statistics

Bose-Einstein statistics apply to particles of integer spin whether they are massless or not. Here we focus on massless bosons, namely photons.

Microstates and macrostates

We now play Bose statistics with a box of one kind of partricles (say, photons of right hand polarization). Thermodynamic equilibrium is assumed.

Cells in phase space

To play Bose statistics, we introduce a *cell* in 6-D phase space ($\boldsymbol{x}, \boldsymbol{p}$), The cell is defined by the magnitude of its "volume" which we call Θ – the shape is immaterial –

$$\Theta = \Delta x\, \Delta p_{\rm x}\, \Delta y\, \Delta p_{\rm y}\, \Delta z\, \Delta p_{\rm z} \tag{4}$$

One would like to cut up the entire phase space of the box into as many cells as possible without violating the spirit of Heisenberg's uncertainty principle. The permissible choice is

$$\begin{aligned} \Delta x\, \Delta p_{\rm x} &= h \\ \Delta y\, \Delta p_{\rm y} &= h \\ \Delta z\, \Delta p_{\rm z} &= h \end{aligned} \tag{5}$$

where h is Planck's constant. The cell volume in phase space is thus,

$$\Rightarrow \qquad \Theta_{\rm cell} = h^3 \qquad \text{(Cell vol. in phase space)} \tag{6}$$

As noted above, in Bose statistics, there is no constraint on how many particles can be placed into one such cell. In Fermi statistics, only zero or one particle of a given type (including spin state) may be placed into one cell.

There are several ways to visualize the momentum portion of the individual cells of volume h^3. In momentum space, they may be (*i*) quasi-cubic microscopic cells distributed over the shell (see Fig. 2 for an example), (*ii*) individual spherical shells associated with different positions in space as in Fig. 1, or (*iii*) spherical subshells of the energy shell of Fig. 2, like the layers of an onion. In all three cases, the individual cells each have volume $h^3/\mathrm{d}^3\boldsymbol{x}$ in momentum space. In the first case, the individual cells contain photons with momenta directed in a single direction. In the latter two cases, the spherical cells contain photons moving in all directions, and the cells would have decreasing radial thicknesses as radius increases in order to keep the 3-D cell volumes constant at $h^3/\mathrm{d}^3\boldsymbol{x}$. In all cases, keep in mind that the cells also have another 3 dimensions in x, y, z space. If unit x, y, z volume is assumed, the 6-D cell volume h^3 equals the volume in momentum space.

The choice among these visualizations is not important; what matters is that there are a number of cells at a given photon energy, and these constitute the subgroup of interest. Hereafter, we will usually refer to an *energy state* rather than to a "shell', but the reader is welcome to visualize a shell consisting of many cells of volume h^3.

Let the number of cells in a specific energy state s be called Z_s; for the German word *zelle* (cell). Then the total number of cells in the 6-D phase space summed over all energy states s is,

$$Z = \sum_s Z_s \qquad \text{(Total no. of cells)} \qquad (7)$$

The summation is over energy states from $u_s = 0$ to $u_s = \infty$, where $u_s = h\nu_s$, the energy of a photon in state s. The number Z_s of cells for a given energy state, depends only upon the volume h^3 of a single cell and the phase space volume available. Hence the Z_s are held fixed as we vary the photon numbers in the cells,

$$Z_s = \text{constant}; \;\; \delta Z_s = 0. \qquad (8)$$

Cell arrangements

Our primary goal is to find the most probable distribution of bosons (photons) over the s energy states, i.e., the energy spectrum. In equilibrium, the bosons are expected to have this most probable distribution. The distribution with the highest probability is the one for which there is the maximum number of photon arrangements among the individual cells that will give rise to that particular distribution.

The proper counting of the number of arrangements, known as *combinatorics,* requires that we define as a *macrostate*. It involves a recipe for forming subgroups of *cells* (not bosons) within a given energy state. As we shall see, each macrostate has a well-defined distribution of bosons among its cells. We will also define a *global macroscopic state* which encompasses all the energy states. It is the most probable global macrostate that we seek by maximizing the number of arrangements that give rise to it. The energy distribution, the Planck function, follows directly from the most probable global macrostate.

The Bose method of combinatorics begins with the consideration of a single energy state s. All bosons in energy state s have energy $u_s = h\nu_s$. Call the number of cells within the energy state that are unoccupied (empty) z_{s0}, the number occupied by one boson z_{s1}, etc. The symbol z_{sm} thus represents the number of cells containing m bosons. It follows from this definition that the number of cells in energy state s is,

$$Z_s = \sum_{m=0}^{\infty} z_{sm} \qquad \text{(Number of cells in state } s\text{)} \tag{9}$$

and that

$$N_s = \sum_{m=0}^{\infty} m\, z_{sm} \qquad \text{(Number of bosons in state } s\text{)} \tag{10}$$

is the number of bosons in the energy state, s. The upper limit is $m = \infty$ because there can be an arbitrarily large number of bosons (e.g., photons) in any cell. The numbers are restricted only by the overall energy content of the entire system.

As an example, consider $Z_s = 4$ cells and $N_s = 3$ bosons, as illustrated in Table 1, and examine the number of ways the particle labels may be arranged. There are a total of 20 possible ways as shown. Take the top group in the table where three cells each contain one boson ($z_{s1} = 3$), and one cell contains none ($z_{s0} = 1$); also $z_{s2} = z_{s3} = 0$. By inspection, there are four arrangements with these values of z_{sm}, if we treat the bosons as indistinguishable. They are $\boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{0}$, $\boxed{0}\,\boxed{1}\,\boxed{1}\,\boxed{1}$, $\boxed{1}\,\boxed{0}\,\boxed{1}\,\boxed{1}$, and $\boxed{1}\,\boxed{1}\,\boxed{0}\,\boxed{1}$.

These arrangements are obtained by interchanging the cell labels (along *with* the particle content) among the four cells. Alternatively, one may consider that the cells themselves (with their particle content and labels) are rearranged among the four positions. The essence of the problem does not change; the combinatorics are identical.

According to Bose, interchanging two cells labeled ‘1’ does *not* yield a distinct arrangement because they are identical. If the cells labeled ‘1’ were distinguishable, say, by color, there would be a different set of distinct arrangements. (See discussion of “Permutations” in the Appendix and in the Boltzmann statistics in the next subsection.)

According to (A9) in the Appendix, the number of distinct permutations of four cell labels, three of which are identical and one of which stands alone, is $W = Z_s!/(z_{s0}!\; z_{s1}!\; z_{s2}!\; z_{s3}!) = 4!/(1!\; 3!\; 0!\; 0!) = 4$. This agrees with the result in Table 1.

Each of the four substates in the first group of Table 1 may be called a *microstate*. That is, there are four microstates which contain no more than one boson in a cell. The four microstates are four ways to construct the *Macrostate* I (see column 1 of Table 1). A macrostate is defined as having a fixed set of z_{sm} (column 2 of Table 1). The Bose probability for Macrostate I is $W_s(1,3,0,0) = 4$ (column 4 of Table 1).

Table 1. Bose statistics for 4 cells and 3 identical objects in energy state, s ($Z_s = 4; N_s = 3$)[a]

		Bosons per cell, each of energy $u_s = h\nu_s$				Bose probability W_s
	Cell:	#1	#2	#3	#4	
Bose	All microstates	1	1	1	0	4
Macrostate[b]	with:	0	1	1	1	
I	$z_{s0} = 1$, $z_{s1} = 3$	1	0	1	1	
	$z_{s2} = 0$, $z_{s3} = 0$	1	1	0	1	
		2	1	0	0	12
		0	2	1	0	
		0	0	2	1	
		1	0	0	2	
Bose	All microstates	1	2	0	0	
Macrostate	with:	0	1	2	0	
II	$z_{s0} = 2$, $z_{s1} = 1$	0	0	1	2	
	$z_{s2} = 1$, $z_{s3} = 0$	2	0	0	1	
		0	1	0	2	
		2	0	1	0	
		0	2	0	1	
		1	0	2	0	
Bose	All microstates	3	0	0	0	4
Macrostate	with:	0	3	0	0	
III	$z_{s0} = 3$, $z_{s1} = 0$	0	0	3	0	
	$z_{s2} = 0$, $z_{s3} = 1$	0	0	0	3	

[a] An "energy state" is defined by the energy of a photon occupying it; $u_s = h\nu_s$. The state may contain any number of photons; i.e., it can have any energy content, subject to an overall energy constraint.

[b] For systems with more than one energy state, a Bose macrostate is defined by a single complete set of values z_{sm},: $z_{1,0}$, $z_{1,1}$, $z_{1,2}$, ...$z_{2,0}$ $z_{2,1}$, $z_{2,2}$, . . .$z_{3,0}$ . . . . . . This is called a *global macrostate*. The number of ways W of obtaining this is maximized to find the most probable (equilibrium) set of z_{sm}.

It is also possible to have 2 bosons in a cell, e.g., $\boxed{2}\boxed{1}\boxed{0}\boxed{0}$, where we still take $Z_s = 4$ and $N_s = 3$ (Table 1). The first and second cells are each different from all the others, and the third and fourth are identical, so the number of arrangements is $Z_s!/(z_{s0}!\, z_{s1}!\, z_{s2}!) = 4!/(2!\ 1!\ 1!) = 12$. There are 12 possible microstates with 2 bosons in one cell. These are shown in the Table and are the 12 ways to form Macrostate II; $W_s(2,1,1,0) = 12$.

Finally, the 4 cells and 3 bosons can be arranged with 3 in a given cell, e.g., $\boxed{3}\,\boxed{0}\,\boxed{0}\,\boxed{0}$. There are $4!/(3!\ 0!\ 0!\ 1!) = 4$ such arrangements; there are four ways to construct Macrostate III; $W_s(3,0,0,1) = 4$. In general, the number of ways W_s to obtain a given macroscopic state in energy state s is, from (A9),

$$W_s = \frac{Z_s!}{z_{s0}!\, z_{s1}!\, z_{s2}!\ldots.} \qquad \text{(Number of ways to get macrostate)} \qquad (11)$$

with the stipulation that the number of states Z_s is the sum of all the z_{sm} (9) in the denominator. This is the number of distinct permutations of the cell labels, i.e. of the particle numbers.

It is important to realize that a given macrostate corresponds to a given set of z_{sm}. Each of the macrostates I, II, and III in Table 1 are distinguished by the different sets of z_{sm} given in Column 2. A different set of z_{sm} corresponds to a different macrostate.

In thermal equilibrium, one might find that one or the other of the three macrostates in Table 1 characterizes the energy state u_s. Alternately, one might also find another macrostate with a different number of bosons, e.g., $N_s = 6$, in the four cells ($Z_s = 4$), say, with $z_{s0} = z_{s1} = z_{s2} = z_{s3} = 1$, of which $\boxed{3}\,\boxed{2}\,\boxed{1}\,\boxed{0}$ is one of 24 permutations. There are additional macrostates with $N_s = 6$, but they will have different z_{sm}. In fact there is an infinite number of possible macrostates at energy u_s because one can put any huge number of bosons in a given state, if there is sufficient energy in the system to allow it.

Boltzmann statistics

In Boltzmann statistics, each line of Table 1 is a macrostate which has an associated probability of occurrence. Consider the top line $\boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{0}$, and distinguish the 3 particles by color, namely red, green, and blue (R, G, B). There are 6 possible arrangements (microstates) that give rise to this macrostate. $\boxed{\text{R}}\,\boxed{\text{G}}\,\boxed{\text{B}}\,\boxed{0}$, $\boxed{\text{R}}\,\boxed{\text{B}}\,\boxed{\text{G}}\,\boxed{0}$, etc. Thus the Boltzmann probability is 6 for each of the four Boltzmann macrostates in the top group. Similarly, the top line of the second group $\boxed{2}\,\boxed{1}\,\boxed{0}\,\boxed{0}$ can be obtained in 3 ways, namely $\boxed{\text{RG}}\,\boxed{\text{B}}\,\boxed{0}\,\boxed{0}$ $\boxed{\text{RB}}\,\boxed{\text{G}}\,\boxed{0}\,\boxed{0}$, and $\boxed{\text{GB}}\,\boxed{\text{R}}\,\boxed{0}\,\boxed{0}$.The interchange of particles within one cell does *not* constitute a separate arrangement. Each line in the middle group thus has a Boltzmann probability of 3.

The number of ways each macrostate can be arranged is given by $W = N!/(N_1!\, N_2!\ldots)$ where N is the total number of particles, N_1 the number in cell 1, N_2 the number in cell 2, etc. For our first example (top line of group 1), this yields 6 arrangements ($3!/1!\ 1!\ 1!\ 0! = 6$). The second example (top line of group 2) yields three arrangements ($3!/(2!\ 1!\ 0!\ 0!) = 3$) as we claimed. Finally each line of the 3rd group, has only one possible arrangement, $3!/(3!\ 0!\ 0!\ 0!) = 1$. The one arrangement of the top line is $\boxed{\text{RGB}}\,\boxed{0}\,\boxed{0}\,\boxed{0}$.

In contrast, Bose statistics count the top row in each group as only one microstate because the particles are indistinguishable; they are all blue, for example.

Global macrostate

One can define a *global macrostate* wherein each energy state u_s has a particular macrostate with its particular set of z_{sm}. The entire collection of z_{sm}, ranging over all energy states, characterizes the global macrostate. There are many possible different global macrostates because each of the several energy states could have any one of many different macrostates with their associated values of z_{sm}.

Consider two energy states, $s = 1$ and $s = 2$, each with a macrostate characterized by its set of z_{sm}, specified as z_{1m} and z_{2m}. For each arrangement of cells in the macrostate of energy state 1, one can have all possible arrangements of the macrostate in state 2. Thus, the total number of cell arrangements is the product $W = W_1\ W_2$. In general, *for a given global set of* z_{sm}, the number of possible cell arrangements in the entire system is the product,

$$\Rightarrow \qquad W(Z, z_{sm}) = \prod_s W_s(z_{sm}) \qquad \text{(Thermodynamic probability)} \qquad (12)$$

Each global set of z_{sm} describes a *global macrostate* which can be obtained in $W = W(Z, z_{sm})$ ways. The overall probability of obtaining this global macrostate is proportional to the number of ways W of obtaining it. Thus, W is known as the *thermodynamic probability*.

As an example, consider again our two-state case. In state 1, if the arrangement is $\boxed{2}\boxed{1}\boxed{0}\boxed{0}$ *or* any of its distinct permutations, we have a specific set of z_{1m}, namely $z_{1,0} = 2$, $z_{1,1} = 1$, $z_{1,2} = 1$. As we learned in Table 1, $W_1 = 12$ is the number of ways to arrange these. Now state 2 may have a different number of cells $Z_2 = 5$ and a different number of bosons $N_2 = 7$, and, of necessity, a different set of z_{2m}, say, the set representing $\boxed{2}\boxed{3}\boxed{0}\boxed{0}\boxed{2}$ *and* its 30 permutations (work it out). The product W thus represents the 360 ways these particular sets of cells can be jointly rearranged, but with each set remaining within its own energy state.

Note that the arrangements $\boxed{2}\boxed{1}\boxed{0}\boxed{0}$ and $\boxed{3}\boxed{0}\boxed{0}\boxed{0}$ in state 1 do *not* have the same z_{sm} even though their total particle count is the same. Similarly, in state 2, the set $\boxed{2}\boxed{3}\boxed{0}\boxed{0}\boxed{2}$ *and* $\boxed{7}\boxed{0}\boxed{0}\boxed{0}\boxed{0}$ do not have the same z_{sm}. Thus, the number of ways W discussed in the example just above does not include the arrangements $\boxed{3}\boxed{0}\boxed{0}\boxed{0}$ and $\boxed{7}\boxed{0}\boxed{0}\boxed{0}\boxed{0}$. They would constitute a different global state which would have its own set of z_{sm} and value of $W(Z; z_{sm})$.

4 Massless bosons

Most probable macrostate

Our next task is to find the most probable global macrostate, that with the highest thermodynamic probability. We accomplish this by varying the z_{sm} to find the maximum W. The values of z_{sm} associated with the maximum W will be those expected for an equilibrium gas of bosons. The result for a given energy state s (photon energy u_s) will give us the number of bosons $N_s = \sum_m m\, z_{sm}$ in that state.

The variation of z_{sm} allows the possibility that m, the number of particles in a given cell can be arbitrarily large (unless we otherwise place independent restrictions on the number). Thus the number of sets of z_{sm} and hence of trial values of W can reach infinity in our quest for the maximum W. In fact, for our problem of a photon gas, we will not restrict the number of photons, but we will put a restriction of fixed energy on the entire system.

The next step of our derivation is primarily mathematics: a maximization problem with constraints.

Maximize probability

To maximize W, introduce the function,

$$\tilde{S} \equiv \ln W = \sum_s \ln W_s \tag{13}$$

The "squiggle" distinguishes this function from the closely related entropy, $S = k \ln W$, we encounter below. Apply the expression (11) for $W_s(Z_z, z_{sm})$, use Stirling's formula (A1), and invoke the relation (9) between Z_s and the z_{sm} to obtain (Prob. 43)

$$\tilde{S} \equiv \sum_s \left[Z_s \ln Z_s - \sum_{m=0}^{\infty} z_{sm} \ln z_{sm} \right] \tag{14}$$

Write the variation of $\tilde{S}$ in terms of the variation of the many z_{sm}, recalling that the number of bins Z_s in an energy state is constant, $\delta Z_s = 0$. Proceed as if taking the total differential, and set the result to zero to locate the maximum of $\tilde{S}$,

$$\delta\tilde{S} \equiv -\sum_s \sum_m (1 + \ln z_{sm})\, \delta z_{sm} = 0 \tag{15}$$

The quantities δz_{sm} in the sum (15) are all independent quantities and thus may be arbitrarily assigned values that need not be equal; for example, the quantity $\delta\tilde{S}$ in (15) will take on the proper value for any assigned (differentially small) values of δz_{sm}. Thus the equation (15) can be satisfied only if each of the parenthetical terms equals zero,

$$1 + \ln z_{sm} = 0 \tag{16}$$

This is a series of equations, one for each z_{sm}, or one for each unknown.

Energy and cell number conservation

Before proceeding to solve for the z_{sm}, we must apply the constraints of fixed total (summed) energy and fixed cell numbers. The energy constraint for all bosons (photons) in the ensemble is

$$U = \sum_s \sum_m u_s m z_{sm} = \sum_s u_s N_s = \text{const.} \tag{17}$$

where $u_s = h\nu_s$ is the energy of a photon in energy state *s*. The variation of this expression is, for fixed energy states u_s,

$$\delta U = \sum_s \sum_m u_s m\, \delta z_{sm} = 0 \qquad \text{(Total energy constant)} \tag{18}$$

This is the first constraint on (15). The constancy of cell number Z_s in each energy state provides additional constraints, one for each energy state *s*. From (9), we have, for each energy state *s*,

$$\delta Z_s = \sum_{m=0}^{\infty} \delta z_{sm} = 0 \qquad \begin{array}{r}\text{(Cell numbers constant}\\ \text{in each energy state)}\end{array} \tag{19}$$

Lagrange multipliers

The equations ofor *U* (17) and Z_s (9) cause the problem to be over determined; there are more equations than unknowns. We have added one energy equation and also one cell number equation for each energy state *s*. The method of *Lagrange multipliers* addresses this by introducing new unknown constants while imposing the constraints we specified.

Multiply the energy constraint (18) by some constant β, and the cell-number constraint (19) by the constant α_s. Add these two expressions to (15),

$$\sum_s \sum_{m=0}^{\infty} \delta z_{sm} \left(\ln z_{sm} + 1 + \alpha_s + \beta u_s m\right) = 0 \tag{20}$$

Since each contributing expression is equal to zero, this new expression also equals zero. Again, because the δz_{sm} are independent and arbitrarily assigned, this equation can be satisfied only if

$$\ln z_{sm} + 1 + \alpha_s + \beta u_s m = 0 \tag{21}$$

This new set of equations (one for each z_{sm}) contains the additional constants. To solve for these and for the z_{sm}, these equations must be supplemented by the expressions for the total energy *U* (17) and the Z_s (9), which are given and conserved quantities.

Most probable z_{sm}

Adopt the abbreviation

$$\gamma_s \equiv \exp\left[-(1+\alpha_s)\right] \tag{22}$$

which can be used to eliminate $1+\alpha_s$ in (21) to yield an expression for z_{sm} in terms of the constants γ_s and β,

$$z_{sm} = \gamma_s\, e^{-m\beta u_s} \tag{23}$$

This is the number of cells z_{sm} containing m bosons of energy $h\nu_s$ (in energy state u_s) when $\tilde{S}$ is maximum.

Now, use the known Z_s to obtain the Lagrange multipliers α_s, or equivalently the γ_s. Sum (23) over all m and use the relation (9) for Z_s as the sum of z_{sm} to write

$$Z_s = \gamma_s\, \sigma_s \tag{24}$$

where, for convenience, we define σ_s to be the summation over the exponentials

$$\sigma_s \equiv \sum_{m=0}^{\infty} e^{-m\beta u_s} \tag{25}$$

Thus,

$$\gamma_s = \frac{Z_s}{\sigma_s} \tag{26}$$

These are the desired Lagrange-multiplier parameters γ_s. The z_{sm} (23) thus become

$$\Rightarrow \qquad z_{sm} = \frac{Z_s}{\sigma_s}\, e^{-m\beta u_s} \qquad \text{(Cell distribution when } \tilde{S} \text{ is maximum)} \tag{27}$$

when $\tilde{S}$ is maximum, Z_s is the number of cells in the energy state s and σ_s is the summation of exponentials (25).

Photon number N_s

The photon number N_s in energy state s is obtained by direct substitution of (27) into the expression for $N_s = \Sigma m z_{sm}$ (10),

$$N_s = Z_s \frac{\tilde{\sigma}_s}{\sigma_s} \tag{28}$$

where, again for convenience, we define $\tilde{\sigma}_s$ as the summation

$$\tilde{\sigma}_s \equiv \sum_{m=0}^{\infty} m\, e^{-m\beta u_s} \tag{29}$$

The ratio $\widetilde{\sigma}_s/\sigma_s$ in (28) involves two geometric series that can be shown to give, from (A17),

$$\Rightarrow \quad \frac{N_s}{Z_s} = \frac{1}{e^{\beta u_s} - 1} \qquad \text{(Av. number of photons per cell when } \widetilde{S} \text{ is maximum)} \qquad (30)$$

The ratio N_s/Z_s in (30) is, by definition, the average number of bosons (photons in our case) per cell. If there are equal amounts of left and right handed photons, we have

$$\Rightarrow \quad \frac{N_s}{Z_s} \equiv \frac{2}{e^{\beta u_s} - 1} \qquad \text{(Most probable number of photons per cell; two spin states)} \qquad (31)$$

where N_s and Z_s are the number of photons and number of cells in energy state s respectively, where $u_s = h\nu_s$ is the energy of a photon in state s, and where β is the constant introduced in the course of restricting the total energy to a fixed value. We will show below that $\beta = (kT)^{-1}$. The quantity Z_s is a known quantity; it follows directly from the partition of phase space into cells; see (5) and next section. This validates our statement after (6.29) of AP that the expression (31) represents the average number of photons per cell at frequency ν.

Phase-space density

The phase-space volume of the energy state s is Θ_s, after (4), and the volume of a single cell is $\Theta_{\text{cell}} = h^3$ (6). Recall that we can represent the energy state in momentum space as a spherical shell; Fig. 2 and equation (3). We thus write

$$\Theta_s = 4\pi\, p_s^2\, \mathrm{d}p_s\, \mathrm{d}V = h^3 Z_s \qquad \text{(Phase-space volume of energy state } s\text{)} \qquad (32)$$

where $\mathrm{d}V = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ is the volume element in x, y, z space, Z_s is the number of cells in the energy state, and $p_s = u_s/c$ for photons.

Distribution function

The density of photons in the phase space (the distribution function) was designated f in the introductory sentences of this Supplement. In the energy state s, the function f is simply the number of photons N_s (31) divided by the volume Θ_s of phase space in the state s (32),

$$\Rightarrow \quad f = \frac{N_s}{\Theta_s} = \frac{2}{h^3\,(e^{\beta u_s} - 1)} \qquad \text{(Distribution function or phase-space density; photons m}^{-3}\text{ (N s)}^{-3}\text{)} \qquad (33)$$

where $u_s = h\nu_s$ and, as we find below, $\beta = 1/kT$.

Planck function

The Planck function I_ν (W m^{-2} Hz^{-1} sr^{-1}) follows directly from the distribution function f(m^{-3} (N s)$^{-3}$) by means of an argument similar to that used in AP to derive I_λ from I_ν ; see (6.13) of AP. In this case, express in terms of I_ν the energy at frequency ν in dν that passes through a surface dA in time dt in some direction θ, ϕ into the solid angle element dΩ. Then express in terms of f the same energy at the corresponding momentum $p = h\nu/c$ and momentum interval dp. Then equate the two expressions for this energy to obtain (Prob. 41),

$$I_\nu = \frac{h^4\nu^3}{c^2}\, f \qquad \text{(Conversion; } f \text{ to } I_\nu) \qquad (34)$$

This is derived in AP in a less direct way (3.26).

Substitute (33) into (34) to obtain the specific intensity,

$$\Rightarrow \qquad I_\nu = \frac{2\,h\nu^3}{c^2}\,\frac{1}{\mathrm{e}^{(\beta u_s)}-1} \qquad \text{(Planck function; W m}^{-2}\text{ Hz}^{-1}\text{ sr}^{-1}) \qquad (35)$$

which is the Planck function given in the text (6.6) if we use $u_s = h\nu$ and set $\beta = (kT)^{-1}$. Now our only remaining task in the derivation is to demonstrate the latter equivalence.

Determination of β

The final unknown, the constant β, is determined in terms of the total given energy U and volume V of the box of photons in thermal equilibrium. The quantity U follows from an integral over the derived phase-space density (33). First, as an example of such an integral, we calculate the total number of particles N in the box. Finally, we relate β to the thermodynamic temperature.

Number of particles N in volume V

The phase-space density f is the number of particles in unit phase space volume. To obtain the number N in volume V of ordinary space, we must multiply f by the phase-space volume Θ_s of an energy state (32), and then integrate over all states (shells) in momentum space and also over the appropriate physical volume. We allow energy states out to infinite energy. Thus,

$$N = \int_0^\infty \frac{2}{h^3\,(\mathrm{e}^{\beta u_s}-1)}\,4\pi\,p_s^2\,\mathrm{d}p_s \int_V \mathrm{d}V \qquad \text{(Number of particles in vol. } V) \qquad (36)$$

Invoke $u_s = p_s c$ for photons, and then change variables, $x \equiv \beta u_s$,

$$N = \frac{8\pi\,V}{h^3c^3}\int_0^\infty \frac{u_s^2\,\mathrm{d}u_s}{(\mathrm{e}^{\beta u_s}-1)} = \frac{8\pi\,V}{h^3c^3\beta^3}\int_0^\infty \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x-1} \qquad (37)$$

The integral is the Riemann zeta function $\zeta(3) = 1.202$ (A11), multiplied by the gamma function $\Gamma(3) = 2!$, or 2.404. . . . The number density of photons in physical (ordinary) space is thus

$$\Rightarrow \quad \frac{N}{V} = 2.404\,\frac{8\pi}{h^3 c^3 \beta^3} \qquad \text{(Number density in } x, y, z \text{ space; m}^{-3}\text{)} \tag{38}$$

If we again adopt $\beta = (kT)^{-1}$, this agrees with the density derived directly from the Planck function; see (6.31) of AP.

Radiative energy content in volume V

To obtain the total radiative energy content in the box, we multiply the phase space density by the energy u_s of each photon and then integrate over the appropriate phase-space volume as above,

$$U = \int_0^\infty u_s \frac{2}{h^3\,(e^{\beta u_s} - 1)}\, 4\pi p_s^2\, dp_s \int_V dV = \frac{8\pi\, V}{h^3 c^3 \beta^4} \int_0^\infty \frac{x^3\, dx}{e^x - 1} \tag{39}$$

where we again invoked $u_s = p_s c$ and $x = \beta u_s$. The integral is seen (A12) to be equal to $\zeta(4)\Gamma(4) = \pi^4/15$. Thus,

$$\Rightarrow \quad \frac{U}{V} = \frac{8\pi^5}{15\, h^3 c^3 \beta^4} \qquad \text{(Radiative energy content in box; J)} \tag{40}$$

Again, if $\beta = (kT)^{-1}$, this result yields energy density $u = U/V = aT^4$ in agreement with (6.25) and (6.26) of AP.

The result (40) demonstrates that β is a function only of the energy density in physical space, U/V. For a given size box V and a given energy content U, the constant β can be determined directly from (40). (Note that one could use this expression for β to eliminate β in (38) to determine N as a function of U and V, two given constants of the system.) We have thus determined the Lagrangian constant β, the final unknown in our problem.

This completes our Lagrangian problem. But in thermodynamics, the energy U is related to temperature. We have yet to connect β to thermodynamic temperature. We do so next after a momentary diversion.

Average energy of a photon

As an aside, the average energy of a photon follows directly from the above. The quotient U/N, from (38) and (40), is

$$\Rightarrow \quad \frac{U}{N} = \frac{\pi^4}{15 \times 2.404}\,\frac{1}{\beta} = 2.701\,\frac{1}{\beta} \qquad \text{(Mean energy of a photon; J)} \tag{41}$$

This is the result obtained in AP (6.32), again if $\beta = (kT)^{-1}$.

Thermodynamic temperature

We still do not know how β connects with the temperature T of the material surrounding or intermingling with the photons, namely, the atoms in the walls of the box or the particle gas sharing the same space with the photons. To find this connection, we invoke two new concepts. First we multiply the function $\tilde{S} = \ln W$ (13) by the Boltzmann constant k and define this to be the *entropy* S

$$\Rightarrow \qquad S \equiv k \ln W \qquad \text{(Entropy; statistical definition)} \qquad (42)$$

Here, the entropy is a function of the number of ways of arranging the cells, and it is maximum for the most probable arrangement, the expected state in thermodynamic equilibrium.

Secondly we introduce the thermodynamic relation for entropy,

$$T\,\mathrm{d}S = \mathrm{d}U + P\,\mathrm{d}V \qquad (43)$$

where $U, P, V, T,$ and S are all state variables: U is the internal energy of the gas, P the pressure, V the volume, and T the temperature. The expression (43) follows from the first law of thermodynamics $\delta Q = \mathrm{d}U + \delta W$, (2.84) of AP, where the heat element is $\delta Q \equiv T\,\mathrm{d}S$ and the work element is $\delta W = P\mathrm{d}V$. (The former serves as the thermodynamic definition of S.) Hold V constant ($\mathrm{d}V = 0$) in (43) to obtain the thermodynamic definition of temperature,

$$\Rightarrow \qquad \frac{1}{T} \equiv \left(\frac{\partial S}{\partial U}\right)_V \qquad \text{(Thermodynamic definition of temperature)} \qquad (44)$$

where $S = k \ln W$ is evaluated at its maximum.

At the maximum of $\tilde{S} = \ln W$, and hence at the maximum of $S = k \ln W$, the values of the z_{sm} were found to be, repeating (27),

$$z_{s\,m} = \frac{Z_s}{\sigma_s}\,\mathrm{e}^{-m\beta u_s} \qquad (45)$$

where Z_s is the number of cells in the energy state s, and σ_s is the summation of exponentials $\exp(-m\beta u_s)$ over all possible m for state s (25). See Prob. 42 for an illustrative example of the cell labels z_{sm} in equilibrium.

Substitute the equilibrium z_{sm} (45) into the expression (14) for $\tilde{S}$, use the definitions of S and $\tilde{S}$ to write $S = k\tilde{S}$, and finally apply the expressions for σ_s (25) and the total energy U (17). Simplify and rearrange terms, and find (Prob. 43) that, at equilibrium,

$$\Rightarrow \qquad S = k\beta\, U + k \sum_s Z_s \ln \sigma_s \qquad \text{(Entropy at its maximum)} \qquad (46)$$

Here σ_s (25) may be written with the aid of the limit of a geometric progression (A15) as

$$\sigma_s \equiv \sum_{m=0}^{\infty} e^{-m\beta u_s} = \frac{1}{1 - e^{-\beta u_s}} \tag{47}$$

Note that these expressions apply to a gas with one kind of photon, e.g., right handed; this was the context within which the Lagrangian constant β was invoked.

Now, evaluate the partial derivative $(\partial S/\partial U)_V$ (44) by taking the partial derivative of (46). Recall from (40) that β is a function of U and V, but that we hold V fixed. Also recall that the number of cells Z_s in a given energy state is fixed as is the energy of a photon in energy state $u_s = h\nu_s$,

$$\left(\frac{\partial S}{\partial U}\right)_V = k\beta + k\left(\frac{\partial \beta}{\partial U}\right)_V \left[U + \sum_s Z_s \frac{1}{\sigma_s}\frac{d\sigma_s}{d\beta}\right] \qquad \text{(Determining } T\text{)} \tag{48}$$

Evaluate the last product in the summation. Use (47) for σ_s, take the derivative with respect to β, and then apply (30) for N_s/Z_s, for a gas with one kind of photon (e.g., right handed),

$$\frac{d\sigma_s}{\sigma_s\, d\beta} = -\frac{u_s}{e^{\beta u_s} - 1} = -\frac{u_s\ N_s}{Z_s} \tag{49}$$

Thus,

$$\sum_s Z_s \frac{d\sigma_s}{\sigma_s\, d\beta} = -\sum_s u_s\ N_s = -U \tag{50}$$

Substitute back into (48), and surprisingly, the term in brackets vanishes,

$$\left(\frac{\partial S}{\partial U}\right)_V = k\beta \tag{51}$$

Finally compare this to the thermodynamic definition of temperature $T^{-1} = (\partial S/\partial U)_V$ (44) to obtain

$$\Rightarrow \qquad \beta = \frac{1}{kT} \qquad \text{(Lagrangian constant determined)} \tag{52}$$

This is the desired relation between the Lagrangian constant β and the thermodynamic temperature.

Another gas of a different kind of photon (left handed) would give rise a similar distribution function and the same value of β. The two gases can coexist and be in thermal equilibrium with one another, and photons of both kinds can occupy the same cells. Each distributes itself as if the other kind did not exist. (The total energy content must of course be conserved.) The combined gas is described by the phase-space density $f(33)$ for which we have now found $\beta(T)$.

Finally, substitute $\beta = 1/kT$ into the specific intensity (35) to obtain the Planck function (specific intensity) given in the text AP (6.6), the function we set out to derive.

5 Massless fermions

The derivation given here can be extended simply to obtain the distribution function for massless fermions in thermal equilibrium. Neutrinos are fermions with spin 1/2. They were long thought to have zero rest mass (like photons), but recent observations suggest otherwise. This distribution function would apply only if they are in thermal equilibrium with their environment and if their energies far exceed their rest energies mc^2.

Since only 0 or 1 particle of a given type (spin state) can reside in a given cell, the summations in the derivation for are limited to $m = 0, 1$. If one works this out (Prob. 51), one obtains an expression identical to that for massless bosons (33) except that it has a plus sign in the denominator,

$$\Rightarrow \quad f_\mathrm{f} = \frac{2}{h^3\left(e^{\beta u_s}+1\right)} = \frac{2}{h^3\left(e^{u/kT}+1\right)} \qquad \text{(Distribution function; massless fermions; two spin states)} \qquad (53)$$

where $u_s = u = pc$ for a continuum of states. One can show here also that $\beta = (kT)^{-1}$ (Prob. 51).

One could go on to derive the specific intensity $I(u_s,T)$ as a function of the neutrino energy $u_s = pc$, and the other functions that follow in turn (Prob. 52).

This distribution has cosmological relevance. Standard cosmology indicates that the early universe ($t \lesssim 1$ s after the big bang) contained high densities of neutrinos in thermal equilibrium with particles (electrons, muons, neutrons, protons, etc.) at temperatures $\gtrsim 10^{10}$ K ($kT \gtrsim 1$ MeV). Since the rest energies of neutrinos are generally considered to be of order 1 eV or less, such neutrinos are highly relativistic; their rest energies are negligible compared to 1 MeV. Hence, our distribution (53) would apply.

After about $t = 1$ s, the universe is transparent to all flavors of neutrinos, and they are free to travel unimpeded through the universe. As massless particles, they would lose energy as the universe expands while preserving their Fermi-Dirac spectrum. [The proof of this statement for photons is in Section 6.3 of AP.] Today, we would find ourselves in a bath of neutrinos with temperature ~2 K ($kT \approx 2 \times 10^{-4}$ eV). If the neutrinos have masses of order 1 eV, they would be non-relativistic at this point. Unfortunately, the low energies of these cosmological neutrinos have precluded successful detection to date.

6 Particles with mass (Bose-Einstein and Fermi-Dirac statistics)

If the particles have a finite mass and there is no particle creation or destruction, one holds constant the total number of particles, $N = \sum N_s$ = constant. This adds another constraint and hence another Lagrangian multiplier α which is distinct from the α_s used above. The result for bosons (Prob. 61) is known as the Bose-Einstein distribution function, for one spin state,

$$\Rightarrow \qquad f_{\text{B-E}} = \frac{1}{h^3\left(e^{\alpha+\beta u_s}-1\right)} \qquad \text{(Bose-Einstein distribution function; bosons of finite mass)} \qquad (54)$$

Similarly, for fermions, the Fermi-Dirac distribution function is,

$$\Rightarrow \qquad f_{\text{F-D}} = \frac{1}{h^3\left(e^{\alpha+\beta u_s}+1\right)} \; . \qquad \text{(Fermi-Dirac distribution function; fermions of finite mass)} \qquad (55)$$

The parameter β again turns out to equal $(kT)^{-1}$, the energy state u_s equals the particle energy E, while α turns out to be a complicated function of particle density and temperature $\alpha(n,T)$. The mathematics required to obtain α (and also β) is quite complex. The expression (55) is given and discussed in AP; see (3.51).

7 Appendix: required mathematics

Here we present some mathematical formulae that are used in this Supplement or are required for the problems.

Stirling's Approximation and Gamma function

The following relation, known as Stirling's Approximation, is valid for a very large integer N:

$$\ln N! \approx N \ln N - N \qquad \text{(Large } N\text{; Stirling's formula)} \qquad (A1)$$

where ln is the natural logarithm and $N!$ is the *factorial* of N,

$$N! \equiv N\,(N-1)\,(N-2)\,.\,.\,.\,.\,.\,1 \text{ where } 0! = 1 \qquad \text{(Factorial)} \qquad (A2)$$

Examples of factorials:

$$\begin{aligned} 4! &= 24 \\ 3! &= 6 \\ 2! &= 2 \\ 1! &= 1 \\ 0! &= 1 \end{aligned} \qquad \text{(A3)}$$

The *generalized factorial function* is known as the *Gamma function,* $\Gamma(N)$

$$\Gamma(N) \equiv (N-1)! \qquad \text{(Gamma function)} \qquad \text{(A4)}$$

Permutations

A permutation is a way of arranging a number of objects into a given number of positions. Often, in statistical mechanics, one needs the total number of possible permutations of a set of objects. Here we consider the cases where (*i*) all available objects are distinguishable from one another and (*ii*) where the objects in each of one or more subgroups are identical to one another.

Distinguishable objects

The number of ways to order N distinguishable objects is obtained as follows. Consider a line of N positions. In the leftmost position one may place any one of the N objects. There are thus N ways to fill the leftmost position. For each of these, one can choose one of the $N–1$ remaining objects to place in the next position (to the right of the first). Thus there are $N(N-1)$ ways to fill the first two positions. For each of these ways, there are $N–2$ objects available for the next position; hence there are $N(N–1)(N–2)$ ways to fill the first three positions. The number of ways to arrange all N objects $W_p(N)$ is thus N! Each such arrangement is called a permutation,

$$W_p(N) = N! \qquad \text{(A5)}$$

As an example, consider four objects a,b,c,d and four positions. The number of permutations is $W_p = 4! = 24$; they are shown in Table 2.

Table 2. Permutations with 4 distinguishable objects

abcd	abdc	acbd	acdb	adbc	adcb
bacd	badc	bcad	bcda	bdac	bdca
cabd	cadb	cbad	cbda	cdba	cdab
dabc	dacb	dbac	dbca	dcab	dcba

Identical subgroups of objects

Now, consider N objects of which a subset N_1 are identical, but let the latter objects have temporary tags so they can be distinguished. Each one of the $N!$ permutations of the entire set of N objects contains the N_1 objects in various locations and ordering. For example, here is a the sequence of $N = 13$ objects where c_{1-3} are identical,

$$\text{a b c}_1\text{ d e f g c}_2\text{ c}_3\text{ h i j k} , \tag{A6}$$

Since there are three identical objects, there are $N_1! = 3! = 6$ ways to rearrange them in the same three positions within the sequence.

$$\begin{array}{l} \text{a b c}_1\text{ d e f g c}_2\text{ c}_3\text{ h i j k} \\ \text{a b c}_1\text{ d e f g c}_3\text{ c}_2\text{ h i j k} \\ \text{a b c}_2\text{ d e f g c}_1\text{ c}_3\text{ h i j k} \\ \text{a b c}_2\text{ d e f g c}_3\text{ c}_1\text{ h i j k} \\ \text{a b c}_3\text{ d e f g c}_1\text{ c}_2\text{ h i j k} \\ \text{a b c}_3\text{ d e f g c}_2\text{ c}_1\text{ h i j k} \end{array} \tag{A7}$$

If one removes the subscripts from the "c" objects, these 6 patterns become identical and would not count as a distinct permutations.

Any other permutation of the 13 objects will contain the same three 'identical' objects c_{1-3}, but in different positions. Again there will be 6 indistinguishable arrangements. The entire set of $N! = 13! = 6.2 \times 10^9$ permutations can thus be grouped in sets of six, each of which has 6 indistinguishable permutations. In other words, $N!$ overcounts the distinct permutations by a factor of six. The number of distinct permutations is thus $N!$ divided by $N_1!$

$$W(N; N_1) = N!/N_1! \tag{A8}$$

or $13!/3! = 1 \times 10^9$ in our example.

If another group of N_2 objects are identical and distinct from the N_1 identical objects, they would occur in all possible arrangements in the $N!/N_1!$ arrangements just arrived at. Since they are identical, again by the same logic, the number of distinct permutations would again be over counted by the factor $N_2!$, and the correct number is $N!/(N_1!\, N_2!)$. In general, the number of distinct permutations of N objects, of which $N_1, N_2, N_3 \ldots$ .are subgroups of identical objects, is,

$$W(N; N_1, N_2, N_3 \ldots) = \frac{N!}{N_1!\, N_2!\, N_3! \ldots} . \qquad \text{(Number of distinct permutations)} \tag{A9}$$

This may be further illustrated with our example of 4 objects (Table 2 above). Let object 'c' be identical to 'd'. Thus the number of distinct permutations should be W(4; 2) = 4!/2! = 12. To test this, replace each 'd' with a 'c' as in Table 3, look for identical pairs, and find that there are 12 of them, and mark one of each of the duplicates with an 'X' to indicate it is redundant. There are 12 remaining distinct permutations as expected.

*Table 3. Permutations with 4 objects of which two are identical (**X** indicates duplicate permutation)*

abcc	abcc **X**	acbc	accb	acbc **X**	accb **X**
bacc	bacc **X**	bcac	bcca	bcac **X**	bcca **X**
cabc	cacb	cbac	cbca	ccba	ccab
cabc **X**	cacb **X**	cbac **X**	cbca **X**	ccab **X**	ccba **X**

Riemann Zeta function

The Riemann zeta function $\zeta(z)$ is defined as

$$\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \text{(Zeta function)} \qquad \text{(A10)}$$

One can find $\zeta(z)$ tabulated in various books,

z	*ζ(z)*
1	diverges
2	1.644 934....≡ $\pi^2/6$
3	1.202 057....
4	1.082 323.. ≡ $\pi^4/90$

(A11)

It can be shown that, for $Re(z) > 1$,

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{x^{z-1}}{e^x - 1}\,dx \qquad \text{(A12)}$$

where $\Gamma(z) \equiv (z-1)!$ is the Gamma function defined above (A4). This enables us to evaluate integrals of the type (A12).

Roots of a transcendental equation.

The following transcendental equation is useful for the determination of the maximum of the blackbody distribution function.

$$\frac{x_n}{1 - e^{-x_n}} = n \qquad (n = 2, 3, 4\ldots) \qquad \text{(A13)}$$

It can be shown to have roots, x_n,

n	x_n
1	no solution
2	1.593 624
3	2.821 439
4	3.920 690
5	4.965 114

(A14)

Geometric series with exponential multiplier

A geometric series is a sum of terms, where each successive term is reduced by a common factor $r < 1$, i.e., $a + ar + ar^2 + ar^3 + \ldots\ldots$ The sum σ of such a series carried out for an infinite number of terms converges if $|r| < 1$ to $\sigma = a/(1-r)$. For the case of $a = 1$ and $r = \exp(-x)$, the sum may be written,

$$\sigma \equiv \sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}} \qquad (x > 0) \qquad \text{(A15)}$$

The partial derivative with respect to x yields the function we call $\tilde{\sigma}$ in (29),

$$\tilde{\sigma} = -\frac{\partial\sigma}{\partial x} = \sum_{n=0}^{\infty} ne^{-nx} = \frac{e^{-x}}{\left(1-e^{-x}\right)^2}\;. \qquad \text{(A16)}$$

The ratio of A15 and A16 is

$$\frac{\tilde{\sigma}}{\sigma} = -\frac{\partial}{\partial x}(\ln\sigma) = \frac{1}{e^x - 1} \qquad \text{(A17)}$$

which was used in (30) above.

Integrals for fermions

Integrals of the type,

$$I_n = \int_0^\infty \frac{x^n}{e^x + 1}\,dx; \quad n = 1, 2, 3, \ldots \qquad \text{(A18)}$$

can be expressed in terms of the Riemann zeta function defined above (A10). It can be demonstrated that

$$I_n = n!\left(1 - \frac{1}{2^n}\right)\zeta(n+1) \quad . \qquad \text{(A19)}$$

Tabulated values of ζ and I are,

n	$\zeta(n+1)$	I_n
1	1.644 934	0.822 467
2	1.202 057	1.803 086
3	1.082 323 2	5.682 197
4	1.036 927 8	23.330 87
5	1.017 343 1	118.366 14

(A20)

Transcendental equation for fermions

Transcendental equations of the type,

$$\frac{x_n}{1 + e^{-x_n}} = n \ , \qquad (n = 2, 3, 4 \ldots) \qquad \text{(A21)}$$

have the following roots.

n	x_n
2	2.217 715
3	3.131 020
4	4.068 418
5	5.032 609
6	6.014 656

(A22)

These are needed for finding the maximum of the distribution function of the massless fermions.

END

Problems

4 Massless bosons

Problem 41. Find the relation (34) between the specific intensity $I_\nu(\nu)$ (W m^{-2} Hz^{-1} sr^{-1}) and the distribution function $f(\boldsymbol{x}, \boldsymbol{p})$ (representative points per unit volume of phase space with units m^{-3} (N s)$^{-3}$). Follow the suggestions in the text. Take the volume element in momentum space to be $p^2\,\mathrm{d}\Omega_\mathrm{p}\,\mathrm{d}p$ where $\mathrm{d}\Omega_\mathrm{p} = \mathrm{d}\Omega$.

Problem 42. The character of the most probable global macrostate is defined by the values of z_{sm} associated with it, i.e., the numbers of cells containing $m = 0, 1, 2\ldots$ photons in energy states $s = 1, 2, \ldots$ Use (27) with $\beta u_s = h\nu_s/kT$ to calculate the numerical values of the z_{sm} for the situation of Fig. 6.6 of AP. For each energy state, use the photon energy $h\nu_s$ and the number of cells Z_s from the figure as inputs. Verify that your answers for each energy state are roughly in accord with the distribution within the energy state and with the total number N_s of photons in the energy state. If you did Problem 6.25 of AP, describe how this problem differs from it.

Problem 43. Show that the entropy $S = k \ln W$ for the most probable global macrostate may be written as given in (46). Pattern your logic after that leading up to (14), invoke the equilibrium values of z_{sm} given in (45), and heed the suggestions preceding (46).

5 Massless fermions

Problem 51. (a) Find the distribution function $f_\mathrm{f}(u_s,\beta)$ (53) for a gas of massless fermions with two spin states, e.g., neutrinos if they were massless, in thermodynamic equilibrium. Follow the derivation of the distribution function for massless bosons to arrive at an expression for N_s, comparable to (31), and then find f_f, comparable to (33). Note that the summations over m are all included in σ_s and $\tilde{\sigma}_s$ and that these summations should include only $m = 0$, and 1 for Fermi-Dirac statistics. (b) Find the relation between the quantity β to the thermodynamic temperature T. Follow the logic in the text to obtain $f_\mathrm{f}(u,T)$ (53). Again it is helpful that the summation over m is contained solely in the quantity σ_s.

Problem 52. According to standard cosmology, the universe is currently filled with a bath of neutrinos with a thermal spectrum of temperature $T\approx 2$ K. Assume (prossibly incorrectly) that these neutrinos may be treated as massless fermions. The energy of a neutrino in state s is thus $u_s = pc$. For a continuum of energy states, let the neutrino energy $u_s = u$. From the distribution function $f_\mathrm{f}(u,T)$ (53), find the requested characteristics of this radiation. You may follow the development in Section 6.2 of AP for massless bosons (photons). (a) Use f_f (53) to obtain the specific intensity $I_\mathrm{u}(u,T)$ in energy units (W m^{-2} J^{-1} sr^{-1}), i.e. the energy flux of neutrinos per unit energy interval (*not* per frequency interval). (b) How does $I_u(u,T)$ depend on energy u at low energies $u \ll kT$ and at high energies $u \gg kT$? Use these results to make qualitative sketches of log $I_u(u,T)$ vs. log u for three temperatures $T_1 < T_2 < T_3$. Can one make a determination of the neutrino temperature by means of a single measure of I_u at some energy $u_0 \ll kT$ as one can for photons? (c) By what numerical factor does the neutrino energy at the maximum of $I_u(u,T)$ differ from the energy kT? Use (A21). (d) Integrate the intensity function $I_u(u,T)$ over energy u to obtain expressions for (*i*) the integrated directional energy flux $I(T)$ (W m^{-2} sr^{-1}) and (*ii*) the integrated directional *particle* flux $\mathcal{N}(T)$ (particles s^{-1} m^{-2} sr^{-1}). (e) From your answers to (d), what is the average neutrino energy u_{av}? (f). For a cosmological neutrino bath at $T = 2$ K, calculate u_{av} in joules and eV, and also the integrated intensity $I(T)$ from part (d) in W m^{-2} sr^{-1}.

[Ans. (a) $\frac{2}{h^3c^2}\;\frac{u^3}{e^{u/kT}+1}$; —; (c) 3.131; (d) $\sim 10\,\frac{(kT)^4}{c^2h^3}$; $\sim 4\,\frac{(kT)^3}{c^2h^3}$; (e) ~3 kT; (f) $\sim 10^{-3}$ eV, $\sim 10^{-7}$ W m^{-2} sr^{-1}]

6 Particles with mass (Bose-Einstein and Fermi-Dirac statistics)

Problem 61. Derive the distribution function $f_{\rm B\text{-}E}$ for bosons of finite mass (54) by following the derivation in the text for photons while requiring that the total number of particles be conserved, $N = \sum N_s$ = constant. This will introduce an additional Lagrange multiplier; designate it α (distinct from α_s). Follow the logic used for massless particles. Do not try to evaluate the Lagrange multipliers α and β.