

A Brief Glossary of Algebraic Structures

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Background material for
Digital Integrated Circuit Design,
from VLSI Architectures to CMOS Fabrication
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Book chapter 4, appendix I, pp130...133

Why this glossary?

Many mathematical texts on abstract algebra explain algebraic structures recursively, e.g.

*A **ring** [with unity] is a set R with two binary operations, usually called addition (+) and multiplication (\cdot), such that R is an **Abelian group** under addition, a **monoid** under multiplication, and such that multiplication is both left- and right-distributive over addition.*

For newcomers to the field such “explanations” just replace one unknown term by several others. So let us break away from that tradition here and present the facts and connections in the most straightforward manner.

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An algebraic structure is defined by

- ▶ a set of elements S
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 - ▶ one or more operations (denoted $\boxplus, \boxminus, \dots$)
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- ▶ The nature of the operations involved determines which axioms are satisfied and which are not.
 - ▶ Each algebraic structure so defined is given a name as a function of the axioms found to hold.

Axioms I

Consider a set of elements S and a first binary operation \boxplus (zweistellig)

1. Closure wrt \boxplus :

if a and b are in S then $a \boxplus b$ is also in S .

2. Associative law wrt \boxplus :

$$(a \boxplus b) \boxplus c = a \boxplus (b \boxplus c).$$

3. Identity element wrt \boxplus (“zero”): (Neutralelement der Addition)

There is a unique element e such that

$$a \boxplus e = e \boxplus a = a \text{ for any } a.$$

4. Inverse element wrt \boxplus :

For every a in S there is an inverse $-a$ such that

$$a \boxplus -a = -a \boxplus a = e.$$

5. Commutative law wrt \boxplus :

$$a \boxplus b = b \boxplus a.$$

Axioms II

Consider a second binary operation \square that takes precedence over \boxplus

6. Closure wrt \square :
if a and b are in S then $a \square b$ is also in S .
7. Associative law wrt \square :
 $(a \square b) \square c = a \square (b \square c)$.
8. Identity element wrt \square (“unity”): (Neutralelement der Multiplikation)
There is a unique element i such that
 $a \square i = i \square a = a$ for any a .
9. Inverse element wrt \square :
For every a in S there is an inverse a^{-1} such that
 $a \square a^{-1} = a^{-1} \square a = i$, the only exception is e
for which no inverse exists.
10. Commutative law wrt \square :
 $a \square b = b \square a$.

Axioms III

... continued

11. Distributive law of
- \square
- over
- \boxplus
- :

$$a \square (b \boxplus c) = a \square b \boxplus a \square c \text{ and} \\ (a \boxplus b) \square c = a \square c \boxplus b \square c.$$

12. Distributive law of
- \boxplus
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$$a \boxplus b \square c = (a \boxplus b) \square (a \boxplus c) \text{ and} \\ a \square b \boxplus c = (a \boxplus c) \square (b \boxplus c).$$

13. Complement:

For every a in S there is a complement \bar{a} such that
 $a \boxplus \bar{a} = i$ and
 $a \square \bar{a} = e.$

Algebraic structures at a glance

Name of algebraic structure	Operations	Axioms satisfied													
		1	2	3	4	5	6	7	8	9	10	11	12	13	
Set (Menge)															
Semigroup (Halbgruppe)	⊞	1	2												
Monoid	⊞	1	2	3											
Group (Gruppe)	⊞	1	2	3	4										
Abelian or commutative group	⊞	1	2	3	4	5									
Abelian semigroup	⊞	1	2			5									
Abelian monoid	⊞	1	2	3		5									
Ring	⊞⊠	1	2	3	4	5	6	7					11		
Ring with unity	⊞⊠	1	2	3	4	5	6	7	8				11		
Division ring aka skew field	⊞⊠	1	2	3	4	5	6	7	8	9			11		
Field (Körper)	⊞⊠	1	2	3	4	5	6	7	8	9	10		11		
Commutative ring	⊞⊠	1	2	3	4	5	6	7			10		11		
Commutative ring with unity	⊞⊠	1	2	3	4	5	6	7	8		10		11		
Semiring (Halbring)	⊞⊠	1	2			5	6	7					11		
Commutative semiring	⊞⊠	1	2			5	6	7			10		11		
Boolean algebra	⊞⊠	1	2	3		5	6	7	8		10		11	12	13

Examples of algebraic structures with one operation

- ▶ The set S_{DNA} of all possible DNA sequences of non-zero length with characters from $\{A,T,C,G\}$ together with the binary operation of string concatenation \smile forms an infinite **semigroup** (S_{DNA}, \smile) .

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- ▶ All permutations of a given number of elements form a **group** when combined with binary composition of such permutations as sole operation.

As an example, consider all six distinct rearrangements of three elements:

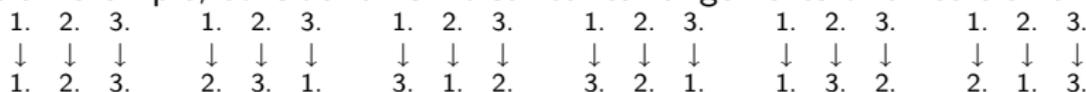
1.	2.	3.	1.	2.	3.	1.	2.	3.	1.	2.	3.	1.	2.	3.	1.	2.	3.
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1.	2.	3.	2.	3.	1.	3.	1.	2.	3.	2.	1.	1.	3.	2.	2.	1.	3.

With S_3 denoting this set of permutations and \circ binary composition, (S_3, \circ) is a finite group.

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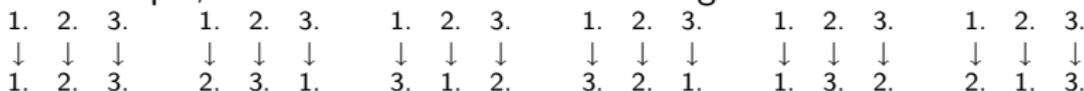
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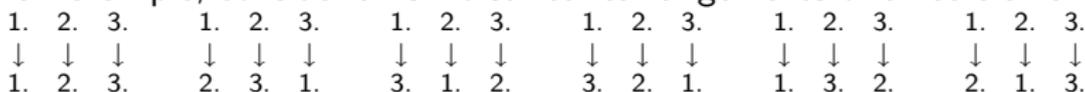
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- ▶ $(\mathbb{Z}, +)$ is an infinite **Abelian group**.

Examples of algebraic structures with two operations I

- ▶ A **commutative ring with unity** $(\mathbb{Z}, +, \cdot)$ results when multiplication is added as a second operation to the Abelian group $(\mathbb{Z}, +)$.

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- ▶ All square matrices $M_{n \times n}$ with coefficients taken from a field together with matrix addition and matrix multiplication form an infinite **ring with unity**.

Examples of algebraic structures with two operations II

- Any subset of integers $S = \{0, 1, \dots, p - 1\}$ forms a field together with addition modulo p and multiplication modulo p iff p is a prime number. Any such **finite field** is called a **Galois field** $\text{GF}(p)$.

$\text{GF}(5)$ ($\{0, 1, 2, 3, 4\}$, $+ \text{ mod } 5$, $\cdot \text{ mod } 5$) is the Galois field for $p = 5$.
Addition and multiplication tables are as follows

$\boxplus = + \text{ mod } 5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\boxdot = \cdot \text{ mod } 5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

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3	3	4	0	1	2
4	4	0	1	2	3

$\boxtimes = \cdot \text{ mod } 5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

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- As opposed to this, ($\{0, 1, \dots, m - 1\}$, $+ \text{ mod } m$, $\cdot \text{ mod } m$) where m is not prime merely forms a finite commutative ring with unity as 0 is not the only element that lacks a multiplicative inverse. In the occurrence of ($\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, $+ \text{ mod } 9$, $\cdot \text{ mod } 9$), this also applies to 3 and 6.

Examples of algebraic structures with two operations III

- ▶ Cardinalities of finite fields are not confined to prime numbers. A so-called **extension field** $\text{GF}(p^n)$ can be defined for any power p^n provided $2 \leq n \in \mathbb{N}^+$.

All polynomials $P(x)$ of degree $0, 1, \dots, n - 1$ with coefficients from $\text{GF}(p)$ constitute the set of elements. The first operation is addition modulo $M(x)$ and the second one multiplication modulo $M(x)$ where $M(x)$ is an irreducible¹ polynomial of degree n with coefficients from $\text{GF}(p)$.

$\text{GF}(3^2)$, for instance, consists of the nine elements
 $0, 1, 2, x, x + 1, x + 2, 2x, 2x + 1, 2x + 2,$
 the operations are $+ \text{ mod } (x^2 + 1)$ and $\cdot \text{ mod } (x^2 + 1)$
 with $M(x) = x^2 + 1$ being an irreducible polynomial².

²Cannot be expressed as a product of non-trivial factors of lower degree. 

Examples of algebraic structures with two operations IV

- ▶ The factors of 30 together with operations least common multiple (lcm) and greatest common divisor (gcd) constitute a **Boolean algebra** of eight elements ($\{1,2,3,5,6,10,15,30\}$, lcm, gcd). Taking the complement \bar{a} is tantamount to computing $\frac{30}{a}$.

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Let $\Omega = \{a,b,c\}$. The set of all sets that can be composed from those elements $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ is called power set $\mathfrak{P}(\Omega)$. $(\mathfrak{P}(\Omega), \cup, \cap)$ forms another Boolean algebra where the empty set \emptyset and the universal set Ω act as identity elements e and i respectively. Each element $x \in \mathfrak{P}(\Omega)$ has a complement $\bar{x} = \Omega - x$.

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- ▶ The well-known **switching algebra** ($\{0,1\}$, \vee , \wedge) is a Boolean algebra with just two elements. The complement of an element is its logic inverse.

Examples of algebraic structures with two operations V

- The class of **semirings** is very broad and encompasses:

Constituent	S	\boxplus	\boxdot
the commutative semiring of natural numbers	\mathbb{N}	$+$	\cdot
the commutative ring with unity of integers	\mathbb{Z}	$+$	\cdot
the "ordinary" fields	$\mathbb{Q}, \mathbb{R}, \text{ or } \mathbb{C}$	$+$	\cdot
all Galois fields, e.g.	$\{0,1\}$	\oplus	\wedge
all other fields, e.g.	$\frac{P(x)}{Q(x)}$	$+$	\cdot
the switching algebra	$\{0,1\}$	\vee	\wedge
other finite Boolean algebras, e.g.	$\{1,2,3,4,6,12\}$	lcm	gcd
all other Boolean algebras, e.g.	$\mathfrak{P}(\Omega)$	\cup	\cap
the path algebras, e.g.	$\{0,1\}$ $\mathbb{R} \cup \{\infty\}$ $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$	max min max	min + ·

Note

Many computational problems can be formulated in a semiring.

Engineering applications

Some of the not-so-common algebraic structures prove extremely helpful for studying real-world phenomena.

- ▶ AC circuits ([field of complex numbers](#)).
- ▶ Digital circuits ([switching algebra](#)).
- ▶ Error correction coding ([Galois fields](#)).
- ▶ Cryptography ([all sorts of algebraic structures](#)).
- ▶ Certain optimization problems ([path algebras](#)).
- ▶ ...