A Flexible Bayesian Ideal Point Estimation Method for Correlated Multidimensional Ideal Points*

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Abstract

It has been known that conventional ideal point estimation methods in social sciences (*e.g.*, the item response theory (IRT) model or the NOMINATE-based methods) fail to recover multidimensional ideal points because of the rotational invariance of multidimensional ideal point estimates. The use of "anchoring" legislators in each dimension to avoid the rotation turns out to be hopeless as the number of dimension increases. Also, misplacement of any of anchoring legislators would be detrimental to inferential outcomes. On the other hand, assuming the orthogonality of each dimension along the principal axes makes it highly difficult to estimate *correlated issue positions* by legislators, parties, coalitions, or countries. In this paper, we present a new Bayesian method to fix the rotational invariance problem in multidimensional ideal point estimation. Our method makes two revisions: (1) we substitute the ℓ_2 norm (Euclidean distance) of existing ideal point estimation models with the ℓ_1 norm (Manhattan distance) and (2) we employ a multivariate slice sampling method to jointly sample multidimensional ideal points (Neal, 2003). Our simulation studies show that the proposed method successfully recovers multidimensional ideal points with a varying degree of correlation. We apply the proposed method to the analysis of the United Nations General Assembly (UNGA) roll-call voting data in which voting coalitions are highly flexible along issue dimensions and no clear dimensional information is known to researchers. The results of our analysis show that the post-Cold War UNGA voting has been highly multidimensional and there is strong evidence to support the U.S. as a "lonely superpower."

Keyword: Ideal point estimation, Bayesian analysis, W-NOMINATE, alpha-NOMINATE, IRT model, slice sampling, UNGA

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1 Introduction

Ideal points are a fundamental political science concept upon which the entire work of explaining political behaviors and workings of political institutions has been built. Since Poole and Rosenthal (1985)'s seminal work on the multidimensional scaling method (MDS) for roll-call data analysis (known as the NOMINATE procedure or the NOMINATE model), numerous scholars have developed statistical methods for ideal point estimation. The subsequent methodological endeavors include a Bayesian item response theory (IRT) model (Clinton, Jackman and Rivers, 2004); dynamic extensions of a Bayesian IRT model (Martin and Quinn, 2002; Park, 2011), incorporating text, social media posts, and contribution data to the model (Slapin and Proksch, 2008; Kim, Londregan and Ratkovic, 2018; Barberá, 2015; Bonica, 2013), and improving the estimation efficiency using Expectation-Maximization (EM) algorithm (Imai, Lo and Olmsted, 2016).

However, previous studies have largely focused on the estimation of ideal points in a low (mostly one)-dimensional space, ignoring rich possibilities of multidimensional ideal point estimation.[1] In fact, it has been well known that conventional ideal point estimation methods (*e.g.*, the Bayesian item response theory (IRT) model or the NOMINATE-based methods) fail to recover multidimensional ideal points because of the rotational invariance of multidimensional ideal point estimates. Two easy methods to fix this problem are (1) using "anchoring" legislators in each dimension and (2) assuming the orthogonality of each dimension along the principal axes. However, the use of anchoring legislators in each dimension turns out to be hopeless as the number of dimension increases. Also, misplacement of any of anchoring legislators would be detrimental to inferential outcomes. On the other hand, the second method (assuming the orthogonality of each dimension along the principal axes) makes it highly difficult to interpret estimated ideal points as political issues are not orthogonal and hence it is almost impossible to show *correlated issue positions* by legislators, parties, coalitions, or countries.

The lack of methodological work on multidimensional ideal point estimation is particularly

[1] While there are some published work on multidimensional ideal point estimation such as Jackman (2001) and Jeong (2008), we could not find a paper that presents a principled solution to the multidimensional ideal point estimation problem. A notable exception is an unpublished work of Sohn (2018).

troubling because empirical demands for multidimensional ideal point estimation methods have existed for a long time and become particularly stronger with the rapid reshaping of party politics around the world. Scholars of legislative politics, party politics, and international relations produce new theories that attempt to explain the rise of new issues, shiftings in coalition structure, and the realignment of political parties (*e.g.*, McCarty, Poole and Rosenthal, 1997; Albright, 2010; Miller and Schofield, 2003; Laver and Schofield, 2004; Voeten, 2004; Miller and Schofield, 2008).

Then, we present a flexible Bayesian method for multidimensional ideal point estimation that aims to uncover correlated ideal points from multidimensional space. The main contribution of this paper is to (1) substitute the ℓ_2 norm (Euclidean distance) of existing ideal point estimation models with the ℓ_1 norm (Manhattan distance) and (2) employ a multivariate slice sampling method to jointly sample multidimensional ideal points (Neal, 2003). The proposed method is "flexible" in the sense that (1) it does not require an arbitrary constraint to resolve rotational invariance, (2) it can be applied to any case where dimensional coordinates of ideal points are correlated, and (3) the number of dimensions and dimension weights can be easily estimated.

We show that the proposed method successfully recovers various types of multidimensional ideal points in simulated data. Then, we analyze the United Nations General Assembly (UNGA) roll-call voting data using the proposed method. The findings show that (1) the North–South divide remains robust after the end of the Cold War and (2) the U.S. has been a "lonely superpower" during the post-Cold War period (Voeten, 2004).

2 Problem Statement

In this section, we report that existing ideal point estimation methods do not properly recover multidimensional ideal points. Then, we discuss underlying causes of the problem.

2.1 A Motivating Example

W-NOMINATE is one of the most widely used methods for estimating multidimensional ideal points in applied researches. For example, Voeten (2000) estimates multidimensional ideal points of the member states of the UN using W-NOMINATE on roll-call votes of the UNGA. Specifically, he divided the roll-call votes during the first forty-three plenary sessions of the UNGA (1946-88) and the forty-sixth through fifty-first session (1991-96) into five periods and implement W-NOMINATE for each period.

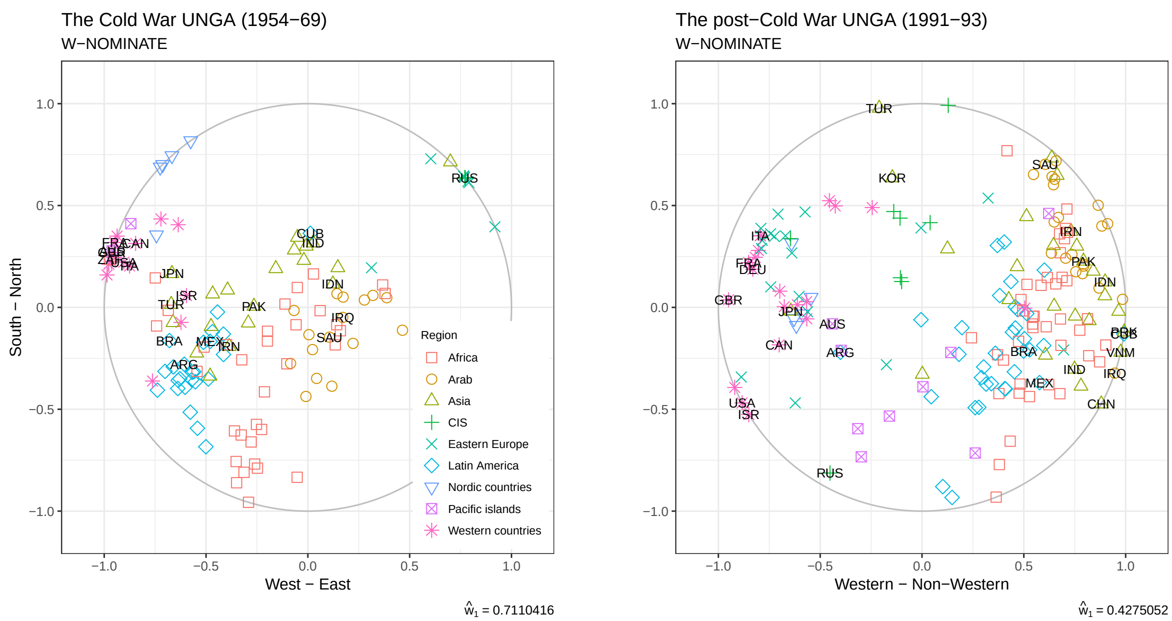

Figure 1: *The UN Member States' Ideal Points Estimated by W-NOMINATE*

Figure 1 replicates Voeten (2000)'s analysis of two of the five periods using R package **wnominate** (Poole et al., 2011). The left plot shows the ideal points of the member states during the Cold War and the right plot shows the ideal points during the post-Cold War. The author argues that the latent dimensions in the Cold War UNGA consist of "East–West" and "North–South" conflict whereas those in the post-Cold War are dominated by "Western–non-Western" division. Based on these results, the study concludes that "the underlying structure of conflict in the post-Cold War UNGA is *one-dimensional*" (Voeten, 2000, 213).

Some critical questions remain about the result. First, how can we interpret the second dimension of the post-Cold War UNGA? The author claims that the second dimension lacks significance in this period; however, lack of significance does not equate to *no-effect*. Second, is East–West cleavage unrelated to North–South conflict? As we will show shortly, NOMINATE-based models fail to recover the true dimensions if they are correlated (*i.e.*, each-dimensional coordinates are correlated). Third, how can we interpret the ideal points of countries on the rim of the unit circle? Regarding this "rim problem" (Rosenthal and Voeten, 2004), some lop-sided votes are removed to reduce the number of ideal points located on the rim, which might be also problematic. These questions are attributed to constraints for identification of NOMINATE-based models.[2]

2.2 Problem

NOMINATE-based models as well as IRT model suffers from rotational invariance (see appendix A). Thus, in order to estimate multidimensional coordinates of the ideal point, some constraints are required to specify the dimensions.

In NOMINATE procedure, the dimensions are arranged in the order of their explanatory power. That is, the first dimension accounts for the largest variability in the data and each succeeding dimension, in turn, accounts for the next highest variability. The method estimates each-dimensional coordinates in turn to implement this constraint. Poole and Rosenthal (1997, 239) describe this procedure as follows: "Our approach bears a family resemblance to eigenvalue/eigenvector decomposition in that we begin by estimating the one-dimensional configuration that best accounts for the data. Keeping this first dimension fixed, we then estimate a second dimension that best accounts for the remaining "variance", and so on, until the desired number of dimensions is estimated."

This constraint bares a hidden assumption: each-dimensional coordinates of ideal points are uncorrelated. In other words, NOMINATE cannot recover true ideal points if each-dimensional coordinates are correlated. In the study of the UNGA, this implies that member states' positions along with East–West and North–South conflict have to be uncorrelated. If a member state that belongs to the Western bloc is likely to be included in the North countries, which is indeed the case in reality, NOMINATE cannot properly recover its ideal point.

Figure 2 illustrates this problem with simulation. Suppose there exist two groups of legislators, group A and B, in two-dimensional space, and each dimension reflects economic and social issue. In this society, the economic issue (*e.g.*, unemployment) is politically entangled with the social issue (*e.g.*, immigration). That is someone who is "conservative" on

[2] The main causes of rim problem are unit circle constraint, perfect legislators and maximum likelihood method. It is beyond the scope of this paper to explain the rim problem in more detail.

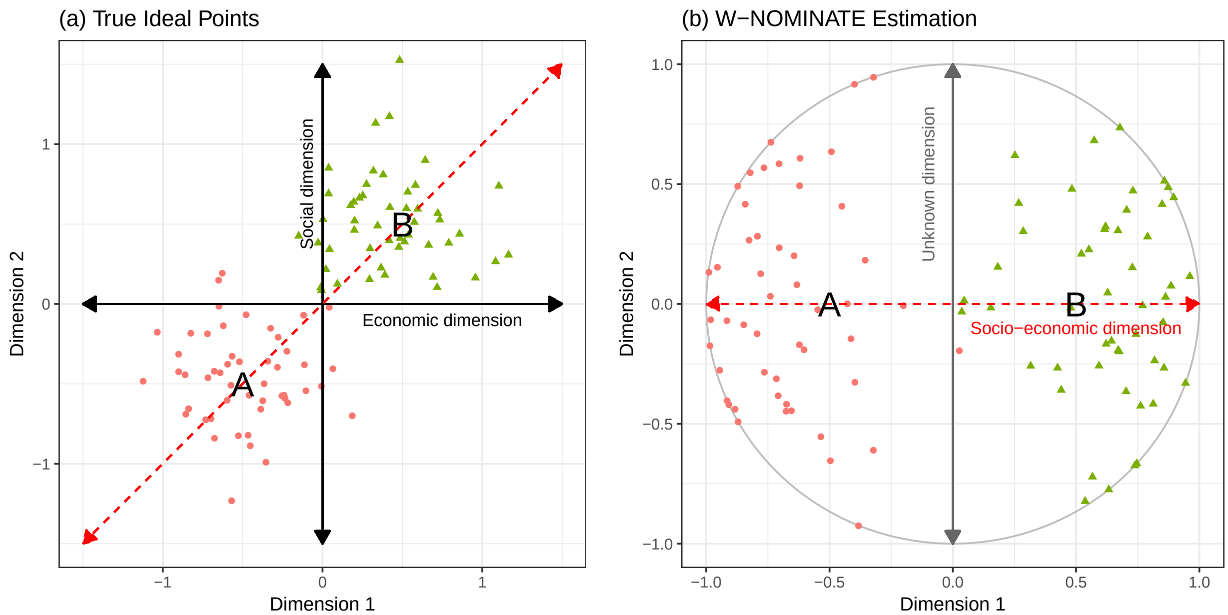

Figure 2: *Description of the Problem of W-NOMINATE*

the economic issue is also "conservative" on the social issue. Accordingly, as we can see in plot (a), the two groups have opposing views on both of the latent dimensions. However, since NOMINATE assumes that the first dimension best accounts for the data and the second dimension accounts for the "remaining variance", the cleavage between group A and B is only reflected in the first dimension of estimates, namely socio-economic dimension (see plot (b)). Consequently, in the case where two dimensions are correlated, three problems arise: (1) the second-dimensional coordinates, as well as the first-dimensional coordinates are highly biased, (2) the two different yet correlated dimensions are lumped together into one dimension, and (3) it is difficult to interpret the underlying issue of second dimension.

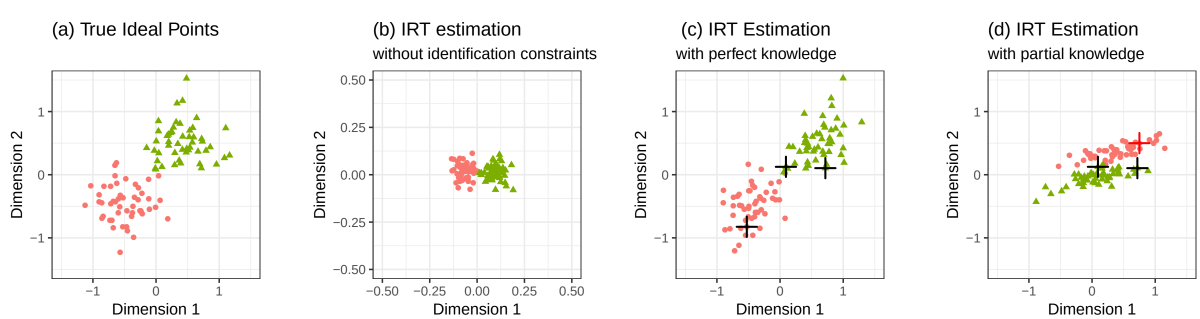

Figure 3: *Description of the Problem of IRT*

In the case of IRT method, the most well-known constraint is to fix the ideal points of $k+1$

legislators in k-dimensional space in advance (Rivers, 2003; Clinton, Jackman and Rivers, 2004). However, if one of $k+1$ legislators is misspecified, the entire result may be distorted. Since fixing the ideological positions of $k+1$ legislator requires sophisticated knowledge about the dimensions, this is also problematic for estimating multidimensional ideal points. Figure 3 illustrates this problem using R package **pscl** (Zeileis, Kleiber and Jackman, 2008). Plot (d) shows that when one of the three legislator is fixed on a wrong position, other legislators' ideal points are also biased. That is, IRT requires an *ex-ante* knowledge of the underlying dimensions to the researcher for the identification.

In conclusion, IRT and NOMINATE methods do not properly recover multidimensional ideal points because they rely on arbitrary constraints to resolve rotational ambiguity. In this paper, we present a new statistical estimation method using Manhattan distance and multivariate slice sampling (MSS) to solve this problem.

3 Multidimensional Ideal Point Method using Manhattan Distance and Multivariate Slice Sampling (MSS)

In this section, we first introduce a new model with the ℓ_1 norm (Manhattan distance) that is exempt from arbitrary constraints for identification. Then, we discuss sampling algorithm of multivariate slice sampling for the model.

3.1 Model

Let N denote the number of legislators and M denote the number of roll-call votes. Each legislator $i = 1, \cdots, N$ votes Yea on roll-call vote $j = 1, \cdots, M$ if the utility of voting Yea (U_{ijy}) is greater than that of voting Nay (U_{ijn}), and vice versa. Let $\mathbf{x}_i$ denote the ideal point of legislator i, $\mathbf{o}_{yj}$ denote the yea position of the roll-call vote j, and $\mathbf{o}_{nj}$ denote the nay position of the roll-call vote j where $\mathbf{x}_i, \mathbf{o}_{yj}, \mathbf{o}_{nj} \in \mathbb{R}^s$. The utility of Yea ($U_{ijy}$) consists of a deterministic part, computed by a weighted ℓ_1 distance between the legislator's ideal point and the yea position, and a stochastic noise ($\epsilon_{ijy} \sim \mathcal{N}(0,1)$) as follows:

$$U_{ijy} = \underbrace{u_{ijy}}_{\text{deterministic part}} + \underbrace{\epsilon_{ijy}}_{\text{stochastic part}} \tag{1}$$

$$u_{ijy} = -\beta \sum_{k=1}^{s} w_k |x_{ik} - o_{yjk}| \tag{2}$$

where β is a signal-to-noise ratio, w_k is a weight of the kth dimension such that $\sum_k w_k = 1$ and $w_k > 0$ for all k, $\cdot$ is inner product, and $|v|$ is an absolute value of v (*i.e.*, the ℓ_1 norm). Note that U_{ijn} is analogous to U_{ijy}.

Let $y_{ij} = 1$ if legislator i votes Yea on the roll-call vote j and $y_{ij} = 0$ otherwise. It follows that

$$\Pr(y_{ij} = 1) = \Pr(U_{ijy} > U_{ijn}) = \Phi(u_{ijy} - u_{ijn}) \tag{3}$$

where $\Phi(\cdot)$ is the cumulative normal function. Then the joint posterior distribution is:

$$p(\cdot|\text{data}) \propto \underbrace{\prod_{i=1}^{N}\prod_{j=1}^{M}\left\{\Phi(u_{ijy} - u_{ijn})^{y_{ij}} \times (1 - \Phi(u_{ijy} - u_{ijn}))^{1-y_{ij}}\right\}}_{\text{likelihood}} \tag{4}$$

$$\underbrace{\times \prod_{i=1}^{N} p(x_i) \times \prod_{j=1}^{M} \{p(o_{yj}) \times p(o_{nj})\} \times p(\beta) \times p(w_1, \cdots, w_s)}_{\text{prior}} \tag{5}$$

We use a Markov chain Monte Carlo (MCMC) method to sample from the joint posterior distribution. we choose normal priors for ideal points, yea positions, and nay positions with diffuse prior for β and w_k's. In the next section, we illustrate the algorithm in more details.

Previous spatial voting models use the ℓ_2 norm as a measure of distance in ideological space.[3] Here, we propose to use the ℓ_1 norm (also known as Manhattan distance, city-block distance, or taxicab geometry) instead of the ℓ_2 norm within the model for the following two reasons.

[3] In W-NOMINATE, the deterministic part of the utility is $u_{ijy} := \beta \exp\left\{-\frac{1}{2}||\mathbf{w} \cdot (\mathbf{x}_i - \mathbf{o}_{jy})||_2^2\right\}$ where $||\mathbf{v}||_2 = \sqrt{\sum_k v_k^2}$ (*i.e.*, the ℓ_2 norm). In IRT model, $u_{ijy} := -||\mathbf{x}_i - \mathbf{o}_{jy}||_2^2$.

First, psychological studies on stimuli show that the ℓ_1 norm is more appropriate than the ℓ_2 norm for similarity scaling with separable dimensions. For example, Shepard (1987) finds that the ℓ_1 norm provides an adequate representation of the "analyzable" stimuli, such as the size and orientation of a shape, whereas the ℓ_2 norm provides that of the "unitary" stimuli, such as the lightness and saturation of a color, with a variety of experiments (see chapter 5 of Garner (2014) for more details). Since "the concept of preference can be reduced to the psychological notion of comparing similarities" (Carroll et al., 2013, 1011) and spatial voting models assume that the legislators may analyze each roll-call vote located in the ideological space with separable dimensions, this result supports the use of the ℓ_1 norm.

Second, the proposed model with the ℓ_1 norm is exempt from arbitrary constraints for identification. The identification problem with rotational invariance in previous models is that when all the ideal points and roll-call votes' positions rotate by the same degree, the ℓ_2 distance between an ideal point and a roll-call vote's position does not change but the order of the ideal points for each dimension changes. To resolve this problem, the existing models required the aforementioned constraints that are also problematic. However, in the proposed model, when all the ideal points and roll-call votes' positions rotate by the same degree, the ℓ_1 distance between an ideal point and a roll-call vote's position changes and so do the likelihood. See appendix B for more details pertaining to identifiability of the proposed model. Consequently, the model does not require arbitrary constraints to specify the dimensions.

3.2 Algorithm

Here, we use the multivariate slice sampling method (Neal, 2003) using hyper-rectangle for the estimation. Slice sampling is a generic method that can be easily implemented even if the full conditional distributions of the parameters do not have a standard form. More importantly, multivariate slice sampling simultaneously updates multiple variables so that it allows to produce an ergodic chain when there are tight dependencies between them.[4] This is

[4] α-NOMINATE (Carroll et al., 2013) uses *univariate* slice sampling which is preferred when the variables are "almost independent."

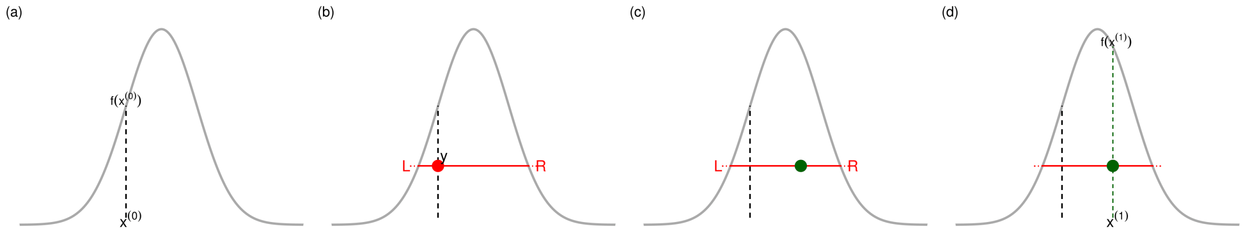

Figure 4: *Single-variable slice sampling*

a key virtue for ideal point estimation since each-dimensional coordinates of an ideal point are dependent.

Figure 4 illustrates the intuition of slice sampling in univariate case.[5] Suppose (a) is the current state to be updated. In (b), a slice (*red line*) of the target distribution is randomly drawn. Then, an interval I is found by gradually expanding a random interval until both ends are outside the slice. Finally, random point is drawn from this interval I, until a point inside the slice is found as shown in (c). This is the new value (*green point*) of current iteration as described in (d), which may be the start point of the next iteration.

Multivariate slice sampling is a straightforward generalization of single-variable slice sampling that updates multiple variables at the same iteration. In MSS method, $(x_{i1}, \cdots, x_{is})$ is sampled at the same time for each i; so are $(o_{jy1}, \cdots, o_{jys})$ and $(o_{jn1}, \cdots, o_{jns})$. For simplicity of notation, suppose $\mathbf{x} = (x_1, \cdots, x_s)$ is a variable to be updated and $\mathbf{x}^{(t+1)}$ is a new sample from the tth iteration. At the tth iteration, the method replaces the current state, $\mathbf{x}^{(t)}$, with a new state, $\mathbf{x}^{(t+1)}$, with the following three-steps (Neal, 2003, 721):

(i) Draw y from $\mathcal{U}(0, f(\mathbf{x}^{(t)}))$, thereby defining a horizontal "slice": $S = \{s : y < f(\mathbf{x})\}$.

(ii) Find a hyper-rectangle $H = (L_1, R_1) \times \cdots \times (L_s, R_s)$ around $\mathbf{x}^{(t)}$ that contains all, or much, of the slice.

(iii) Draw the new point, $\mathbf{x}^{(t+1)} \in A = \{\mathbf{x} : \mathbf{x} \in S \cap H \text{ and } P(\text{Select } H|\mathbf{x}) = P(\text{Select } H|\mathbf{x}^{(t)})\}$.

[5]Single-variable slice sampling can be conducted in the case when variables are "almost independent". Let x denote the single variable being updated and $f(x)$ denote the full conditional distribution of x. This sampling method replace the current value, $x^{(t)}$, with a new value, $x^{(t+1)}$, with the following three-steps at the tth iteration (Neal, 2003, 712-713):

(i) Draw y from $\mathcal{U}(0, f(x^{(t)}))$, thereby defining a horizontal "slice": $S = \{s : y < f(x)\}$.

(ii) Find an interval $I = (L, R)$ around $x^{(t)}$ that contains all, or much, of the slice.

(iii) Draw the new point $x^{(t+1)} \in A = \{x : x \in S \cap I \text{ and } P(\text{Select } I|\text{At state } x) = P(\text{Select } I|\text{At state } x^{(t)})\}$.

The tth iteration with these three-steps can be done as illustrated in the Algorithm 1.

Algorithm 1: tth Iteration of MSS

input : $f(x) =$ the full conditional distribution of x
$\mathbf{x}^{(t)} =$ the current point
$c =$ estimate of the typical size of a slice
output : $\mathbf{x}^{(t+1)} =$ the new point
initialization: $y \leftarrow f(\mathbf{x}^{(t)}) - \text{rexp}(1)$
For $i = 1, \cdots, s$:
$\ell_i \leftarrow x_i^{(t)} - c \times \text{runif}(0, 1)$
$R_i \leftarrow L_i + c$

repeat
  for $i = 1, \cdots, s$ **do**
    $x_i^{(t+1)} \leftarrow L_i + \text{runif}(0, 1) \times (R_i - L_i)$
  if $y < f(\mathbf{x}^{(t+1)})$ **then**
    exit loop
  for $i = 1, \cdots, s$ **do**
    if $x_i^{(t+1)} < x_i^{(t)}$ **then**
      $\ell_i \leftarrow x_i^{(t+1)}$
    else
      $R_i \leftarrow x_i^{(t+1)}$

4 Simulations

Through simulation studies, we show that MSS method successfully recovers the multidimensional ideal points.

4.1 Data Generating Process

We generate the data for simulation as follows. First, we draw the ideal points of legislators from the multivariate normal distribution. Emphasized here is that we assume latent blocs of legislators on the ideological space. This assumption is made because legislators might hold particular combinations of ideologies for several reasons: *e.g.*, correlation of the issues, the majority of the elected holding typical issue positions, parties contrasting against each other to win elections. By adjusting the number of blocs ($\#g$), their center ($\boldsymbol{\mu}_g$), and the degree of dispersion (Σ_g), we conduct simulations based on three possible settings: non-partisanship, two-party system, and multi-party system.

Second, we randomly draw each-dimensional coordinates of yea and nay positions of each roll-call vote from the uniform distribution. We carefully adjust the lower and upper bounds of the distribution so that it coincides with the maximum and minimum of ideal points' coordinates. We made no further assumption on roll-call votes' positions.

Third, based on these ideal points and roll-call votes' positions, we compute each legislator's possibility of voting yea for each roll-call vote. Then, we conduct inverse transform sampling to randomly generate voting with the computed possibility. The entire simulation setup can be summarized as follows:

1. Draw ideal points $x_{g[i]}$ from $\mathcal{N}(\boldsymbol{\mu}_{g[i]}, \Sigma_{g[i]})$, where $g[i] \in \{1, 2, \cdots, \#g\}$ denotes a bloc to which the legislator i belongs and $\Sigma_{g[i]} = \begin{pmatrix} \sigma^2_{g[i]1} & 0 \\ 0 & \sigma^2_{g[i]2} \end{pmatrix}$.
2. Draw each-dimensional coordinates of yea and nay positions o_{yjk}, o_{njk} from $\mathcal{U}(-\theta, \theta)$.
3. Compute $\Pr(y_{ij} = 1)$ for all i and j with $w_1 = w_2 = 0.5, \beta = 1$.
4. Sample γ_{ij} from $\mathcal{U}(0, 1)$. If $\gamma_{ij} < \Pr(y_{ij} = 1)$ set $y_{ij} = 1$, otherwise set $y_{ij} = 0$.

4.2 Simulation Results

In all settings, the number of legislators is fixed at 50 and the number of roll-call votes is set at 500. All samples are obtained from 20,000 MCMC iterations with 10,000 burn-in trials and thinning every 10th draw. Posterior draws passed conventional measures of convergence diagnostics.

We generate synthetic ideal point data from three different settings: “non-partisanship," “two-party system," and “multi-party system." Non-partisanship indicates a case where legislators have uncorrelated ideal points across dimensions. Two-party system indicates a case where two parties strongly pull legislators' vote choices in the opposite direction and hence party medians are clearly identified in both dimensions. Multi-party system indicates a case where two polarized main parties coexist with two additional minor parties that are also distinguished by their own issue positions in both dimensions. The two-party system approximates American politics while the multi-party system represents European politics.

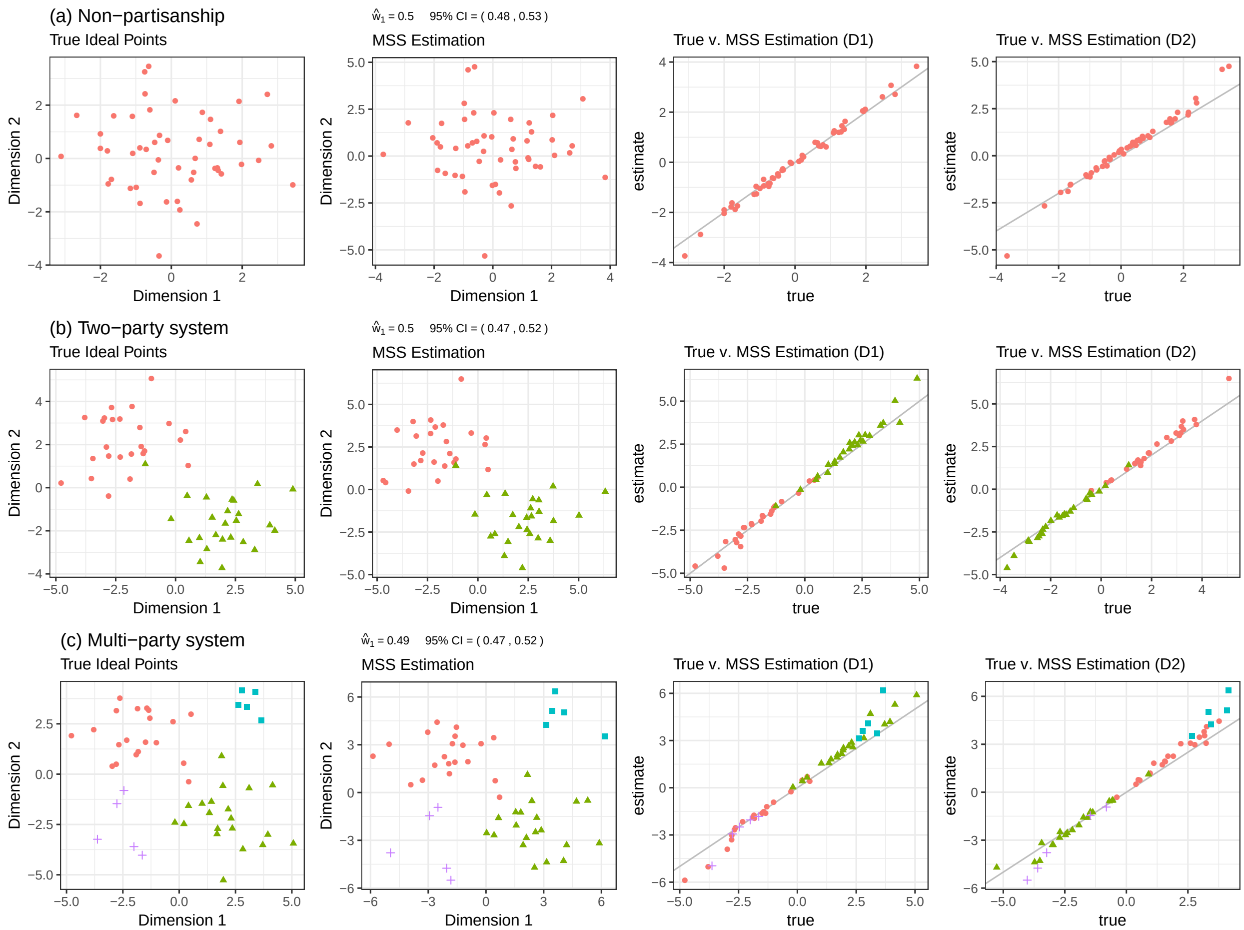

Figure 5: *Simulation Results of MSS*

Figure 5 shows the result of simulation studies.[6] Each point of the first plots is the product of an ideal point and signal-to-noise ratio ($\beta \mathbf{x}_i$); so is each point of the second plots ($\hat{\beta}\hat{\mathbf{x}}_i$). We present these products because the latent ideal points and the size of their effect are at the core of the spatial voting model; also, no additional constraint on the scale of ideal points for identification is required for these products.

MSS method successfully recovers various types of multidimensional ideal points in simulations. In all three cases, the fourth and fifth plots comparing the true ideal points and their estimates on each dimension show a remarkable result; the order of estimates matches well with that of the true values. In particular, the simulation study of two-party system shows that MSS method can be applied in the case where each-dimensional coordinates are

[6] If all the sign of estimated ideal points have flipped (*e.g.*, $(x_{i1}, x_{i2}) \rightarrow (\hat{x}_{i1}, -\hat{x}_{i2})$ for all i) or the order of dimension has changed ((*e.g.*, $(x_{i1}, x_{i2}) \rightarrow (\hat{x}_{i2}, \hat{x}_{i1})$ for all i)), we have adjusted these for ease of comparison. Note that the sign of ideal points and the order of dimensions are meaningless and can be ignored.

correlated. Moreover, the weight of each dimension is successfully recovered as noted in the bottom of second plots. We also conduct simulations with different weights of dimensions ($(w_1, w_2) = (0.7, 0.3)$) and one-dimensional case (*i.e.*, $(w_1, w_2) = (1, 0)$); MSS method still performs well in these cases (see appendix C).

5 Application

In this study, we use the roll-call voting data collected by Voeten (2013). The data consists of all adopted resolutions put to a roll-call vote during the seventy-two sessions of the UNGA (1946-2017). For purpose of comparison with previous studies, we have divided the entire sessions into five periods and have treated abstention as signs of not accepting a resolution as Voeten (2000) did. We conduct two-dimensional MSS estimation to analyze the underlying conflict of the UNGA and its change over time. Here, due to the space constraint, we only state the result of the Cold War period (1954-69) and the post-Cold War period (1991-93).

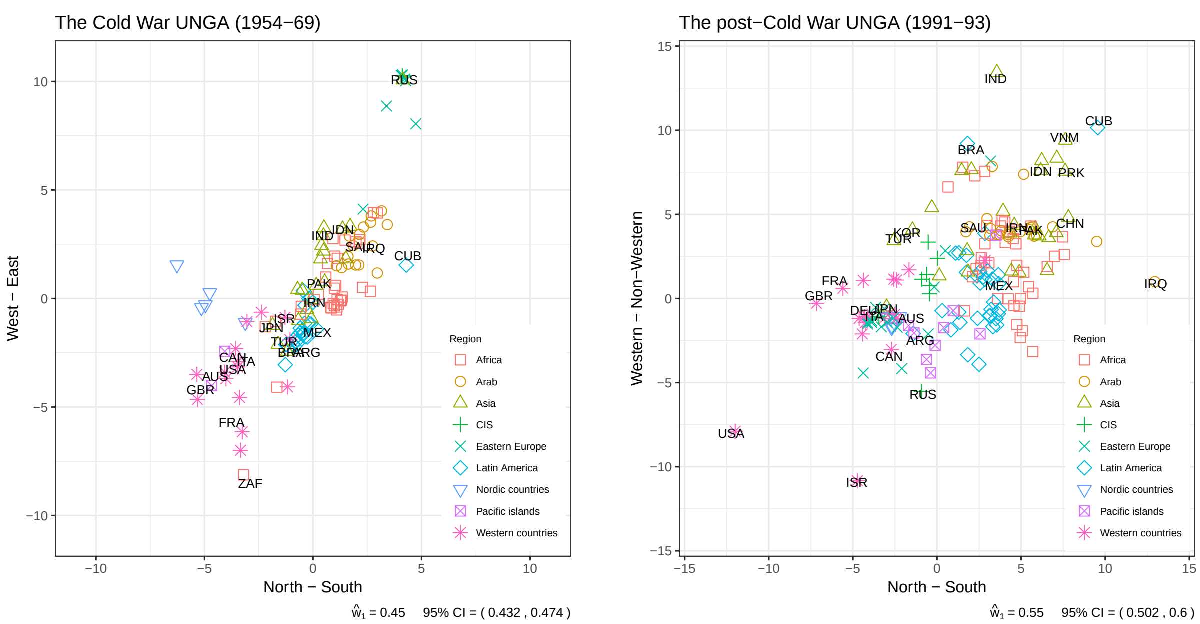

Figure 6: *The UN Member States' Ideal Points Estimated by MSS*

Figure 6 shows the member states' ideal points estimated by MSS for each period. Three points must be emphasized. First, in contrast to Voeten (2000)'s finding, the underlying structure of conflict in the post-Cold War UNGA is multidimensional. As stated in section

2.1, Voeten (2000) argues that the post-Cold War UNGA has been dominated by the Western–non-Western conflict. However, as the right plot of figure 6 shows, the ideal points of member states hardly be mapped into one-dimensional space. If the ideological space is one-dimensional, one of the two weights must be zero in two-dimensional estimation. On the contrary, the estimated weights are $(\hat{w}_1, \hat{w}_2) = (0.55, 0.45)$, which supports multidimensional ideological space.

Second, North–South and Western–non-Western divisions jointly construct conflict of the post-Cold War UNGA. As shown in the right plot of figure 6, Western countries are located in the left and bottom; African countries are located in the right; Asian countries are located in the top. Through these locations, one may interpret the cleavage of the first dimension as North–South conflict and that of the second dimension as Western–non-Western conflict. Moreover, figure 7 shows that the votings related to the Middle East issue are divided horizontally whereas those of other issues such as colonialism or economic development are divided vertically, which supports the interpretation of dimensions.[7] This result refutes the prominence of North–South cleavage (Kim and Russett, 1996) and the dominance of Western–non-Western clash (Voeten, 2000).

Third, the U.S. has become a "lonely superpower" after the Cold War periods. Voeten (2004) demonstrates the notion of the lonely superpower by selected UNGA voting and one-dimensional multilevel IRT model. MSS estimation further elaborates this result by mapping the ideal points into the multidimensional space. In figure 6, the U.S., once located in the middle of the ideological space at the Cold War period, has become isolated in the post-Cold War UNGA. This isolation stands out in MSS estimation compared to W-NOMINATE (figure 1) where all the extremists are on the rim of a unit circle.

6 Conclusion

In this study, we proposed a flexible Bayesian method for multidimensional ideal point estimation using the ℓ_1 norm and multivariate slice sampling. Existing ideal point estimation methods based on the item response theory (IRT) model or the NOMINATE procedure

[7] Voeten (2013) coded related issue(s) of each voting.

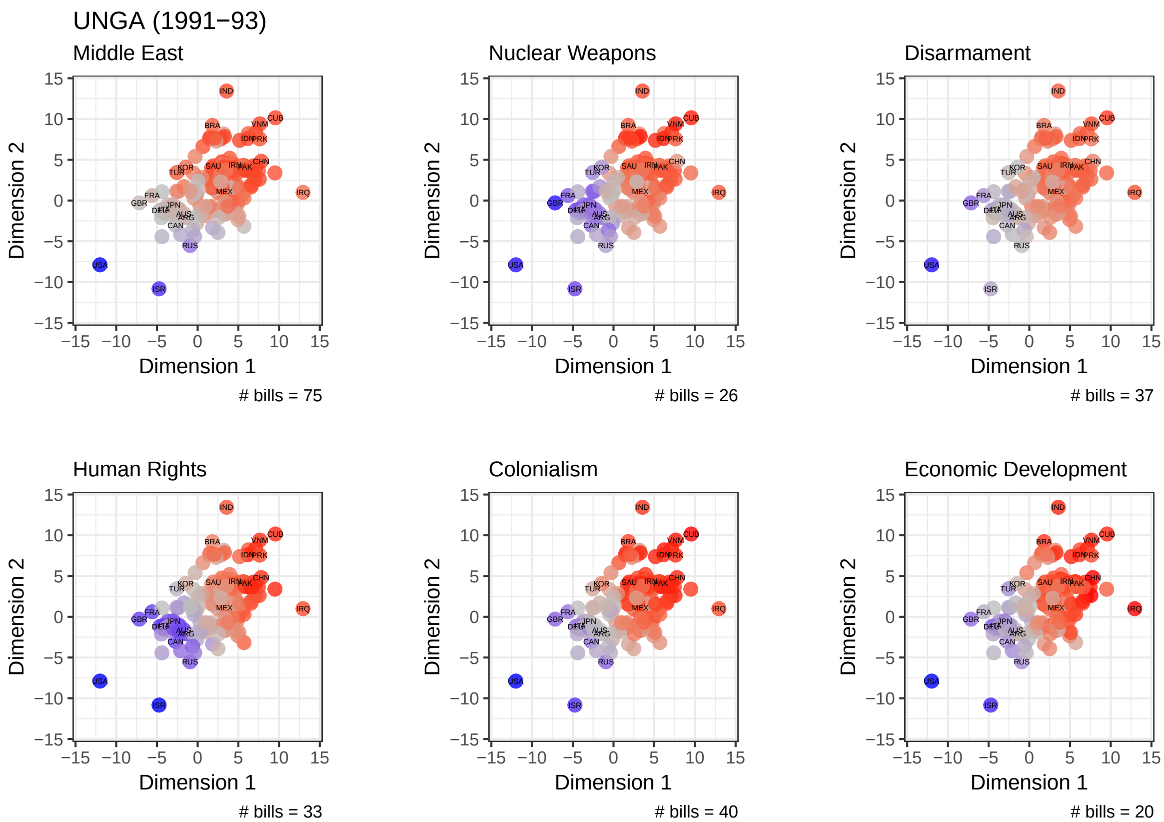

Figure 7: *The Degree of Voting Along with the U.S. by Issue*

do not properly recover multidimensional ideal points because of arbitrary identification constraints to resolve the rotational invariance problem. These constraints either require sophisticated knowledge about the locations of multidimensional ideal points (in the case of IRT models) or assume no cross-dimensional correlation of ideal points (in the case of the NOMINATE procedure). As a result, conventional ideal point estimation methods fail to uncover *correlated issue positions* by legislators, parties, coalitions, or countries.

The proposed method is "flexible" in the sense that (1) it does not require an arbitrary constraint to resolve rotational invariance, (2) it can be applied to any case where dimensional coordinates of ideal points are correlated, and (3) the number of dimensions and dimension weights can be easily estimated. MSS method successfully recovers various types of multidimensional ideal points in simulations. In particular, it successfully recovered multidimensional ideal points in the case where each-dimensional coordinates are highly correlated. Furthermore, it successfully estimates the weight of dimension even when one of two weights is zero. That is, MSS method successfully identifies one-dimensional nature of data even when the two-dimensional model is used.

We applied MSS method to the UNGA roll-call voting data. The result shows that the latent voting space of the UNGA is highly multidimensional during the post-Cold War period. The North–South and Western–non-Western divides characterize latent voting dimensions of the UNGA, the findings of which contrast with Voeten (2000). Also, the findings show strong evidence for the isolation of the U.S. during the post-Cold War period, which is consistent with in Voeten (2004).

In sum, MSS method is a highly useful and flexible method to investigate correlated multidimensional ideal points in political science. The authors are in the process of publishing an open-source software to implement MSS method in R. Future developments of MSS method include testing multidimensional theories of legislative politics using MSS method and applying MSS method to the study of European Parliament data and the U.S. roll-call voting data over a long history.

Appendix

A The Rotational Invariance Problem

IRT and NOMINATE-based models suffer from rotational invariance since these models incorporate the ℓ_2 norm in the utility form. IRT model assumes quadratic utility and NOMINATE-based models assume Gaussian (Normal) utility. Here, we illustrate each model of IRT and W-NOMINATE and the problem of rotational invariance.

IRT

$$
\begin{aligned}
U_{ijy}^{\text{Quad}} &= u_{ijy}^{\text{Quad}} + \epsilon_{ijy} && (6)\\
u_{ijy}^{\text{Quad}} &= -\sum_{k=1}^{s} (x_{ik} - o_{yjk})^2 && (7)\\
&= -(\mathbf{x}_i - \mathbf{o}_{yj})^\top (\mathbf{x}_i - \mathbf{o}_{yj}) && (8)
\end{aligned}
$$

where $\mathbf{x}_i = (x_{i1}, \cdots, x_{is})^\top$ and $\mathbf{o}_{yj} = (o_{yj1}, \cdots, o_{yjs})^\top$. Suppose a s by s matrix R such that $R^\top = R^{-1}$ and $\det(R) = 1$. R is a rotation matrix by definition. Then multiplying $R^\top R = I$ in the middle of two vectors of equation (8) does not change the value; *i.e.*, $u_{ijy}^{\text{Quad}} = -(\mathbf{x}_i - \mathbf{o}_{yj})^\top R^\top R(\mathbf{x}_i - \mathbf{o}_{yj})$. Let each vector of $\mathbf{x}_i$, $\mathbf{o}_{yj}$, and $\mathbf{o}_{nj}$ for all i and j be multiplied by R to create rotated ideal points, yea positions, and nay positions: $\mathbf{x}'_i = R\mathbf{x}_i$, $\mathbf{o}'_{yj} = R\mathbf{o}_{yj}$, and $\mathbf{o}'_{nj} = R\mathbf{o}_{nj}$ for all i and j. It follows that,

$$
\begin{aligned}
u_{ijy}^{\text{Quad}} &= -(\mathbf{x}_i - \mathbf{o}_{yj})^\top (\mathbf{x}_i - \mathbf{o}_{yj}) && (9)\\
&= -(\mathbf{x}_i - \mathbf{o}_{yj})^\top R^\top R (\mathbf{x}_i - \mathbf{o}_{yj}) && (10)\\
&= -(R\mathbf{x}_i - R\mathbf{o}_{yj})^\top (R\mathbf{x}_i - R\mathbf{o}_{yj}) && (11)\\
&= -(\mathbf{x}'_i - \mathbf{o}'_{yj})^\top (\mathbf{x}'_i - \mathbf{o}'_{yj}) && (12)
\end{aligned}
$$

and so do $u_{ijn}^{\text{Quad}} = -(\mathbf{x}'_i - \mathbf{o}'_{nj})^\top (\mathbf{x}'_i - \mathbf{o}'_{nj})$ for all i and j. That is, the rotated ideal points and yea/nay positions yields the same value of u_{ijy} and u_{ijn} (*i.e.*, *rotation invariance*).

Accordingly, the likelihood does not change when all the ideal points and yea/nay positions are rotated with the same degree, which causes an identification problem.

W-NOMINATE

$$U_{ijy}^{\text{Norm}} = u_{ijy}^{\text{Norm}} + \epsilon_{ijy} \tag{13}$$

$$u_{ijy}^{\text{Norm}} = \beta \exp\left\{-\frac{1}{2}\sum_{k=1}^{s} w_k^2 (x_{ik} - o_{yjk})^2\right\} \tag{14}$$

$$= \beta \exp\left\{-\frac{1}{2}(\mathbf{x}_i - \mathbf{o}_{yj})^\top (\mathbf{x}_i - \mathbf{o}_{yj})\right\} \tag{15}$$

where $\mathbf{x}_i = (w_1 x_{i1}, \cdots, w_s x_{is})^\top$ and $\mathbf{o}_{yj} = (w_1 o_{yj1}, \cdots, w_s o_{yjs})^\top$, each of which denote a weighted ideal point and yea position for simplicity. In analogy with previous case of IRT, W-NOMINATE also suffers from rotational invariance.

$$u_{ijy}^{\text{Norm}} = \beta \exp\left\{-\frac{1}{2}(\mathbf{x}_i - \mathbf{o}_{yj})^\top (\mathbf{x}_i - \mathbf{o}_{yj})\right\} \tag{16}$$

$$= \beta \exp\left\{-\frac{1}{2}(\mathbf{x}_i - \mathbf{o}_{yj})^\top R^\top R (\mathbf{x}_i - \mathbf{o}_{yj})\right\} \tag{17}$$

$$= \beta \exp\left\{-\frac{1}{2}(R\mathbf{x}_i - R\mathbf{o}_{yj})^\top (R\mathbf{x}_i - R\mathbf{o}_{yj})\right\} \tag{18}$$

$$= \beta \exp\left\{-\frac{1}{2}(\mathbf{x}'_i - \mathbf{o}'_{yj})^\top (\mathbf{x}'_i - \mathbf{o}'_{yj})\right\} \tag{19}$$

Consequently, the rotational invariance of W-NOMINATE causes an identification problem. Note that DW-NOMINATE also suffers from the same problem since it uses the same utility form with the ℓ_2 norm.

B Identifiability of the Proposed Model

Below, we discuss the identifiability of the likelihood function of the model, which is the only data dependent part of the model. We remark that even the likelihood function is not identifiable, the full Bayesian model can be identifiable by the specification of the prior distribution.

Let, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$,

$$\begin{aligned} g_1\big(\mathbf{w} \circ \mathbf{x}_i, \mathbf{w} \circ \mathbf{o}_{yj}, \mathbf{w} \circ \mathbf{o}_{nj}, \beta\big) &= \beta\big\{\|\mathbf{x}_i - \mathbf{o}_{yj}\|_1^{\mathbf{w}} - \|\mathbf{x}_i - \mathbf{o}_{nj}\|_1^{\mathbf{w}}\big\} \\ &= \beta\big\{\|\mathbf{w} \circ \mathbf{x}_i - \mathbf{w} \circ \mathbf{o}_{yj}\|_1 - \|\mathbf{w} \circ \mathbf{x}_i - \mathbf{w} \circ \mathbf{o}_{nj}\|_1\big\} \\ &= \beta \sum_{k=1}^{s} \big\{|w_k x_{ik} - w_k o_{yjk}| - |w_k x_{ik} - w_k o_{njk}|\big\}, \end{aligned}$$

where $\mathbf{a} \circ \mathbf{b}$ is the Hadamard product of $s-$dimensional vectors $\mathbf{a}$ and $\mathbf{b}$. The likelihood function is monotonic to $g_1\big(\mathbf{w} \circ \mathbf{x}_i, \mathbf{w} \circ \mathbf{o}_{yj}, \mathbf{w} \circ \mathbf{o}_{nj}, \beta\big)$.

The identifiability of $g_1\big(\mathbf{w} \circ \mathbf{x}_i, \mathbf{w} \circ \mathbf{o}_{yj}, \mathbf{w} \circ \mathbf{o}_{nj}, \beta\big)$ is about $\mathbf{x}'_i = \mathbf{w} \circ \mathbf{x}_i$, $\mathbf{o}'_{yj} = \mathbf{w} \circ \mathbf{o}_{yj}$ and $\mathbf{o}'_{nj} = \mathbf{w} \circ \mathbf{o}_{nj}$. We use the notation $\mathbf{x}_i$, $\mathbf{o}_{yj}$ and $\mathbf{o}_{nj}$ instead of $\mathbf{x}'_i$, $\mathbf{o}'_{yj}$ and $\mathbf{o}'_{nj}$ for notational simplicity.

Theorem 1. *With the centering constraint (i)* $\mathbf{o}_{yj} + \mathbf{o}_{nj} = 0$ *and (ii) the scaling constraint* $\|\mathbf{o}_{yj}\|_2 = \|\mathbf{o}_{nj}\|_2 = 1$ *for* $j = 1, 2, \ldots, m$*, the likelihood function (the function of* $\mathbf{w} \circ \mathbf{x}_i, \mathbf{w} \circ \mathbf{o}_{yj}, \mathbf{w} \circ \mathbf{o}_{nj}$ *and* β*) is identifiable up to the permutation of axes (i.e. basis vectors) of (s−dimensional) projected space.*

Proof. Without the constraints, the function $g_1\big(\mathbf{x}_i, \mathbf{o}_{yj}, \mathbf{o}_{nj}, \beta\big)$ is identifiable only up to location-shift, scale, and permutation transformation. That is,

$$g_1\big(\mathbf{x}_i, \mathbf{o}_{yj}, \mathbf{o}_{nj}, \beta\big) = g_1\big(c\mathrm{A}\mathbf{x}_i + \Delta, c\mathrm{A}\mathbf{o}_{yj} + \Delta, c\mathrm{A}\mathbf{o}_{nj} + \Delta, (1/c)\beta\big), \tag{20}$$

where $c \in \mathcal{R}^+$, $\mathrm{A} = \big(a_{kl}, 1 \le k, l \le s\big)$ is a permutation matrix that is a_{kl}s are 0 or 1 and $\sum_{k=1}^{s} a_{kl} = \sum_{l=1}^{s} a_{kl} = 1$ for every $k, l = 1, 2, \ldots, s$. The centering constraint (i) $\mathbf{o}_y + \mathbf{o}_n = 0$ resolves the non-identifiability with respect to the location-shift transformation; the equivalent relationship (20) implies that $\Delta = \mathbf{0}_{s\times 1}$. Now, with the condition $\mathbf{o}_y + \mathbf{o}_n = 0$, the function $g_1\big(\mathbf{x}, \mathbf{o}_y, \mathbf{o}_n, \mathbf{w}\big)$ is identifiable up to the scale and permutation transformation:

$$g_1\big(\mathbf{x}_i, \mathbf{o}_{yj}, \mathbf{o}_{nj}, \beta\big) = g_1\big(c\mathrm{A}\mathbf{x}_i, c\mathrm{A}\mathbf{o}_{yj}, c\mathrm{A}\mathbf{o}_{nj}, (1/c)\beta\big),$$

where $c \in \mathcal{R}^+$ and A is a $s-$dimensional permutation matrix. Further, by noting that $\|c\mathrm{A}\mathbf{o}_{yj}\|_2 = c\|\mathbf{o}_{yj}\|_2$, the scaling constraint (ii) identifies the model with respect to the scale transformation. Finally, with both constraints (i) and (ii), we have the invariance up to the permutation transformation:

$$g_1\big(\mathbf{x}_i, \mathbf{o}_{yj}, \mathbf{o}_{nj}, \beta\big) = g_1\big(\mathrm{A}\mathbf{x}_i, \mathrm{A}\mathbf{o}_{yj}, \mathrm{A}\mathbf{o}_{nj}, \beta\big) \tag{21}$$

for any permutation matrix A. □

Our remark on the proposed ℓ_1 norm based multidimensional scaling (MDS) is that it makes an objective and data dependent choice on axes (*i.e.* coordinate vectors) of MDS compared to ℓ_2 norm based MDS. To understand this better, let us briefly review the ℓ_2 norm based MDS procedure.

Similarly to the ℓ_1 norm based MDS, the ℓ_2 norm based MDS relies on, for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$,

$$g_2\big(\mathbf{x}_i, \mathbf{o}_{yj}, \mathbf{o}_{nj}, \mathbf{w}, \beta\big) = \beta\big\{\|\mathbf{x}_i - \mathbf{o}_{yj}\|_2^2 - \|\mathbf{x}_i - \mathbf{o}_{nj}\|_2^2\big\}. \tag{22}$$

For each i and j, the function $g_2\big(\mathbf{x}, \mathbf{o}_y, \mathbf{o}_n, \mathbf{w}, \beta\big)$ is identifiable up to location-shift, scale, and rotation transformation as

$$g_2\big(\mathbf{x}_i, \mathbf{o}_{yj}, \mathbf{o}_{nj}, \beta\big) = g_2\big(c\mathrm{T}^\top\mathbf{x}_i + \Delta, c\mathrm{T}^\top\mathbf{o}_{yj} + \Delta, c\mathrm{T}^\top\mathbf{o}_{nj} + \Delta, (1/c)\beta\big), \tag{23}$$

for any $\Delta \in \mathcal{R}^s$ and $s-$dimensional orthonormal matrix T, $\mathrm{T}\mathrm{T}^\top = \mathrm{T}^\top\mathrm{T} = \mathrm{I}_s$.

To make $g_2(\cdot)$ be identifiable, we assume scaling constraint (ii) and (iii) the scores of $s+1$ legislatives, say $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{s+1}$, be fixed as $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{s+1}$. As in the ℓ_1-based MDS, the scaling constraint makes the $g_2(\cdot)$ be identifiable to the scale transformation. Now, let us see why (iii) resolves the identifiability problem related with the location-shift and rotation transformation. Let $\mathbf{x}_i^*$s, $\mathbf{o}_{yj}^*$s, and $\mathbf{o}_{nj}^*$s be one set of estimates for n legislatives and m roll calls. Suppose we subjectively fix the ideal points of the first $s+1$ legislatives as $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{s+1}$, which have equivalent relationship with $\mathbf{x}_1^*, \ldots, \mathbf{x}_{s+1}^*$ in (23) with $c = 2$, so that, for $i = 1, 2, \ldots, s+1$,

$$\mathbf{f}_i = \mathrm{T}\mathbf{x}_i^* + \Delta \tag{24}$$

for some orthonormal matrix T and $\Delta \in \mathcal{R}^s$. The identity (24) induces

$$\mathrm{T}\left[\ \mathbf{x}_2^* - \mathbf{x}_1^*, \mathbf{x}_3^* - \mathbf{x}_1^*, \cdots, \mathbf{x}_{s+1}^* - \mathbf{x}_1^*\ \right] = \left[\ \mathbf{f}_2 - \mathbf{f}_1, \mathbf{f}_3 - \mathbf{f}_1, \cdots, \mathbf{f}_{s+1} - \mathbf{f}_1\ \right] \text{ and } \Delta = \mathbf{f}_1 - \mathrm{T}\mathbf{x}_1^*,$$

and these decide the orthonormal matrix T and Δ uniquely. Thus, the assumption or constraint (iii) points out the unique solution among infinitely many equivalent solutions generated by location-shift and rotation transformation.

Here, in the ℓ_2-based MDS, the set of axes (s coordinate vectors) for $\mathrm{T}\mathbf{x}_i$ is changing and dependendent on T; it is not equal to that of $\mathbf{x}_i$ and not invariant to the rotation transformation. Thus, the axes for ideal points $\mathbf{x}_i$ are decided by the subjective selection of $s+1$ legislatives and the constants $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{s+1}$. Further, the numbers $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{s+1}$ will decide how each axis is composed by fundamental units (economic, social, ideology, ...); for example, one choice of the numbers makes the first axis be 80% of economic and 20% of social, and another choice makes it be 60% of economic and 40% of social. The arbitrary fixed scores $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{s+1}$ for the subjectively chosen $s+1$ legislatives are much influential to the MDS results.

Unlike the ℓ_2-based MDS, in the ℓ_1-based MDS, for any permutation matrix A, the set of axes (coordinate vectors) of $\mathbf{x}_i$ and $\mathrm{A}\mathbf{x}_i$ is equal, invariant to the permutation transformation A. Thus, the ℓ_1-based MDS has a finite number ($s!$) of equivalent solutions which are based on the same set of coordinate vectors. In addition, it decides the axes fully based on data information free from any subjective choice.

C Simulation Results

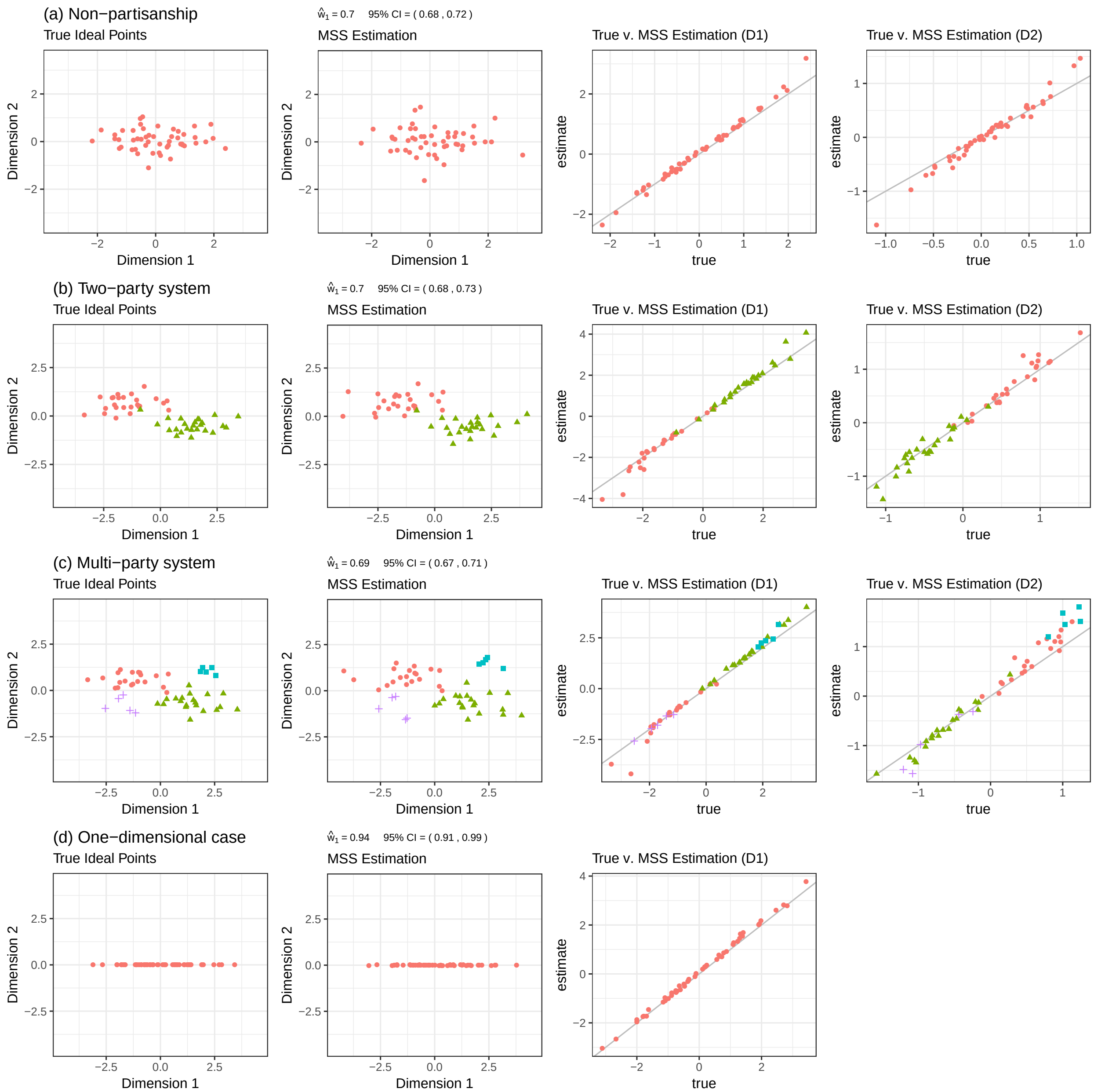

Figure 8: *Two-dimensional MSS Estimation*

Each row is the result of four different simulation studies: (a) non-partisanship with weights $(w_1, w_2) = (0.7, 0.3)$; (b) two-party system with weights $(w_1, w_2) = (0.7, 0.3)$; (c) multi-party system with weights $(w_1, w_2) = (0.7, 0.3)$; (d) one-dimensional case — *i.e.*, weights $(w_1, w_2) = (1, 0)$. MSS method successfully recovers each ideal point and weight for all cases.

D MCMC Diagnostics

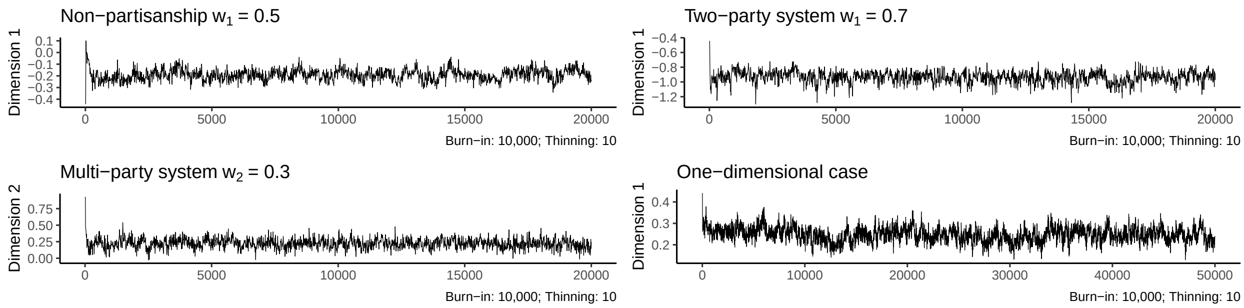

Figure 9: *Trace Plots of Randomly Chosen Ideal Points (Simulation Results)*

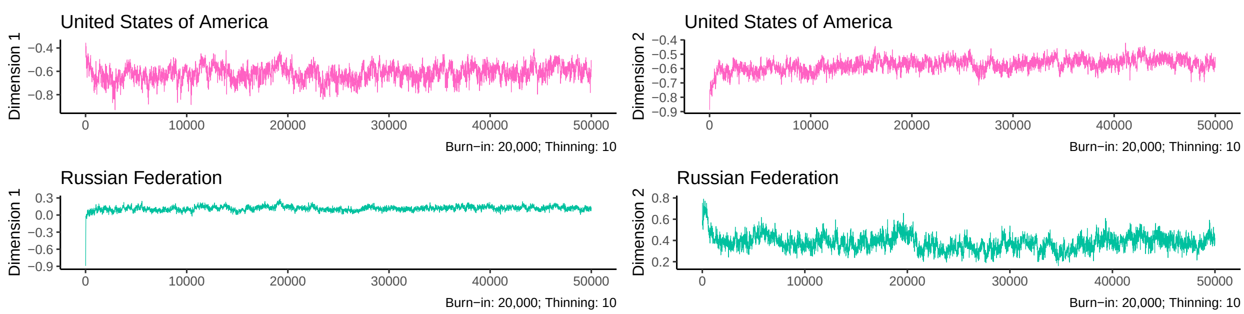

Figure 10: *Trace Plots of Ideal Points of the U.S. and Russian Federation (1954-69 UNGA)*

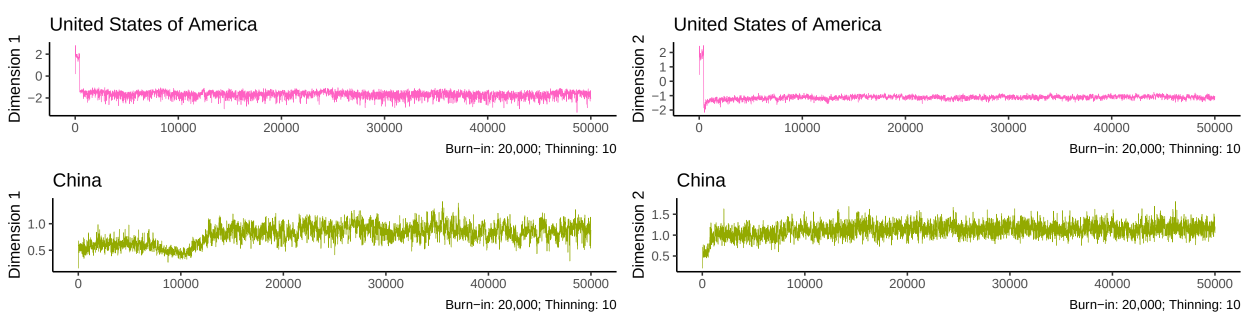

Figure 11: *Trace Plots of Ideal Points of the U.S. and China (1991-93 UNGA)*

E Application Results

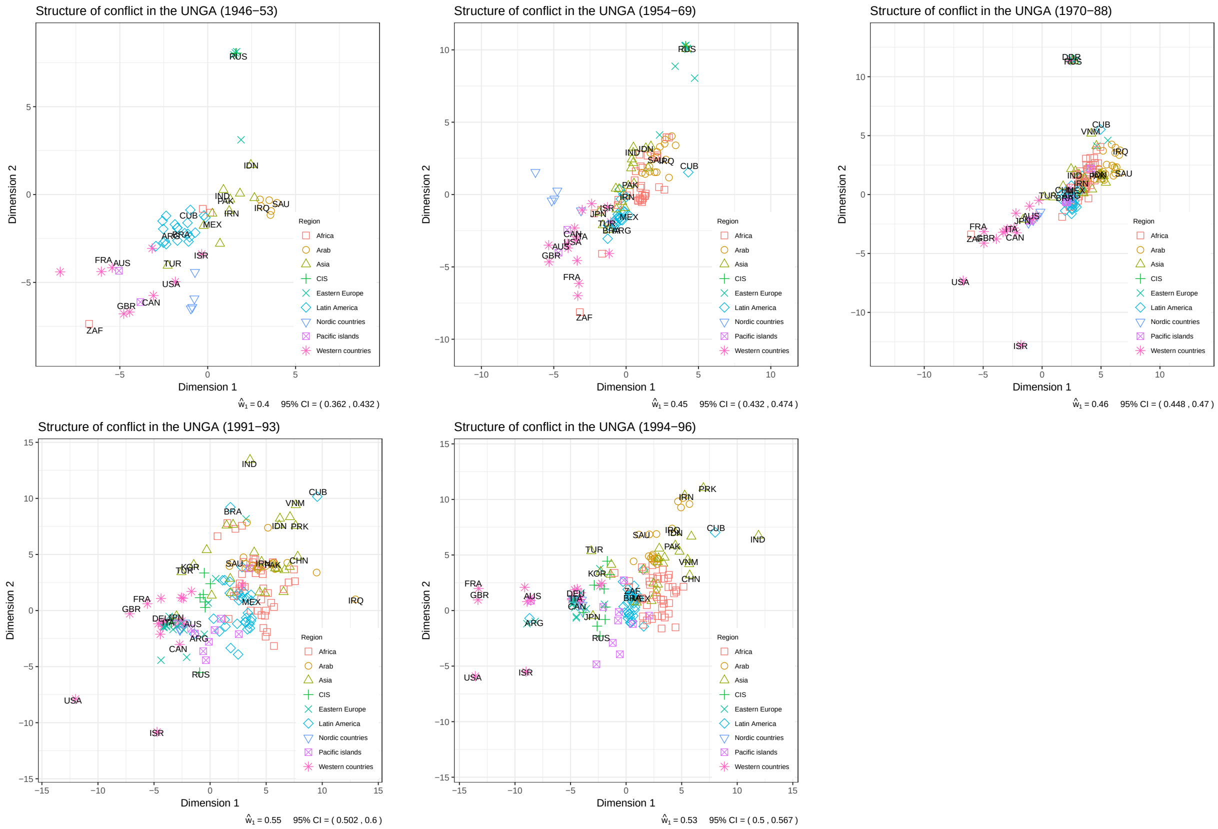

Figure 12: *The UN Member States' Ideal Points Estimated by MSS (1946-1996)*

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