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A New Method for High-Degree Spline Interpolation: Proof of Continuity for Piecewise Polynomials

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Abstract. Effective and accurate high-degree spline interpolation is still a challenging task in today's applications. Higher degree spline interpolation is not so commonly used, because it requires the knowledge of higher order derivatives at the nodes of a function on a given mesh.

In this article, our goal is to demonstrate the continuity of the piecewise polynomials and their derivatives at the connecting points, obtained with a method initially developed by Beaudoin (1998, 2003) and Beauchemin (2003). This new method, involving the discrete Fourier transform (DFT/FFT), leads to higher degree spline interpolation for equally spaced data on an interval $[0, T]$. To do this, we analyze the singularities that may occur when solving the system of equations that enables the construction of splines of any degree. We also note an important difference between the odd-degree splines and even-degree splines. These results prove that Beaudoin and Beauchemin's method leads to spline interpolation of any degree and that this new method could eventually be used to improve the accuracy of spline interpolation in traditional problems.

1 Introduction

Spline interpolation is now widely used in industry and research. It has proved to be a very efficient tool, particularly for data interpolation or curve smoothing applications. It is typically preferred to polynomial interpolation, as it avoids the problem of Runge's phenomenon where oscillations occur at the edges of the interval when using higher degree polynomials. Cubic spline interpolation, which is the most popular type of spline used in practice due to its accuracy and low computational cost, has been investigated in [1,5]. A more general approach for higher odd-degree spline interpolation was analyzed in [1].

In [2,3], a new method to obtain, from a discrete function, an accurate approximation of the continuous Fourier transform was developed. It was also mentioned that this method led to polynomial splines of any odd degree. This statement was based on many observations, and no formal proof was provided. The goal of this paper is therefore to formally demonstrate that the piecewise continuous polynomials defined in [2,3] form a spline function for both odd degree and even degree polynomials. By assuming that θ is the degree of spline polynomials obtained via the numerical Fourier

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transform, we show that the derivatives of the resulting piecewise polynomials up to order $\theta - 1$ are continuous at every internal node.

Several authors [7,11,14,15] mentioned that the discrete Fourier transform (DFT) is far from being adequate to act in place of the Fourier transform. One of the advantages of using the approach proposed by Beaudoin and Beauchemin in [2,3] to numerically compute the Fourier transform of a digitized function is that it performs better than the classical DFT for both smooth and rough functions, all the while providing additional information not obtainable with the DFT. Other than giving polynomial splines, it also yields accurate numerical derivatives and numerical integration. The other advantage of the method is that the resulting polynomial splines can be obtained easily for higher degrees, something that becomes more difficult when using the standard approach to calculate splines. More accurate results can therefore be obtained when desired without significantly increasing the computational cost. In fact, the numerical accuracy of the method increases much more rapidly than the computational cost [3].

This contribution is organized as follows. Section 2 gives a brief summary of Beaudoin and Beauchemin's method and includes numerical examples to support their conclusion that their method leads to spline interpolation. Section 3 states important properties needed to show under what conditions we are able to build piecewise polynomials of even and odd degrees, while Section 4 is devoted to the formal proof of continuity of the resulting interpolation functions, thus proving that they are indeed spline functions. Let us note that one-dimensional functions are considered herein.

2 Mathematical Background

Let $t_0, t_1, \dots, t_N$ be $N+1$ equidistant interpolation nodes on an interval $[0, T]$, $T > 0$, such that $t_j = j\Delta t$ for $j = 0, 1, \dots, N$ with $\Delta t = T/N$. Knowing the values of an unknown continuous and indefinitely derivable real-valued function g at these nodes only, and denoting them by $g(t_j) = g_j$, the goal is to approximate g on the interval $[0, T]$ by constructing an interpolation polynomial of degree θ, denoted as g_θ. The approximation of order θ of the ℓ-th derivative of g at node t_j is denoted $g_{j,\theta}^{(\ell)}$. Note that $g_{j,\theta}$ is independent of the approximation order, since it corresponds to the function g evaluated at the interpolation node t_j.

To compute g_θ, as well as its derivatives, Beaudoin and Beauchemin [2] developed a method based on the Fourier transform and the Taylor expansion of the unknown function g. By using the properties of both of these tools, they obtained the following equation:

$$-i2\pi f_k I_{0,k} F_{n,k} + \sum_{p=1}^{\infty} (I_{p-1,k} - i2\pi f_k I_{p,k}) F_{p+n,k} = b_n, \tag{2.1}$$

where $f_k = k/T$, $F_{\ell,k}$ is the k-th term of the DFT[1] of the sequence $g_j^{(\ell)}$, where $g_j^{(\ell)}$ corresponds to the ℓ-th derivative of g at the node t_j, b_n is defined in terms of values

[1]The DFT can be computed with FFT algorithms.

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at the boundaries of the interval as follows:

$$b_n = g_N^{(n)} - g_0^{(n)},$$

and where $I_{p,k}$ is given by

$$I_{p,k} = \frac{1}{\Gamma(p+1)} \int_0^{\Delta t} \tau^p \exp(-i2\pi f_k \tau)\, d\tau \tag{2.2}$$

for $k = 0, 1, \ldots, N-1$, where Γ is the gamma function.

In the particular case where p is a positive integer, we can write the last equation as

$$I_{p,k} = \frac{1}{p!} \int_0^{\Delta t} \tau^p \exp(-i2\pi f_k \tau)\, d\tau \tag{2.3}$$

for $k = 0, 1, \ldots, N-1$, since $\Gamma(p+1) = p!\ \forall\ p \in \mathbb{N}$.

It can easily be shown that equation (2.3) is equivalent to

$$I_{p,k} = \begin{cases} \frac{1}{(p+1)!}(\Delta t)^{p+1}, & \text{if } f_k = 0, \\ \frac{1}{(i2\pi f_k)^{p+1}} - \exp(-i2\pi k/N) \sum_{q=1}^{p+1} \frac{1}{(i2\pi f_k)^q} \frac{(\Delta t)^{p-q+1}}{(p-q+1)!}, & \text{if } f_k \neq 0, \end{cases}$$

for all $p \in \mathbb{N}$. In the case where p is a negative integer, we have $I_{p,k} = 0$ by equation (2.2).

By introducing a truncating parameter θ and by expanding equation (2.1) for a specific range of values for n and p, Beaudoin and Beauchemin were then able to extract a part of the resulting system and show that in order to approximate the DFT of the derivatives of g up to an order θ at the nodes, the following system needed to be solved:

$$M_b^\theta F_b^\theta = B^\theta + C^\theta, \tag{2.4}$$

where F_b^θ is the vector of unknowns defined by $F_{\ell,k,\theta}$ for $\ell = 1, 2, \ldots, \theta$, with

$$F_{\ell,k,\theta} = \sum_{j=0}^{N-1} g_{j,\theta}^{(\ell)} \exp(-i2\pi kj/N).$$

Then $F_{\ell,k,\theta}$ denotes the k-th term of the DFT of the sequence $g_{j,\theta}^{(\ell)}$. It is important to note that the case $\ell = 0$ corresponds to the k-th term of the DFT of the sequence $g_{j,\theta}$, which can be computed directly for $k = 0, 1, \ldots, N-1$, since the sequence $g_{j,\theta}$ is entirely known for $j = 0, 1, \ldots, N-1$. As for M_b^θ, it is a square matrix of dimension θ defined by

$$(M_b^\theta)_{\mu,\nu} = \begin{cases} 0, & \text{if } \nu - \mu + 1 < 0, \\ J_{\nu-\mu+1,k}, & \text{otherwise}, \end{cases}$$

for $\mu, \nu = 1, 2, \ldots, \theta$, and where

$$J_{p,k} = I_{p-1,k} - i2\pi f_k I_{p,k},$$

which is also equivalent to

$$J_{p,k} = \begin{cases} \exp(-i2\pi k/N) - 1, & \text{if } p = 0, \\ \frac{(\Delta t)^p}{p!} \exp(-i2\pi k/N), & \text{if } p > 0. \end{cases}$$

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Explicitly, M_b^θ can be written as:

$$M_b^\theta = \begin{bmatrix} J_{1,k} & J_{2,k} & \dots & J_{\theta,k} \\ J_{0,k} & J_{1,k} & \dots & J_{\theta-1,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{1,k} \end{bmatrix}.$$

The vector B^θ, which is unknown, is called the *boundary conditions vector*, while vector C^θ is defined by

$$(C^\theta)_\mu = \begin{cases} -J_{0,k}F_{0,k}, & \text{if } \mu = 1, \\ 0, & \text{if } 1 < \mu \leq \theta. \end{cases}$$

To compute B^θ, the reader can refer to [3] where an algorithm is given. We note that other algorithms could also be used. However, as the values of this vector only influence the accuracy of the results and not the continuity of the resulting interpolation functions, we will not focus on its calculation at this time. This issue, which is key to improving the accuracy of the results, will be treated in a forthcoming publication.

Once $F_{\ell,k,\theta}$ is known for $\ell = 1, 2, \dots, \theta$, the inverse discrete Fourier transform (iDFT) can be used to approximate the derivatives of g, which are initially unknown, at the interpolation nodes:

$$g_{j,\theta}^{(\ell)} = \frac{1}{N} \sum_{k=0}^{N-1} F_{\ell,k,\theta} \exp(i2\pi kj/N). \tag{2.5}$$

Using these values, the polynomials and their derivatives on each interval $[t_j, t_{j+1}[$ are then constructed by a Taylor expansion as follows:

$$[g^{(\ell)}]_j^\theta(t) = \sum_{p=0}^{\theta-\ell} \frac{(t-t_j)^p}{p!} g_{j,\theta}^{(p+\ell)}, \qquad t \in [t_j, t_{j+1}[$$

for $\ell = 0, 1, \dots, \theta$.

Let us note that even though the Taylor series are truncated, the way the numerical derivatives are computed leads to interpolating functions that are continuous on the whole interval $[0, T]$.

Beaudoin and Beauchemin made this observation when they numerically noticed the continuity of the interpolation functions obtained from their new method. They arrived at this conclusion by analyzing the differences between the values at every interior node between the $(j-1)$-th and the j-th polynomial, and up to the θ-th derivative. They observed that this difference, up to the $(\theta-1)$-th derivative, was nearly zero, which led them to believe that their method led to spline functions of any (odd) degree. They also noticed that while the choice of boundary conditions did not influence the continuity of the spline, it had an impact on the accuracy of the results.

In order to illustrate this, let us consider fifty ($N = 49$) randomly generated values ranging from -1 to 1. We will compute the spline passing through those equally distributed points on the interval $[0, 2]$ for two, arbitrary chosen, sets of boundary conditions. This illustrates, firstly, that the continuity of the spline of degree θ obtained from this method is independent of the chosen boundary conditions. Secondly, Figures 1 and 2 illustrate the differences created by using splines of different degrees.

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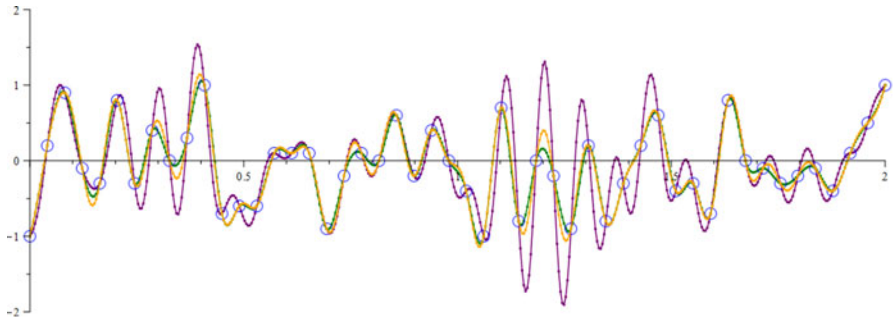
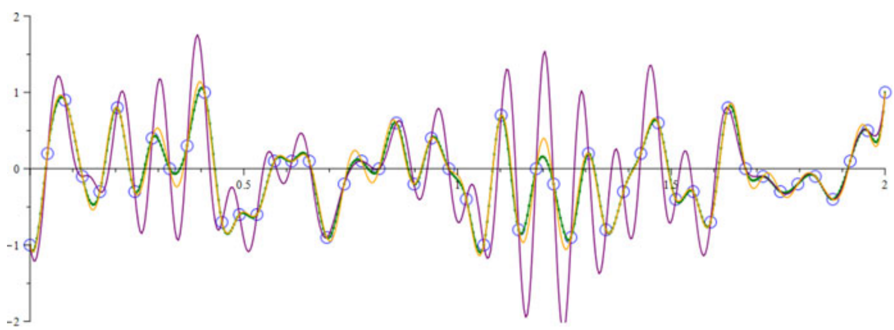


Green: $\theta = 3$ Purple: $\theta = 6$ Orange: $\theta = 9$

Figure 1: Plots of different interpolation splines, with boundary conditions $b_0 = g_N - g_0$ and $b_n = 0$, $n = 1, 2, \ldots, \theta - 1$. Blue circles represent the interpolation nodes. θ is the degree of the spline polynomial. (Colour online).



Green: $\theta = 3$ Purple: $\theta = 6$ Orange: $\theta = 9$

Figure 2: Plots of different interpolation splines, with boundary conditions $b_0 = g_N - g_0$ and $b_n = 50n$, $n = 1, 2, \ldots, \theta - 1$. Blue circles represent the interpolation nodes. (Colour online).

Let us note that, in Figures 1 and 2, there is a significant difference between the splines of odd and even degree. However, as the interpolation degree θ increases, these differences are less significant. Finally, in Figure 3, we observe the consequence of choosing different boundary conditions on the spline.

By defining a maximum absolute error on the continuity of the ℓ-th derivative by

$$\textbf{Maximum Absolute Error} = \max_{j=1,2,\ldots,N-1} \left| g_j^{(p+\ell)} - \Delta t \sum_{p=0}^{\theta-\ell} g_{j-1}^{(p+\ell)} \right|,$$

and an average absolute error on the continuity of the ℓ-th derivative by

$$\textbf{Average Absolute Error} = \frac{1}{N-1} \sum_{j=1}^{N-1} \left| g_j^{(p+\ell)} - \Delta t \sum_{p=0}^{\theta-\ell} g_{j-1}^{(p+\ell)} \right|;$$

we can then compare the values from the $(j-1)$-th and j-th polynomials at every interior nodes for $\ell = 0, 1, \ldots, \theta$. These results are shown in Tables 1 and 2.

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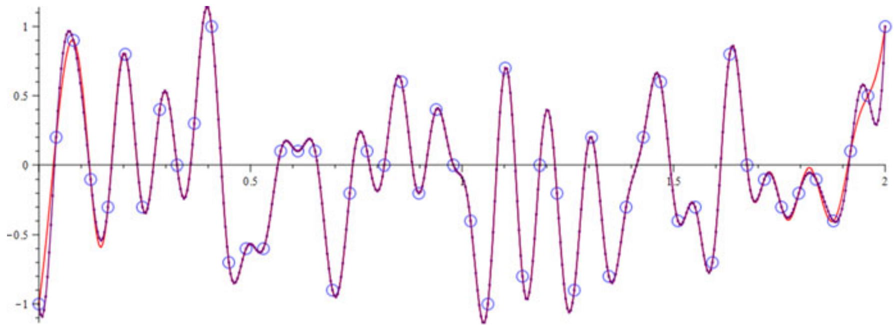

Red: $b_0 = g_N - g_0$ and $b_n = 0$ Purple: $b_0 = g_N - g_0$ and $b_n = 50n$

Figure 3: Comparisons of the spline of degree θ = 9 using two different sets of boundary conditions. Blue circles represent the interpolation nodes. (Colour online).

Derivative Order	Maximum Absolute Error	Average Absolute Error
0	6.53×10^{-99}	3.13×10^{-99}
1	3.20×10^{-97}	1.37×10^{-97}
2	1.58×10^{-95}	6.28×10^{-96}
3	9.64×10^{-94}	4.45×10^{-94}
4	9.06×10^{-92}	3.53×10^{-92}
5	6.58×10^{-90}	2.76×10^{-90}
6	5.06×10^{-88}	2.14×10^{-88}
7	3.40×10^{-86}	1.27×10^{-86}
8	1.50×10^{-84}	5.66×10^{-85}
9	4.39×10^{16}	2.74×10^{16}

Table 1: Errors at the interior interpolation nodes of an arbitrary function and its derivatives when using a spline of degree $\theta = 9$. Computations were performed with Maple, using a precision of 100 digits. Boundary conditions were chosen as $b_0 = g_N - g_0$ and $b_n = 0$, for $n = 1, 2, \ldots, 8$. Of course, the error for the derivative of order $\theta = 9$ is not 0, since a spline of degree θ is continuous up to the $(\theta - 1)$-th derivative.

The continuity of the interpolation function and its derivatives are obvious in this example. This is a convincing observation, but not a formal proof of the continuity of our interpolation functions on the interval $[0, T]$. In Section 4, we will therefore formally demonstrate the continuity of these polynomials at every internal node for $\ell = 0, 1, \ldots, \theta - 1$. To achieve this goal, some important properties need to be stated, which is the object of the next section.

3 Important Properties

The numerical tests completed by Beaudoin and Beauchemin led them to believe that their method could only produce polynomial splines of odd degrees. Evidently, since

Derivative Order	Maximum Absolute Error	Average Absolute Error
0	8.22×10^{-99}	4.05×10^{-99}
1	4.09×10^{-97}	1.58×10^{-97}
2	1.87×10^{-95}	8.89×10^{-96}
3	1.40×10^{-93}	5.50×10^{-94}
4	1.10×10^{-91}	4.66×10^{-92}
5	8.58×10^{-90}	4.11×10^{-90}
6	5.66×10^{-88}	2.85×10^{-88}
7	3.89×10^{-86}	1.92×10^{-86}
8	1.61×10^{-84}	9.08×10^{-85}
9	4.39×10^{16}	2.91×10^{16}

Table 2: Errors at the interior interpolation nodes of an arbitrary function and its derivatives when using a spline of degree $\theta = 9$. Computations were performed with Maple, using a precision of 100 digits. Boundary conditions were chosen as $b_0 = g_N - g_0$ and $b_n = 50n$, for $n = 1, 2, \ldots, 8$. As mentioned in Table 1, the error for the derivative of order $\theta = 9$ is not 0 since a spline of degree θ is continuous up to the $(\theta - 1)$-th derivative.

their method requires solving (2.4), the problem observed for even values of θ could be due to singularities in the matrix M_b^θ. One of the goals of this section is therefore to establish a formula for the determinant of M_b^θ in order to find the conditions under which this determinant becomes null. To do so, we first need to define Eulerian numbers and polynomials and state some of their properties. Let us note that the definitions presented can be found in [12].

3.1 Eulerian Numbers and Polynomials

Definition 3.1 Let m and n be two integers such that $0 \le m \le n - 1$; the Eulerian number $\left\langle {n \atop m} \right\rangle$ is defined by

$$\left\langle {n \atop m} \right\rangle = \sum_{k=0}^{m} (-1)^k C_k^{n+1} (m + 1 - k)^n,$$

where

$$C_k^{n+1} = \frac{(n+1)!}{(n+1-k)!k!}.$$

For example, when $n = 4$, the four Eulerian numbers for $0 \le m \le 3$ are 1, 11, 11, and 1, respectively. For $n = 5$, the five Eulerian numbers for $0 \le m \le 4$ are 1, 26, 66, 26, and 1, respectively.

As can be observed from this last example, the Eulerian numbers for $0 \le m \le n - 1$ are symmetric around the $\lceil n/2 \rceil$-th number when n is odd, where $\lceil \cdot \rceil$ denotes the ceiling operator. In a similar way, when n is even, the Eulerian numbers for $0 \le m \le n - 1$ are symmetric around the $(n/2)$-th and $(n/2 + 1)$-th numbers. More generally,

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from these symmetric properties, [8] gives the following relation:

$$\left\langle {n \atop m} \right\rangle = \left\langle {n \atop n-m-1} \right\rangle. \tag{3.1}$$

Definition 3.2 Let $A_n : \mathbb{C} \to \mathbb{C}$ be the Eulerian polynomial of degree $n-1$; it is defined in terms of Eulerian numbers as follows:

$$A_n(z) = \sum_{q=0}^{n-1} \left\langle {n \atop q} \right\rangle z^q, \tag{3.2}$$

where $n \in \mathbb{N}\backslash\{0\}$ and $z \in \mathbb{C}$.

As an example, the first five Eulerian polynomials are

$$\begin{aligned} A_1(z) &= 1, \\ A_2(z) &= 1 + z, \\ A_3(z) &= 1 + 4z + z^2, \\ A_4(z) &= 1 + 11z + 11z^2 + 1z^3, \\ A_5(z) &= 1 + 26z + 66z^2 + 26z^3 + z^4. \end{aligned}$$

Generally, finding all the roots of these polynomials is not easy. From [13], it is known that all $n-1$ roots of the Eulerian polynomial A_n are real numbers. This resulted from a theorem demonstrated by Frobenius in 1910. Additionally, it is clear that these roots are negative, since $A_n(z) > 0$ for all real number $z \geq 0$. This property will be essential to analyze the determinant of M_b^θ.

Now, by assuming $q = n - q' - 1$ and by using property (3.1), equation (3.2) can be written as

$$A_n(z) = \sum_{q'=n-1}^{0} \left\langle {n \atop n-q'-1} \right\rangle z^{n-q'-1} = \sum_{q=0}^{n-1} \left\langle {n \atop q} \right\rangle z^{n-q-1}. \tag{3.3}$$

Let us note that equation (3.3) will be used to simplify some equations.

Other properties of Eulerian polynomials can also be derived. Here are three that will be useful in what there is to follow:

- From [6], we have:

$$A_n(z) = \begin{cases} 1, & \text{if } n = 0, \\ \sum_{k=0}^{n-1} C_k^n A_k(z)(z-1)^{n-1-k}, & \text{if } n \geq 1. \end{cases} \tag{3.4}$$

- For $n \in \mathbb{N}\backslash\{0\}$ and $z \neq 0$, we can demonstrate that:

$$A_n(z) = z^{n-1} A_n(z^{-1}). \tag{3.5}$$

In fact, by factoring z^{n-1} from equation (3.2), we obtain

$$A_n(z) = z^{n-1} \sum_{q=0}^{n-1} \left\langle {n \atop q} \right\rangle z^{q-(n-1)},$$

which is equivalent to

$$A_n(z) = z^{n-1}\sum_{q=0}^{n-1}\left\langle {n \atop q} \right\rangle \frac{1}{z^{n-q-1}} = z^{n-1}\sum_{q=0}^{n-1}\left\langle {n \atop q} \right\rangle \left(\frac{1}{z}\right)^{n-q-1} = z^{n-1}A_n(z^{-1})$$

by property (3.3).

- For $n \in \mathbb{N}\backslash\{0\}$ and $z \neq 0$, we have

$$A_n(z) = (1-z)^{n-1} + z\sum_{k=1}^{n-1} C_k^n A_k(z)(1-z)^{n-1-k}. \tag{3.6}$$

In fact, from (3.5) and (3.4), it follows that

$$A_n(z) = z^{n-1}\sum_{k=0}^{n-1} C_k^n A_k(z^{-1})(z^{-1}-1)^{n-1-k},$$

which is equivalent to

$$\begin{aligned} A_n(z) &= z^{n-1}\sum_{k=0}^{n-1} C_k^n A_k(z^{-1})\left(\frac{1-z}{z}\right)^{n-1-k} \\ &= \sum_{k=0}^{n-1} C_k^n A_k(z^{-1})(1-z)^{n-1-k} z^k. \end{aligned}$$

Now, since $A_k(z^{-1})z^k$ can be written as

$$\begin{aligned} A_k(z^{-1})z^k &= \sum_{q=0}^{k-1}\left\langle {k \atop q} \right\rangle \left(\frac{1}{z}\right)^q z^k = \sum_{q=0}^{k-1}\left\langle {k \atop q} \right\rangle z^{k-q} \\ &= \frac{z}{z}\sum_{q=0}^{k-1}\left\langle {k \atop q} \right\rangle z^{k-q} = z\sum_{q=0}^{k-1}\left\langle {k \atop q} \right\rangle z^{k-q-1} = z\sum_{q=0}^{k-1}\left\langle {k \atop q} \right\rangle z^{q} = zA_k(z) \end{aligned}$$

for $k \geq 1$, it follows that

$$A_n(z) = (1-z)^{n-1} + z\sum_{k=1}^{n-1} C_k^n A_k(z)(1-z)^{n-1-k}$$

3.2 Determinant of M_b^θ

Let us first note that matrix M_b^θ, which is given by

$$M_b^\theta = \begin{bmatrix} J_{1,k} & J_{2,k} & \dots & J_{\theta,k} \\ J_{0,k} & J_{1,k} & \dots & J_{\theta-1,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{1,k} \end{bmatrix},$$

is an upper Toeplitz–Hessenberg matrix, which is a class of almost upper triangular matrices (see [10]). Its determinant is given by

$$\det(M_b^\theta) = J_{1,k}\det(M_b^{\theta-1}) + \sum_{r=1}^{\theta-1}(-1)^{\theta-r} J_{\theta+1-r,k}(J_{0,k})^{\theta-r}\det(M_b^{r-1}), \tag{3.7}$$

where $\det(M_b^0) = 1$ by convention. We refer the reader to [4] for a complete proof of this property.

Theorem 3.3 *The determinant of matrix M_b^θ is given by*

$$\det(M_b^\theta) = \frac{(\Delta t)^\theta}{\theta!} \sum_{q=1}^{\theta} \left\langle {\theta \atop q-1} \right\rangle \big(\exp(-i2\pi k/N)\big)^{\theta+1-q}. \tag{3.8}$$

Proof To demonstrate this, we proceed by complete induction on θ.

- For $\theta = 1$, we obtain $\det(M_b^1) = \Delta t \exp(-i2\pi k/N) = J_{1,k}$, which is also the result obtained from equation (3.7).
- Let us now assume that equation (3.8) holds for all integers up to $m-1$:

$$\det(M_b^\ell) = \frac{(\Delta t)^\ell}{\ell!} \sum_{q=1}^{\ell} \left\langle {\ell \atop q-1} \right\rangle (\exp(-i2\pi k/N))^{\ell+1-q} \tag{3.9}$$

  for $1 \le \ell \le m-1$.
- To prove that equation (3.8) also holds for the integer m, let us first note that, from equations (3.7) and (3.9),

$$\det(M_b^m) = J_{1,k} \frac{(\Delta t)^{m-1}}{(m-1)!} \sum_{q=1}^{m-1} \left\langle {m-1 \atop q-1} \right\rangle z_k^{m-q} + \sum_{r=1}^{m-1} (-1)^{m-r} J_{m+1-r,k} (J_{0,k})^{m-r} \frac{(\Delta t)^{r-1}}{(r-1)!} \sum_{q=1}^{r-1} \left\langle {r-1 \atop q-1} \right\rangle z_k^{r-q},$$

  where $z_k = \exp(-i2\pi k/N)$.

  By replacing $J_{0,k} = z_k - 1$ and $J_{n,k} = \frac{(\Delta t)^n}{n!} z_k$ in this last equation, we obtain

$$\begin{aligned}\det(M_b^m) &= \frac{(\Delta t)^m}{(m-1)!} z_k \sum_{q=1}^{m-1} \left\langle {m-1 \atop q-1} \right\rangle z_k^{m-q} \\ &\quad + \sum_{r=1}^{m-1} (-1)^{m-r} \frac{(\Delta t)^{m+1-r}}{(m+1-r)!} z_k (z_k - 1)^{m-r} \\ &\quad \times \frac{(\Delta t)^{r-1}}{(r-1)!} \sum_{q=1}^{r-1} \left\langle {r-1 \atop q-1} \right\rangle z_k^{r-q} \\ &= \frac{m(\Delta t)^m}{m!} z_k \sum_{q=1}^{m-1} \left\langle {m-1 \atop q-1} \right\rangle z_k^{m-q} \\ &\quad + \frac{(\Delta t)^m}{m!} z_k \sum_{r=1}^{m-1} C_{r-1}^m (1-z_k)^{m-r} \sum_{q=1}^{r-1} \left\langle {r-1 \atop q-1} \right\rangle z_k^{r-q} \\ &= \frac{(\Delta t)^m}{m!} z_k \sum_{r=1}^{m} C_{r-1}^m (1-z_k)^{m-r} \sum_{q=1}^{r-1} \left\langle {r-1 \atop q-1} \right\rangle z_k^{r-q}.\end{aligned}$$

  By performing a change of variables on both summations and by using property (3.3), we obtain

$$\det(M_b^m) = \frac{(\Delta t)^m}{m!} z_k \sum_{r=0}^{m-1} C_r^m (1-z_k)^{m-r-1} \sum_{q=0}^{r-1} \left\langle {r \atop q} \right\rangle z_k^{q+1}.$$

Since $\sum_{q=0}^{r-1} \left\langle {r \atop q} \right\rangle z_k^{q+1} = 1$ when $r = 0$ (as $\det(M_b^0) = 1$) and using equation (3.6), we obtain

$$\det(M_b^m) = \frac{(\Delta t)^m}{m!} z_k \Big((1 - z_k)^{m-1} + z_k \sum_{r=1}^{m-1} C_r^m (1 - z_k)^{m-r-1} \sum_{q=0}^{r-1} \left\langle {r \atop q} \right\rangle z_k^q \Big)$$

$$= \frac{(\Delta t)^m}{m!} z_k A_m(z_k) = \frac{(\Delta t)^m}{m!} z_k \sum_{q=0}^{m-1} \left\langle {m \atop q} \right\rangle z_k^q$$

Finally, by applying equation (3.3) and by performing another change of variables on the summation index, we obtain

$$\det(M_b^m) = \frac{(\Delta t)^m}{m!} \sum_{q=1}^{m} \left\langle {m \atop q-1} \right\rangle z_k^{m+1-q},$$

which proves that equation (3.8) also holds for $\theta = m$.

From the complete induction principle, it follows that equation (3.8) holds for all $\theta \in \mathbb{N} \backslash \{0\}$. ∎

3.3 Singularities of M_b^θ

From equation (3.8), we study cases when $\det(M_b^\theta) = 0$ (or when M_b^θ is singular). When this condition is met, system (2.4) cannot be solved, and we are unable to compute the numerical derivatives of the given digitized function g. Evidently, $\det(M_b^\theta) = 0$ implies that

$$\sum_{q=1}^{\theta} \left\langle {\theta \atop q-1} \right\rangle \big(\exp(-i2\pi k/N)\big)^{\theta+1-q} = \sum_{q=1}^{\theta} \left\langle {\theta \atop q-1} \right\rangle z_k^{\theta+1-q} = 0.$$

Since $z_k \neq 0$, we are therefore searching for values of z_k satisfying

$$\sum_{q=0}^{\theta-1} \left\langle {\theta \atop q} \right\rangle z_k^{\theta-q} = \sum_{q=0}^{\theta-1} \left\langle {\theta \atop q} \right\rangle z_k^{\theta-q-1} = 0,$$

which is equivalent to

$$\sum_{q=0}^{\theta-1} \left\langle {\theta \atop q} \right\rangle z_k^q = 0 \tag{3.10}$$

by property (3.3).

From (3.2), we notice that the left-hand side of equation (3.10) corresponds to an Eulerian polynomial of degree $\theta - 1$. Generally, the zeros of Eulerian polynomials are not easy to compute. However, they are all negative real numbers. This implies that $\Im\{z_k\} = 0$. Furthermore, since $z_k = \exp(-i2\pi k/N)$ has an euclidean norm of 1 (*i.e.*, $\|z_k\|_2^2 = 1$), the only possible root corresponds to $z_k = -1$, which occurs when

$$\cos\left(\frac{2\pi k}{N}\right) = -1, \qquad k \in \{0, 1, \ldots, N-1\},$$

or equivalently when

$$\frac{2\pi k}{N} = \gamma\pi,$$

where γ is an odd integer and $k \in \{0, 1, \ldots, N-1\}$. This implies $k = N/2$.

In order for $k = N/2$ to be an integer, N must be chosen even. If we also choose θ as an even integer, we obtain

$$\begin{aligned}\sum_{q=0}^{\theta-1} \left\langle {\theta \atop q} \right\rangle z_k^q &= \sum_{q=0}^{\theta-1} \left\langle {\theta \atop q} \right\rangle (-1)^q \\ &= \sum_{q=0}^{\frac{\theta}{2}-1} \left\langle {\theta \atop q} \right\rangle (-1)^q + \sum_{q=\frac{\theta}{2}}^{\theta-1} \left\langle {\theta \atop q} \right\rangle (-1)^q \\ &= \sum_{q=0}^{\frac{\theta}{2}-1} \left\langle {\theta \atop q} \right\rangle (-1)^q + \sum_{q=\frac{\theta}{2}}^{\theta-1} \left\langle {\theta \atop \theta - q - 1} \right\rangle (-1)^q \\ &= \sum_{q=0}^{\frac{\theta}{2}-1} \left\langle {\theta \atop q} \right\rangle (-1)^q + \sum_{q=\frac{\theta}{2}-1}^{0} \left\langle {\theta \atop q} \right\rangle (-1)^{q-\theta+1} \\ &= \sum_{q=0}^{\frac{\theta}{2}-1} \left\langle {\theta \atop q} \right\rangle (-1)^q + (-1)^{1-\theta} \sum_{q=0}^{\frac{\theta}{2}-1} \left\langle {\theta \atop q} \right\rangle (-1)^q \\ &= 0,\end{aligned}$$

which implies $\det(M_b^\theta) = 0$ when both θ and N are even integers. If θ is chosen as an odd integer, we can easily show that $\det(M_b^\theta) \neq 0$.

In [2,3], it is pointed out that there are issues when choosing θ as an even integer, but no further analysis was given, because, having to cope with large numbers of data points, they were always using a power of 2 for N. We now have demonstrated that the issues come from the singularity of matrix M_b^θ when θ and N are both chosen as even integers. This therefore implies that splines of even degrees can be constructed, as long as N is chosen as an odd integer. Then, as a rule, when θ is odd, N can be odd or even. When θ is even, then N must be odd.

4 Polynomial Splines

To demonstrate that the resulting piecewise polynomials form a spline function, we must show that the interpolation functions $g_\theta^{(\ell)}$, $\ell = 0, 1, \ldots, \theta - 1$, are continuous on $[0, T]$. Let us therefore establish a general equation for the continuity of these functions.

4.1 Continuity Conditions

As already stated, a polynomial of degree $\theta - \ell$ was defined on each interval $[t_j, t_{j+1}[$, for $j = 0, 1, \ldots, N-1$, by

$$[g^{(\ell)}]_j^\theta(t) = \sum_{p=0}^{\theta-\ell} \frac{(t-t_j)^p}{p!} g_{j,\theta}^{(p+\ell)}, \qquad t \in [t_j, t_{j+1}[. \tag{4.1}$$

Clearly, for $\ell = 0, 1, \ldots, \theta - 1$, $[g^{(\ell)}]_j^\theta$ is continuous on every interval $[t_j, t_{j+1}[$. The only points that could be problematic are at $t = t_{j+1}$, for $j = 0, 1, \ldots, N-2$, where there might be first kind discontinuities. Therefore, for the functions to be continuous, we must have

$$\lim_{t \to t_{j+1}^-} \left([g^{(\ell)}]_j^\theta(t) \right) = g_{j+1,\theta}^{(\ell)} \tag{4.2}$$

for $j = 0, 1, \ldots, N-2$ and for $\ell = 0, 1, \ldots, \theta - 1$. By replacing (4.1) in (4.2), we obtain

$$g_{j+1,\theta}^{(\ell)} = \sum_{p=0}^{\theta-\ell} \frac{(\Delta t)^p}{p!} g_{j,\theta}^{(\ell+p)}, \tag{4.3}$$

which is an elegant continuity condition that is dependent of the degree θ of the polynomial spline and the derivative order ℓ. In order for all the polynomial pieces to connect smoothly, we must verify that equation (4.3) is satisfied.

4.2 Proof of Continuity

Theorem 4.1 *The numerical derivatives $g_{j,\theta}^{(\ell)}$ obtained from solving*

$$M_b^\theta F_b^\theta = B + C$$

are computed in such a way that

$$[g^{(\ell)}]_j^\theta(t) = \sum_{p=0}^{\theta-\ell} \frac{(t - t_j)^p g_{j,\theta}^{(p+\ell)}}{p!}, \qquad t \in [t_j, t_{j+1}]$$

satisfy the property

$$\left[g^{(\ell)}\right]_{j-1}^\theta(t_j) = \left[g^{(\ell)}\right]_j^\theta(t_j)$$

for all $j = 1, 2, \ldots, N-1$ and for all $\ell = 0, 1, \ldots, \theta - 1$.

Hence, g_θ is a spline function of degree θ on the interval $[0, T]$.

Proof Let us first note that system (2.4), which needs to be solved to obtain the piecewise polynomials, can be expressed as

$$\sum_{p=0}^{\theta-\ell} J_{p,k} F_{p+\ell,k,\theta} = b_\ell$$

for $\ell = 0, 1, \ldots, \theta - 1$.

By expanding $J_{p,k}$ in the last equation, we can write it as

$$\sum_{p=0}^{\theta-\ell} \frac{(\Delta t)^p}{p!} F_{p+\ell,k,\theta} \exp(-i2\pi k/N) = b_\ell + F_{\ell,k,\theta}$$

or in an equivalent manner as

$$\sum_{p=0}^{\theta-\ell} \frac{(\Delta t)^p}{p!} F_{p+\ell,k,\theta} = (b_\ell + F_{\ell,k,\theta}) \exp(i2\pi k/N). \tag{4.4}$$

To prove that the continuity condition (4.3) is satisfied, we begin by multiplying equation (4.4) by $\exp(i2\pi kj/N)/N$ and then taking the sum over all values of k

(*i.e.*, $k = 0, 1, \ldots, N-1$). From this, it follows that:

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{p=0}^{\theta-\ell}\frac{(\Delta t)^p}{p!}F_{p+\ell,k,\theta}\exp(i2\pi kj/N) = \frac{1}{N}\sum_{k=0}^{N-1}(b_\ell + F_{\ell,k,\theta})\exp(i2\pi k(j+1)/N).$$

By interchanging the summations on the left-hand side, we obtain

$$\frac{1}{N}\sum_{p=0}^{\theta-\ell}\frac{(\Delta t)^p}{p!}\sum_{k=0}^{N-1}F_{p+\ell,k,\theta}\exp(i2\pi kj/N) = \frac{1}{N}\Big(b_\ell\sum_{k=0}^{N-1}\exp(i2\pi k(j+1)/N) + \sum_{k=0}^{N-1}F_{\ell,k,\theta}\exp(i2\pi k(j+1)/N)\Big).$$

Since

$$\sum_{k=0}^{N-1}\exp(i2\pi k(j+1)/N) = \frac{1-\exp(i2\pi(j+1))}{1-\exp(i2\pi(j+1)/N)} = 0$$

for $j = 0, 1, \ldots, N-2$ (see [9] for properties on complex exponential series), and using equation (2.5), we obtain

$$\sum_{p=0}^{\theta-\ell}\frac{(\Delta t)^p}{p!}g_{j,\theta}^{(p+\ell)} = g_{j+1,\theta}^{(\ell)}$$

for $\ell = 0, 1, \ldots, \theta-1$.

This proves that for any chosen degree θ, the derivatives are such that the piecewise polynomials are continuous at every break point, which makes the resulting functions continuous over the interval $[0, T]$. ∎

5 Conclusions

The initial goal of Beaudoin and Beauchemin was to develop a method that would approximate the Fourier transform of a digitized function more accurately than the DFT. By doing so, they realized that their method not only led to more accurate results for the Fourier transform, but also seemed to enable spline interpolation of any odd degree.

In our work, we have not only formally demonstrated that the resulting interpolation functions are spline functions, but also that we are able to obtain these spline functions for both even and odd degree approximations. The only condition to creating even-degree splines is to make sure to use an odd number N of intervals. Otherwise, matrix M_b^θ becomes singular, and the numerical derivatives at the nodes cannot be calculated. This same result for even-degree splines was mentioned in [1].

The advantage of our method to building splines of any degree is that it enables us to obtain high-degree splines very easily. However, the issue of effectively approximating the boundary conditions still persists. In [3], Beaudoin and Beauchemin initially proposed a method to accurately compute these boundary conditions for any given order, though the method lacks robustness. The next step is to find an efficient and robust algorithm to accurately compute the boundary condition vector B^θ. With such a method, which is the goal of a forthcoming contribution, higher degree spline interpolation could be used for numerical problems necessitating high accuracy.

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