SOME RESULTS FOR THE GENERALIZED LOTOTSKY TRANSFORM

V. F. COWLING AND C. L. MIRACLE

1. Introduction. Let $A = (a_{nk})$ and $x = \{s_n\}$ $(n,k = 0,1,2,\ldots)$ be a matrix and a sequence of complex numbers, respectively. We write formally

$$(1.1)\qquad t_n = \sum_{k=0}^{\infty} a_{nk}s_k,$$

and say that the sequence x is summable A to the sum t or that the A matrix sums the sequence x to the value t if the series in (1.1) converges and

$$\lim_{n\to\infty} t_n$$

exists and equals t. We say that the matrix A is regular provided it sums every convergent sequence to its limit. Well-known necessary and sufficient conditions in order that a matrix A be regular are:

$$(1.2)\qquad \sum_{k=0}^{\infty} |a_{nk}| \leqslant M \qquad (n = 0, 1, \ldots),$$

$$(1.3)\qquad \lim_{n\to\infty} a_{nk} = 0 \qquad (k = 0, 1, \ldots),$$

$$(1.4)\qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1,$$

where M is a constant independent of n.

Let $\{d_n\}$ $(n = 1, 2, 3, \ldots)$ be a given sequence of complex numbers with $d_n \neq -1$. To simplify the notation and computations of this paper we will also assume that $d_n \neq 0$. Using the given sequence $\{d_n\}$, we define the elements P_{nk} of the $[F, d_n]$ matrix by the relations

$$(1.5)\qquad \begin{aligned} &P_{00} = 1 \\ &P_{0k} = 0 \qquad k \neq 0 \\ &\prod_{j=1}^{n} \frac{\theta + d_j}{1 + d_j} = \sum_{k=0}^{\infty} P_{nk}\theta^k \qquad (n \geqslant 1). \end{aligned}$$

Notice that P_{nk} are undefined for $-\infty < k < 0$ and $P_{nk} = 0$ for $n < k < \infty$. If needed, one can define $P_{nk} = 0$ for $-\infty < k < 0$. Jakimovski **(3)** has shown that the $[F, d_n]$ matrix so defined is regular provided the following three conditions hold: (1) d_n is real, (2) $d_n \geqslant 0$ for $n \geqslant n_0$ where n_0 is some integer, and (3) $\sum_{n=1}^{\infty} d_n^{-1}$ diverges. He has also shown that if the above three conditions hold and in addition that $\sum_{n=1}^{\infty} d_n^{-2}$ converges, then the $[F, d_n]$ matrix sums the sequence of partial sums of the geometric series to the value $(1-z)^{-1}$

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for $\mathrm{Re}(z) < 1$, and uniformly in any bounded domain inside the half-plane $\mathrm{Re}(z) < 1$. One of the results of this paper is that the above conclusion follows from the first three hypotheses without requiring the extra assumption that $\sum_{n=1}^{\infty} d_n^{-2}$ converges.

This paper falls naturally into four sections. First, we obtain several necessary conditions bearing on a complex sequence $\{d_n\}$ in order that the corresponding $[F, d_n]$ matrix be regular. Secondly, with these conditions in mind, we obtain one set of sufficient conditions bearing on a complex sequence $\{d_n\}$ in order that the corresponding $[F, d_n]$ matrix be regular. This matrix has the regular Euler matrix as studied by Agnew **(1)**, the Lototsky matrix as studied by Lototsky **(5)** and Agnew **(2)**, and the $[F, d_n]$ matrix as studied by Jakimovski **(3)** as special cases. Thirdly, we find (if $\lim |d_n| = \infty$ or $\lim d_n$ exists) the domain in which this regular $[F, d_n]$ matrix sums the geometric series to the value $(1 - z)^{-1}$. The domain in which this matrix sums any power series then follows from a theorem of Okada **(7)**. Finally, it was shown by Agnew **(1)** that the Euler transform provides a method of analytic continuation for cases in which the method is not regular. We study certain non-regular $[F, d_n]$ matrices from this point of view.

2. Necessary conditions in order that the $[F, d_n]$ matrix be regular. Throughout this paper we will assume that the sequence $\{d_n\}$ $(n = 1, 2, 3, \ldots)$ is a given sequence of complex numbers subject perhaps to certain conditions. Using this sequence we then define the elements P_{nk} of the $[F, d_n]$ matrix by (1.5). Also we will always use the notation, $|d_n| = \rho_n$ and $\arg d_n = \theta_n$ $(-\pi < \theta_n \leqslant \pi)$.

THEOREM 2.1. *Let $d_n = x_n + iy_n$. A necessary condition in order that the $[F, d_n]$ matrix be regular is that there exists a monotone increasing sequence of natural numbers $\{n_k\}$ such that*

$$\sum_{k=1}^{\infty} \frac{1 + 2x_{n_k}}{|1 + d_{n_k}|^2} = +\infty.$$

Proof. Assume that the $[F, d_n]$ matrix is regular. Upon setting $\theta = 0$ in (1.5), we get

$$P_{n0} = \prod_{j=1}^{n} \frac{d_j}{1 + d_j}.$$

Since the numbers P_{n0} must satisfy (1.3), it follows that

$$\lim_{n\to\infty} \prod_{j=1}^{n} \frac{d_j}{1 + d_j} = 0.$$

However,

$$\prod_{j=1}^{\infty} \frac{d_j}{1 + d_j} = 0$$

if and only if

$$\prod_{j=1}^{\infty} \frac{|d_j|^2}{|1+d_j|^2} = 0.$$

But

$$\prod_{j=1}^{\infty} \frac{|d_j|^2}{|1+d_j|^2} = 0$$

if and only if there exists a monotone increasing sequence of natural numbers $\{n_k\}$ such that

$$\sum_{k=1}^{\infty} \left(-1 + \left|\frac{d_{nk}}{1+d_{nk}}\right|^2\right) = - \sum_{k=1}^{\infty} \frac{1+2x_{nk}}{|1+d_{nk}|^2}$$

diverges to minus infinity.

COROLLARY 2.1. *A necessary condition in order that the* $[F, d_n]$ *matrix be regular is that* $x_n \geqslant -\frac{1}{2}$ *for infinitely many values of* n.

COROLLARY 2.2. *A necessary condition in order that the* $[F, d_n]$ *matrix be regular is that* $\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty$.

Proof. Assume that $\sum_{n=1}^{\infty} \rho_n^{-1}$ is convergent. Since $\rho_n > 0$ this implies that

$$\lim_{n\to\infty} \rho_n = +\infty.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1+x_n}{|1+d_n|^2}$$

is absolutely convergent. This in turn implies that if $\{n_k\}$ is any sequence of integers, then

$$\sum_{k=1}^{\infty} \frac{1+x_{nk}}{|1+d_{nk}|^2}$$

is also convergent. Since the $[F, d_n]$ matrix is regular, this is a contradiction to Theorem 2.1.

Meir **(6)** has constructed an example of a real sequence $\{d_n\}$ which is regular and does not satisfy Jakimovski's **(3)** sufficient conditions for regularity. Hence these conditions are not necessary. Meir's sequence is an interesting example showing that Corollary 2.1 is the strongest theorem of this type one can obtain.

THEOREM 2.2. *Let* α *be given where* $0 < \alpha < \pi/2$. *If there is a positive integer* N *such that* $\alpha < \theta_n$ *for all* $n \geqslant N$, *then the* $[F, d_n]$ *matrix is not regular.*

Proof. Assume the $[F, d_n]$ matrix is regular. Define the terms λ_n of the sequence $\{\lambda_n\}$ by $\lambda_n = \rho_n e^{i\beta_n}$, where $\beta_n = \theta_n - \alpha$. Define the elements b_{nk} of the $[F, \lambda_n]$ matrix by the relation

$$b_{00} = 1$$

$$b_{0k} = 0 \qquad k \neq 0$$

$$\prod_{j=1}^{n} \frac{\theta + \lambda_j}{1 + \lambda_j} = \sum_{k=0}^{\infty} b_{nk}\theta^k. \tag{2.1}$$

Using the Cauchy integral formula, the numbers P_{nk} in (1.5) are given by

$$P_{nk} = \frac{1}{2\pi i} \int_c \prod_{j=1}^{n} \left(\frac{t + d_j}{1 + d_j}\right) \frac{dt}{t^{k+1}},$$

where c is any closed curve containing the origin. Integrating we get

$$P_{nk} = \frac{1}{\prod_{j=1}^{n} (1 + d_j)} \sum_{s_1+s_2+\ldots+s_n+k=n} d_1^{s_1} d_2^{s_2} \ldots d_n^{s_n}, \tag{2.2}$$

where $s_i = 0$ or 1 and the sum is taken over all possible values of s_i such that $s_1 + s_2 + \ldots + s_n = n - k$. Similarly, the numbers b_{nk} as defined by (2.1) are given by

$$b_{nk} = \frac{1}{\prod_{j=1}^{n} (1 + \lambda_j)} \sum_{s_1+s_2+\ldots+s_n+k=n} \lambda_1^{s_1} \lambda_2^{s_2} \ldots \lambda_n^{s_n},$$

where the sum is the same type as in (2.2). Now since $\lambda_n = d_n e^{-i\alpha}$, it follows that

$$b_{nk} = \frac{1}{\prod_{j=1}^{n} (1 + \lambda_j)} \sum_{s_1+s_2+\ldots+s_n+k=n} d_1^{s_1} d_2^{s_2} \ldots d_n^{s_n}\, e^{i(n-k)\alpha},$$

or that

$$\left| \sum_{s_1+s_2+\ldots+s_n+k=n} d_1^{s_1} d_2^{s_2} \ldots d_n^{s_n} \right| = |b_{nk}| \prod_{j=1}^{n} |1 + \lambda_j|. \tag{2.3}$$

Inserting absolute values in (2.2), summing over k from 1 to n, and making use of (2.3), we have

$$\sum_{k=0}^{n} |P_{nk}| = \prod_{j=1}^{n} \left| \frac{1 + \lambda_j}{1 + d_j} \right| \sum_{k=0}^{n} |b_{nk}|. \tag{2.4}$$

Substituting $\theta = 1$ in (2.1) yields $\sum_{k=0}^{n} b_{nk} = 1$. Hence $\sum_{k=0}^{n} |b_{nk}| \geqslant |\sum_{k=0}^{n} b_{nk}| = 1$. Using this inequality, (2.4) becomes

$$\sum_{k=0}^{n} |P_{nk}| \geqslant \prod_{j=1}^{n} \left| \frac{1 + \lambda_j}{1 + d_j} \right|. \tag{2.5}$$

Since the $[F, d_n]$ matrix is by assumption regular, it satisfies (1.2). Hence (2.5) implies that

$$\prod_{j=1}^{n}\left(\frac{1+\lambda_j}{1+d_j}\right)$$

is a bounded function of n.

From the hypothesis of the theorem and the fact that $-\pi < \theta_n \leqslant \pi$, there exist N such that $\alpha < \theta_n \leqslant \pi$ for all $n \geqslant N$. This implies that $\theta_n > \beta_n > 0$ and $\cos\theta_n < \cos\beta_n$ for $n \geqslant N$. It follows that $|1+\lambda_n|^2 > |1+d_n|^2$ for $n \geqslant N$. This means that

$$\prod_{j=1}^{n}\left|\frac{1+\lambda_j}{1+d_j}\right|$$

is a monotone function of n for $n \geqslant N$. Therefore, a necessary condition in order that

$$\prod_{j=1}^{n}\left|\frac{1+\lambda_j}{1+d_j}\right|$$

be bounded for all n is that

$$\prod_{j=1}^{\infty}\left|\frac{1+\lambda_j}{1+d_j}\right|^2$$

be convergent. From a well-known theorem on infinite products,

$$\prod_{j=1}^{\infty}\left|\frac{1+\lambda_j}{1+d_j}\right|^2$$

is convergent if and only if

$$\sum_{j=1}^{\infty}\left(-1+\left|\frac{1+\lambda_j}{1+d_j}\right|^2\right)$$

is convergent. If $n \geqslant N$, $\alpha < \beta_n + \theta_n < 2\pi - \alpha$ and so

$$\sin\left(\frac{\beta_n+\theta_n}{2}\right) \geqslant \sin\frac{\alpha}{2}.$$

Hence if $n \geqslant N$, then $\cos\beta_n - \cos\theta_n \geqslant 2\sin^2\alpha/2$. Using this fact, we have that if $n \geqslant N$, then

$$\left(-1+\left|\frac{1+\lambda_n}{1+d_n}\right|^2\right) = \frac{2\rho_n(\cos\beta_n-\cos\theta_n)}{1+2\rho_n\cos\theta_n+\rho_n^2} \geqslant \frac{4\rho_n\sin^2\left(\dfrac{\alpha}{2}\right)}{(1+\rho_n)^2}.$$

By assumption the $[F, d_n]$ matrix is regular and so by Corollary 2.2 $\sum_{n=1}^{\infty}\rho_n^{-1} = +\infty$. Hence

$$\sum_{n=1}^{\infty}\left(-1+\left|\frac{1+\lambda_n}{1+d_n}\right|^2\right)$$

and thus

$$\prod_{n=1}^{\infty}\left|\frac{1+\lambda_n}{1+d_n}\right|^2$$

diverge to plus infinity. Consequently (2.5) implies that $\sum_{k=0}^{n}|P_{nk}|$ is not uniformly bounded for all n, and thus (1.2) is not satisfied. This is a contradiction to the assumption that the $[F, d_n]$ matrix is regular.

THEOREM 2.3. *Let α be given where $-\pi/2 < \alpha < 0$. If there is a positive integer N such that $\theta_n < \alpha$ for all $n \geqslant N$, then the $[F, d_n]$ matrix is not regular.*

Proof. Suppose the $[F, d_n]$ matrix is regular. Define the terms λ_n of the sequence $\{\lambda_n\}$ by $\lambda_n = \rho_n e^{-i\theta_n}$. The elements A_{nk} of the $[F, \lambda_n]$ matrix are defined by the relation

$$\prod_{j=1}^{n} \frac{\theta + \lambda_j}{1 + \lambda_j} = \sum_{k=0}^{\infty} A_{nk}\theta^k, A_{00} = 1, A_{0k} = 0, k \neq 0. \tag{2.6}$$

Since $\lambda_j = \bar{d}_j$, where the bar denotes complex conjugate, it follows from (1.5) and (2.6) that

$$\sum_{k=0}^{n} \overline{A}_{nk}\theta^k = \sum_{k=0}^{n} P_{nk}\theta^k$$

for all θ. This implies that

$$\sum_{k=0}^{n} |P_{nk}| = \sum_{k=0}^{n} |A_{nk}|. \tag{2.7}$$

But the sequence $\{\lambda_n\}$ and the numbers A_{nk} defined by (2.6) satisfy exactly the same conditions as did the sequence $\{d_n\}$ and the numbers P_{nk} in the proof of Theorem 2.2. Consequently, using the same method as in Theorem 2.2, we may conclude that $\sum_{k=0}^{n}|A_{nk}|$ is not a bounded function of n. Hence (2.7) implies that $\sum_{k=0}^{n}|P_{nk}|$ is not a bounded function of n and so (1.2) is not satisfied. This contradicts the assumption that the $[F, d_n]$ matrix is regular.

The theorem to be proved next is included here largely because its proof best displays the central idea used in the proofs of Theorem 2.2 and Theorem 2.3. It is also much simpler to apply.

THEOREM 2.4. *Suppose that $\theta_n = \alpha$ $(n = 1, 2, \ldots)$, and that the corresponding $[F, d_n]$ matrix is regular, then $\theta_n = \alpha = 0$.*

Proof. Recall that $|d_n| = \rho_n$, and let a_{nk} denote the elements of the $[F,\rho_n]$ matrix. Note that $|a_{nk}| = a_{nk}$. Then using the same method that was employed to arrive at the inequality (2.5), we can obtain the equation

$$\sum_{k=0}^{n} |P_{nk}| = \prod_{j=1}^{n} \frac{1 + \rho_j}{|1 + d_j|}.$$

Hence the regularity condition (1.2) implies that

$$\prod_{j=1}^{\infty} \frac{1 + \rho_j}{|1 + d_j|}$$

is convergent. However, the terms of this product are real and greater than

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or equal to one, and so by a well-known theorem this product is convergent if and only if

$$\sum_{n=1}^{\infty}\left\{-1+\left(\frac{1+\rho_n}{|1+d_n|}\right)^2\right\}=\sum_{n=1}^{\infty}\frac{2\rho_n(\cos\alpha-1)}{1+2\rho_n\cos\theta_n+\rho_n^2}$$

is convergent. But from Corollary 2.2 $\sum_{n=1}^{\infty}\rho_n^{-1}=+\infty$, so this series is divergent unless $\alpha=0$. Hence in order for the regularity condition (1.2) to be satisfied, we must have $\alpha=0$.

3. A regular $[F, d_n]$ **matrix.** In view of the theorems in § 2, one need only consider two types of non-real sequences $\{d_n\}$ in attempting to construct a regular $[F, d_n]$ matrix. Firstly, those sequences for which

$$\lim_{n\to\infty}\arg d_n=0 \qquad (-\pi<\arg d_n\leqslant\pi).$$

Secondly, those sequences $\{d_n\}$ which satisfy both the following conditions: (1) there exists a subsequence $\{n_k\}$ of the sequence of natural numbers such that $\mathrm{Re}(d_{n_k})>-\frac{1}{2}$ and $\mathrm{Im}(d_{n_k})>0$, and (2) there exists a subsequence $\{m_k\}$ of the sequence of natural numbers such that $\mathrm{Re}(d_{m_k})>-\frac{1}{2}$ and $\mathrm{Im}(d_{m_k})<0$. In this section we will give an example of a $[F, d_n]$ matrix of the first type which is regular and for which $\theta_n\neq 0$. We leave the following as an open problem: to find a sequence $\{d_n\}$ of the second type such that

$$\lim_{n\to\infty}\arg d_n\neq 0$$

and such that the $[F, d_n]$ matrix is regular; or to show that no such sequence exists.

THEOREM 3.1. *Suppose that* $\sum_{n=1}^{\infty}\rho_n^{-1}$ *is divergent and* $\sum_{n=1}^{\infty}\theta_n^2\rho_n^{-1}$ *is convergent, then the* $[F, d_n]$ *transform is regular.*

Proof. Substituting $\theta=1$ in the defining relationship (1.5), we get $\sum_{k=0}^{n}P_{nk}=1$ and so condition (1.4) is satisfied.

Note that the elements a_{nk} of the $[F, \rho_n]$ matrix also satisfy $\sum_{k=0}^{n}|a_{nk}|=\sum_{k=0}^{n}a_{nk}=1$. From this fact and a relationship of the type (2.2) for the numbers a_{nk}, it follows that

$$\sum_{k=0}^{n}\sum_{s_1+s_2+\ldots+s_n+k=n}\rho_1^{s_1}\rho_2^{s_2}\ldots\rho_n^{s_n}=\prod_{j=1}^{n}(1+\rho_j),\ (n\geqslant 1). \tag{3.1}$$

Inserting absolute value signs on both sides of (2.2) and making use of (3.1), we obtain

$$\begin{aligned}\sum_{k=0}^{n}|P_{nk}| &\leqslant \frac{1}{\prod_{j=1}^{n}|1+d_j|}\sum_{k=0}^{n}\sum_{s_1+s_2+\ldots+s_n+k=n}|d_1^{s_1}d_2^{s_2}\ldots d_n^{s_n}| \\ &\leqslant \prod_{j=1}^{n}\frac{1+\rho_j}{|1+d_j|}.\end{aligned} \tag{3.2}$$

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Note that the convergence of $\sum_{n=1}^{\infty}\theta_n^2\rho_n^{-1}$ and the divergence of $\sum_{n=1}^{\infty}\rho_n^{-1}$ implies that

$$\lim_{n\to\infty}\theta_n = 0.$$

Hence there exists an integer N such that $|\theta_n| < \pi/3$ for $n \geqslant N$. Thus for $n \geqslant N$,

$$-1 + \left(\frac{1+\rho_n}{|1+d_n|}\right)^2 = \frac{2\rho_n(1-\cos\theta_n)}{1+2\rho_n\cos\theta_n+\rho_n^2} \leqslant \frac{\rho_n\theta_n^2}{\rho_n^2} = \frac{\theta_n^2}{\rho_n}.$$

Now since $1 + x \leqslant e^x$ for x real and $1 + \rho_n \geqslant |1 + d_n|$, it follows upon applying the above inequality that

$$\frac{1+\rho_n}{|1+d_n|} \leqslant \left(\frac{1+\rho_n}{|1+d_n|}\right)^2 \leqslant \exp\left\{-1 + \left(\frac{1+\rho_n}{|1+d_n|}\right)^2\right\} \leqslant \exp\,(\theta_n^2\rho_n^{-1})$$

for $n \geqslant N$. In view of this inequality, (3.2) may be written as

$$\sum_{k=0}^{n} |P_{nk}| \leqslant \prod_{j=1}^{N-1}\left(\frac{1+\rho_j}{|1+d_j|}\right)\exp\left\{\sum_{j=N}^{n}\theta_j^2\rho_j^{-1}\right\} \qquad (n \geqslant N).$$

But by hypothesis $\sum_{k=1}^{\infty}\theta_k^2\rho_k^{-1}$ is convergent. Therefore, $\sum_{k=1}^{n}|P_{nk}|$ is uniformly bounded for all n and condition (1.2) is satisfied.

In order to show that the $[F, d_n]$ transform is regular it remains only to show that (1.3) is satisfied. Using the Cauchy integral formula, (1.5) implies that

$$P_{nk} = \frac{1}{2\pi i}\int_c \prod_{j=1}^{n}\left(\frac{t+d_j}{1+d_j}\right)\frac{dt}{t^{k+1}}, \tag{3.4}$$

where c is a circle with centre at the origin and radius $1/4$. Recall that N is an integer such that $|\theta_n| < \pi/3$ for $n \geqslant N$. Since $1 + z < e^z$ for real z, it follows with $t = x + iy$ that

$$\begin{aligned}
\left|\frac{t+d_n}{1+d_n}\right|^2 &\leqslant \exp\left\{-1 + \left|\frac{t+d_n}{1+d_n}\right|^2\right\} \\
&\leqslant \exp\left\{\frac{-\frac{15}{16} + 2\rho_n\cos\theta_n(x-1) + 2\rho_n y\sin\theta_n}{1+2\rho_n\cos\theta_n+\rho_n^2}\right\} \\
&\leqslant \exp\left\{\frac{-2\rho_n\cos\theta_n + \frac{1}{2}\rho_n}{1+2\rho_n\cos\theta_n+\rho_n^2}\right\} \\
&\leqslant \exp\left\{\frac{-\rho_n}{2(1+\rho_n)^2}\right\}
\end{aligned}$$

for $n \geqslant N$. Taking the positive square root of both sides of this inequality, we obtain the inequality

$$\left|\frac{t+d_n}{1+d_n}\right| \leqslant \exp\left\{\frac{-\rho_n}{4(1+\rho_n)^2}\right\}. \tag{3.5}$$

Hence upon inserting absolute values on both sides of (3.4) and using the inequality (3.5), we get

$$|P_{nk}| \leqslant \exp\left\{-\sum_{j=N}^{n} \frac{\rho_j}{4(1+\rho_j)^2}\right\} \frac{4^k}{2\pi} \int_0^{2\pi} \prod_{j=1}^{N-1} \left|\frac{t+d_j}{1+d_j}\right| d\theta.$$

But

$$\prod_{j=1}^{N-1} \left|\frac{t+d_j}{1+d_j}\right|$$

is the absolute value of a polynomial in t of degree $N-1$, and such a polynomial has a maximum value M on $|t| = \frac{1}{4}$. Hence

$$|P_{nk}| \leqslant M \cdot 4^k \exp\left\{-\sum_{j=N}^{n} \frac{\rho_j}{4(1+\rho_j)^2}\right\} \tag{3.6}$$

for $n \geqslant N$. By assumption $\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty$, and so

$$\lim_{n\to\infty} \sum_{j=N}^{n} \frac{-\rho_j}{4(1+\rho_j)^2} = -\infty.$$

Hence, (3.6) implies that

$$\lim_{n\to\infty} |P_{nk}| = 0$$

for all k.

4. Summation of power series. Suppose we have a function $g(z)$ defined by a power series with a non-zero radius of convergence, then we would like to know in what domain the $[F, d_n]$ matrix of Theorem 3.1 sums the sequence of partial sums of this power series to its analytic continuation by radial extension. In view of a theorem by Okada **(7)**, one need only consider the domain in which the geometric series with partial sums

$$s_n(z) = (1-z)^{-1} - z^{n+1}(1-z)^{-1} \tag{4.1}$$

is summed to its analytic continuation $(1-z)^{-1}$.

THEOREM 4.1. *Suppose* $\sum_{n=1}^{\infty} \rho_n^{-1}$ *is divergent,* $\sum_{n=1}^{\infty} \theta_n^2 \rho_n^{-1}$ *is convergent, and*

$$\lim_{n\to\infty} \rho_n = +\infty,$$

then the $[F, d_n]$ *matrix sums the sequence whose terms are given by* (4.1) *to* $(1-z)^{-1}$ *if* $\mathrm{Re}(z) < 1$.

Proof. Let $\{\sigma_n(z)\}$ denote the $[F, d_n]$ transform of the sequence $\{s_n(z)\}$ given by (4.1), then

$$\sigma_n(z) = (1-z)^{-1} \sum_{k=0}^{n} P_{nk} - z(1-z)^{-1} \sum_{k=0}^{n} P_{nk} z^k. \tag{4.2}$$

Using the fact that $\sum_{k=0}^{n} P_{nk} = 1$ plus the relation (1.5), (4.2) can be written as

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$$\sigma_n(z) = (1 - z)^{-1} - z(1 - z)^{-1} \prod_{j=1}^{n} \frac{z + d_j}{1 + d_j}. \tag{4.3}$$

Let $z = x + iy$. Since $1 + w \leqslant e^w$ for w real, we have

$$\left|\frac{z + d_n}{1 + d_n}\right|^2 \leqslant \exp\left\{\frac{2\rho_n[(x - 1)\cos\theta_n + y\sin\theta_n]}{1 + 2\rho_n\cos\theta_n + \rho_n^2} + \frac{2\epsilon_n}{\rho_n}\right\}, \tag{4.4}$$

where

$$2\epsilon_n = \frac{|z|^2 - 1}{|1 + d_n|^2}\rho_n.$$

Note that

$$\lim_{n\to\infty} \epsilon_n = 0$$

since

$$\lim_{n\to\infty} \rho_n = \infty.$$

Taking the positive square root of both sides of (4.4), we get

$$\left|\frac{z + d_n}{1 + d_n}\right| \leqslant \exp\left\{\frac{\rho_n[(x - 1)\cos\theta_n + y\sin\theta_n]}{1 + 2\rho_n\cos\theta_n + \rho_n^2} + \frac{\epsilon_n}{\rho_n}\right\}. \tag{4.5}$$

Since

$$\lim_{n\to\infty} \theta_n = 0$$

and $\sum_{n=1}^{\infty}\rho_n^{-1} = +\infty$, (4.5) implies that

$$\prod_{j=1}^{\infty} \frac{z + d_j}{1 + d_j} = 0,$$

if $x - 1 < 0$. Therefore (4.3) implies that

$$\lim_{n\to\infty} \sigma_n(z) = (1 - z)^{-1}$$

if $\operatorname{Re}(z) < 1$. This completes the proof of the theorem.

Corollary 4.1. *Suppose* $\sum_{n=1}^{\infty}\rho_n^{-1} = +\infty$, $\sum_{n=1}^{\infty}\rho_n^{-1}\theta_n = +\infty$, $\sum_{n=1}^{\infty}\rho_n^{-1}\theta_n^2 < +\infty$, *and*

$$\lim_{n\to\infty} \rho_n\theta_n = +\infty,$$

then the $[F, d_n]$ *matrix sums the sequence* $\{s_n(z)\}$ *whose terms are given by* (4.1) *to its analytic continuation* $(1 - z)^{-1}$ *for all* z *such that* $\operatorname{Re}(z) < 1$ *and also for all* z *such that* $\operatorname{Re}(z) = 1$ *and* $\operatorname{Im}(z) < 0$.

Proof. Since

$$\lim_{n\to\infty} \theta_n = 0 \text{ and } \lim_{n\to\infty} \rho_n = +\infty,$$

this corollary follows directly from the theorem except for z such that

$\operatorname{Re}(z) = 1$ and $\operatorname{Im}(z) < 0$. As in the previous theorem let $z = x + iy$. Placing $x = 1$ in (4.4), we get

$$\left|\frac{z + d_n}{1 + d_n}\right|^2 \leqslant \exp\left\{\frac{2\rho_n y \sin\theta_n + y^2}{1 + 2\rho_n \cos\theta_n + \rho_n^2}\right\}.$$

Since

$$\lim_{n\to\infty} \rho_n\theta_n = +\infty,$$

there exists N such that $2\rho_n \sin\theta_n > -y + \rho_n\theta_n$ if $n \geqslant N$. Since $y < 0$, we have

$$\left|\frac{z + d_n}{1 + d_n}\right| \leqslant \exp\left\{\frac{\rho_n\theta_n y}{2(1 + \rho_n)^2}\right\} \text{ whenever } n \geqslant N.$$

Since $y < 0$ and $\sum_{n=1}^{\infty}\rho_n^{-1}\theta_n$ diverges to plus infinity the corollary follows.

If we take $\rho_n = n^{\alpha}$ and $\theta_n = n^{-1+\alpha}$, the hypotheses of Corollary 4.1 are satisfied for $\frac{1}{2} < \alpha < 1$.

COROLLARY 4.2. *Suppose* $\sum_{n=1}^{\infty}\rho_n^{-1} = +\infty$, $\sum_{n=1}^{\infty}\rho_n^{-1}\theta_n = -\infty$. $\sum_{n=1}^{\infty}\rho_n^{-1}\theta_n^2 < +\infty$, *and*

$$\lim_{n\to\infty} \rho_n\theta_n = +\infty,$$

then the $[F, d_n]$ *matrix sums the the sequence* $\{s_n(z)\}$ *whose terms are given by* (4.1) *to its analytic continuation* $(1 - z)^{-1}$ *for all* z *such that* $\operatorname{Re}(z) < 1$ *and also for all* z *such that* $\operatorname{Re}(z) = 1$ *and* $\operatorname{Im}(z) > 0$.

The proof of this corollary is similar to the proof of Corollary 4.1.

THEOREM 4.2. *Suppose* $\sum_{n=1}^{\infty}\rho_n^{-1}$ *is divergent,*

$$\lim_{n\to\infty} \theta_n = 0,$$

and

$$\lim_{n\to\infty} \rho_n = +\infty,$$

then the sequence $\{\sigma_n(z)\}$, *whose terms are given by* (4.2), *is divergent if* $\operatorname{Re}(z) > 1$.

Proof. Assume z is given such that $\operatorname{Re}(z) > 1$. From (4.3) it follows that the sequence $\{\sigma_n(z)\}$ is divergent if the product

$$\prod_{n=1}^{\infty} \left|\frac{z + d_n}{1 + d_n}\right|^2$$

is divergent to infinity. Since $\operatorname{Re}(z) > 1$, we have $|z + d_n| > |1 + d_n|$ for n sufficiently large. Therefore, using a well-known theorem, the product

$$\prod_{n=1}^{\infty} \left|\frac{z + d_n}{1 + d_n}\right|^2$$

is divergent to infinity if and only if the series

$$\sum_{n=1}^{\infty} \left(-1 + \left|\frac{z+d_n}{1+d_n}\right|^2\right) \tag{4.6}$$

diverges to infinity. Thus to prove the theorem we need only show that (4.6) diverges to infinity. A typical term of the series (4.6) is given by

$$-1 + \left|\frac{z+d_n}{1+d_n}\right|^2 = \frac{|z|^2 - 1 + 2(x-1)\rho_n \cos\theta_n + 2y\rho_n \sin\theta_n}{|1+d_n|^2} \tag{4.7}$$

where $z = x + iy$. Let α $(0 < \alpha < \pi/2)$ be given. Since $x > 1$,

$$\lim_{n\to\infty} \theta_n = 0,$$

and

$$\lim_{n\to\infty} \rho_n = \infty,$$

there exists N such that $2\rho_n > 1 + \rho_n$, $|y \sin\theta_n| < \frac{1}{2}(x-1)\cos\theta_n$, and $\cos\theta_n > 4\cos\alpha$ whenever $n \geqslant N$. Also, we have $|z|^2 > 1$ and $|1 + d_n| \leqslant 1+\rho_n$. Using these relations (4.7) becomes

$$-1 + \left|\frac{z+d_n}{1+d_n}\right|^2 > \frac{(x-1)\cos\alpha}{\rho_n} \tag{4.8}$$

for $n \geqslant N$. Since $\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty$ by hypothesis, (4.8) implies that the series (4.6) diverges to plus infinity.

THEOREM 4.3. *Suppose that*

$$\lim_{n\to\infty} \rho_n = \rho$$

and that $\sum_{n=1}^{\infty} \theta_n^2$ *converges, then the* $[F, d_n]$ *matrix sums the sequence* $\{s_n(z)\}$, *whose terms are given by* (4.1), *to its analytic continuation for all* z *such that* $|z + \rho| < 1 + \rho$.

Proof. Let $\{\sigma_n(z)\}$ denote the $[F, d_n]$ transform of the sequence $\{s_n(z)\}$ whose terms are given by (4.1), then using the same argument as was employed to obtain (4.3) and (4.5) we obtain

$$\sigma_n(z) = (1-z)^{-1} - z(1-z)^{-1} \prod_{j=1}^{n} \frac{z+d_j}{1+d_j}, \tag{4.9}$$

and

$$\left|\frac{z+d_n}{1+d_n}\right| \leqslant \exp\left\{\frac{|z|^2 - 1 + 2\rho_n \cos\theta_n (x-1) + 2\rho_n y \sin\theta_n}{2(1 + 2\rho_n \cos\theta_n + \rho_n^2)}\right\}. \tag{4.10}$$

However, since

$$\lim_{n\to\infty} \theta_n = 0 \text{ and } \lim_{n\to\infty} \rho_n = \rho,$$

we have

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$$\frac{|z|^2 - 1 + 2\rho_n(x-1)\cos\theta_n + 2\rho_n y \sin\theta_n}{1 + 2\rho_n \cos\theta_n + \rho_n^2} = \frac{|z|^2 - 1 + 2\rho(x-1) + \epsilon_n}{(1+\rho)^2}$$

where

$$\lim_{n\to\infty} \epsilon_n = 0.$$

Using this equality, (4.10) can be written as

$$\left|\frac{z+d_n}{1+d_n}\right| \leqslant \exp\left\{\frac{|z|^2 - 1 + 2\rho(x-1) + \epsilon_n}{2(1+\rho)^2}\right\}. \tag{4.11}$$

Suppose z such that $|z+\rho| < 1+\rho$ is given. Then there exists $\alpha > 0$ such that $|z+\rho|^2 \leqslant (1+\rho)^2 - \alpha$. Since $\alpha > 0$ and

$$\lim_{n\to\infty} \epsilon_n = 0,$$

there exists an integer N such that $|\epsilon_n| < \alpha$ for $n \geqslant N$. Since $|z|^2 - 1 + 2\rho(x-1) = |z+\rho|^2 - (1+\rho)^2$, (4.11) becomes

$$\left|\frac{z+d_n}{1+d_n}\right| \leqslant \exp\left\{\frac{|z+\rho|^2 - (\rho+1)^2 + \alpha}{2(1+\rho)^2}\right\} \tag{4.12}$$

for $n \geqslant N$. But since $\{|z+\rho|^2 - (\rho+1)^2 + \alpha\}(1+\rho)^{-2}$ is a negative constant, (4.12) implies that

$$\prod_{j=1}^{\infty} \frac{z+d_j}{1+d_j} = 0$$

for all z such that $|z+\rho| < 1+\rho$. Therefore, (4.9) implies that

$$\lim_{n\to\infty} \sigma_n(z) = (1-z)^{-1}$$

for any z such that $|z+\rho| < 1+\rho$. This completes the proof of the theorem.

Note that the convergence of $\{\sigma_n(z)\}$ to $(1-z)^{-1}$ is uniform for all z inside a circle concentric with the circle $|z+\rho| = 1+\rho$ and having a radius less than $1+\rho$.

Notice that in the proofs of Theorem 4.1 and Theorem 4.3 we require only that

$$\lim_{n\to\infty} \theta_n = 0.$$

We made the assumption that $\sum_{n=1}^{\infty}\theta_n^2\rho_n^{-1}$ is convergent since in Theorem 3.1 we proved the $[F, d_n]$ matrix regular in this case only. Notice also that both Theorem 4.2 and Theorem 4.4 contain the hypothesis

$$\lim_{n\to\infty} \theta_n = 0.$$

This would seem to indicate that there is some possibility of proving Theorem 3.1 with the hypothesis $\sum_{n=1}^{\infty}\theta_n^2\rho_n^{-1} < +\infty$ replaced by

$$\lim_{n\to\infty} \theta_n = 0.$$

We leave this as an open problem.

If $|z + \rho| = 1 + \rho$, (4.10) can be written as

$$\left|\frac{z + d_n}{1 + d_n}\right| \leqslant \exp\left\{\frac{2(x - 1)(\rho_n \cos\theta_n - \rho) + 2\rho_n y \sin\theta_n}{2(1 + 2\rho_n \cos\theta_n + \rho_n^2)}\right\}. \tag{4.13}$$

Suppose that $\rho_n \cos\theta_n \geqslant \rho$, $\theta_n > 0$, and $2\sin\theta_n \geqslant \theta_n$ for $n \geqslant N$; and that $y < 0$, then (4.13) reduces to

$$\left|\frac{z + d_n}{1 + d_n}\right| \leqslant \exp\left\{\frac{\rho_n y \theta_n}{2(1 + \rho_n)^2}\right\} \tag{4.14}$$

for $n \geqslant N$ since $x \leqslant 1$. From the inequality (4.14) and Theorem 4.2, we get the following

COROLLARY 4.3. *Suppose*

$$\lim_{n\to\infty} \rho_n = \rho,$$

$\sum_{n=1}^{\infty}\theta_n = +\infty$, $\sum_{n=1}^{\infty}\theta_n^2 < +\infty$, *and that* $\rho_n \cos\theta_n \geqslant \rho$ *for n sufficiently large, then the* $[F, d_n]$ *matrix is regular and sums the sequence* $\{s_n(z)\}$, *whose terms are given by* (4.1), *to* $(1 - z)^{-1}$ *for all z which satisfy the condition* $|z + \rho| < 1+\rho$ *and for all z which satisfy both the condition that* $|z + \rho| = 1 + \rho$ *and the condition* $\operatorname{Im}(z) < 0$.

Similarly we have another corollary which is the same as Corollary 4.3 except that $\sum_{n=1}^{\infty}\theta_n = +\infty$ is replaced by $\sum_{n=1}^{\infty}\theta_n = -\infty$ in the hypothesis, and $\operatorname{Im}(z) < 0$ is replaced by $\operatorname{Im}(z) > 0$ in the conclusion.

THEOREM 4.4. *Suppose*

$$\lim_{n\to\infty} \rho_n = \rho, \lim_{n\to\infty} \theta_n = 0,$$

and z is given such that $|z + \rho| > 1 + \rho$, *then the sequence* $\{\sigma_n(z)\}$, *whose terms are given by* (4.9), *is divergent.*

Proof. Suppose z such that $|z + \rho| > 1 + \rho$ is given. Then there exists $0 < \alpha < 8$ such that

$$|z + \rho|^2 \geqslant (1 + \rho)^2(1 + 2\alpha).$$

Hence we have

$$|z|^2 - 1 = -2\rho(x - 1) + |z + \rho|^2 - (1 + \rho)^2 \geqslant -2\rho(x - 1) + 2\alpha(1 + \rho)^2.$$

Using this inequality, we get

$$\begin{aligned} |z + d_n|^2 - |1 + d_n|^2 &= |z|^2 - 1 + 2\rho_n(x - 1)\cos\theta_n + 2\rho_n y \sin\theta_n \\ &\geqslant 2(x - 1)(\rho_n \cos\theta_n - \rho) + 2y\rho_n \sin\theta_n + 2\alpha(1+\rho)^2. \end{aligned} \tag{4.15}$$

But since

$$\lim_{n\to\infty} \rho_n = \rho \text{ and } \lim_{n\to\infty} \theta_n = 0,$$

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there exists an integer N such that if $n \geqslant N$, then

$$|2(x-1)(\rho_n \cos\theta_n - \rho) + 2y\rho_n \sin\theta_n| < \alpha(1+\rho)^2.$$

Therefore if $n \geqslant N$, (4.15) can be written as

$$|z+d_n|^2 - |1+d_n|^2 \geqslant \alpha(1+\rho)^2.$$

Dividing through by $|1+d_n|^2$, we get

$$\left|\frac{z+d_n}{1+d_n}\right|^2 \geqslant 1 + \alpha\left(\frac{1+\rho}{|1+d_n|}\right)^2 \tag{4.16}$$

whenever $n \geqslant N$. Since

$$\lim_{n\to\infty} |d_n| = \rho,$$

(4.16) implies that

$$\lim_{n\to\infty} \left|\frac{z+d_n}{1+d_n}\right| \geqslant 1 + \frac{\alpha}{4} \neq 1.$$

But this is a sufficient condition in order that

$$\prod_{j=1}^{\infty} \left|\frac{z+d_n}{1+d_n}\right|$$

diverge to plus infinity. Hence (4.9) implies that the sequence $\{\sigma_n(z)\}$ is divergent.

5. Analytic continuation. Upon setting $d_n = (1-r)r^{-1}$, the $[F,d_n]$ matrix becomes the classical Euler matrix of order r. Since Agnew **(1)** has shown that the Euler matrix provides the analytic continuation of a power series for cases in which the matrix is not regular, it is natural to raise the following question. Does the $[F, d_n]$ matrix also provide the analytic continuation of power series for some sequences $\{d_n\}$ for which it is not regular other than the above-mentioned cases studied by Agnew? By Corollary 2.2 the $[F, d_n]$ matrix is not regular if $\sum_{n=1}^{\infty} \rho_n^{-1}$ converges. The proposed question is answered for these non-regular matrices by observing the results of the following theorem.

THEOREM 5.1. *Let* $\sum_{n=1}^{\infty} \rho_n^{-1} < \infty$, $S_n(z) = \sum_{k=0}^{n} a_k z^k$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be regular in a neighbourhood of the origin. Let* $\{\sigma_n^{(\nu)}(z)\}$ *denote the* $[F, d_n]$ *transform of the sequence* $0, 0, \ldots, 0, S_0(z), S_1(z), \ldots,$ *where* $-\nu$ *is an integer denoting the number of zeros in the sequence preceding the term* $S_0(z)$. *Let* $\{\sigma_n^{(\nu)}(z)\}$, *where* ν *is a positive integer, denote the* $[F, d_n]$ *transform of* $S_\nu(z), S_{\nu+1}(z), S_{\nu+2}(z)\ldots.$ *Then* $\{\sigma_n^{(\nu)}(z)\}$ *converges uniformly for all* z *in any bounded domain except that* z *must be outside some neighbourhood of the origin when* $\nu < -1$.

Proof. Since $f(z)$ is regular in a neighbourhood of the origin

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{t^{n+1}}$$

where c is a small circle with centre at the origin. Replacing a_n by this integral, we obtain

$$
\begin{aligned}
S_n(z) &= \sum_{k=0}^{n} \frac{1}{2\pi i} \int_c \frac{f(t)dt}{t^{k+1}} z^k \\
(5.1) \qquad &= \frac{1}{2\pi i} \int_c \sum_{k=0}^{n} \left(\frac{z}{t}\right)^k \frac{f(t)}{t}\, dt \\
&= \frac{1}{2\pi i} \int_c \frac{f(t)}{t-z} \left\{1 - \left(\frac{z}{t}\right)^{n+1}\right\} dt.
\end{aligned}
$$

Since $f(t)$ is regular inside the circle c, it follows by the Cauchy integral theorem that

$$
(5.2) \qquad \frac{1}{2\pi i} \int_c \frac{f(t)}{t-z} \left\{1 - \left(\frac{z}{t}\right)^{m}\right\} dt = 0
$$

for $m \leqslant 0$ and $z \neq 0$. If the elements of the $[F, d_n]$ matrix are denoted by P_{nk}, then it follows from (5.1) and (5.2) that

$$
\begin{aligned}
(5.3) \quad \sigma_n^{(\nu)}(z) &= \frac{1}{2\pi i} \int_c \frac{f(t)}{t-z} \sum_{k=0}^{n} P_{nk} \left\{1 - \left(\frac{z}{t}\right)^{k+\nu+1}\right\} dt \\
&= \frac{1}{2\pi i} \int_c \frac{f(t)}{t-z}\, dt \sum_{k=0}^{n} P_{nk} \\
&\qquad - \frac{1}{2\pi i} \int_c \frac{f(t)}{t-z} \left(\frac{z}{t}\right)^{\nu+1} \sum_{k=0}^{n} P_{nk} \left(\frac{z}{t}\right)^k dt.
\end{aligned}
$$

But using the fact that $\sum_{k=0}^{n} P_{nk} = 1$ and the relation (1.5), (5.3) reduces to

$$
\sigma_n^{(\nu)}(z) = \frac{1}{2\pi i} \int_c \frac{f(t)dt}{t-z} - \frac{1}{2\pi i} \int_c \frac{f(t)}{t-z} \left(\frac{z}{t}\right)^{\nu+1} \prod_{j=1}^{n} \left(\frac{\frac{z}{t} + d_j}{1 + d_j}\right) dt.
$$

Hence it follows that

$$
(5.4) \quad \sigma_{n+1}^{(\nu)}(z) - \sigma_n^{(\nu)}(z) = \frac{1}{2\pi i} \int_c \frac{f(t)}{t} \left(\frac{z}{t}\right)^{\nu+1} (1 + d_{n+1})^{-1} \prod_{j=1}^{n} \left(\frac{\frac{z}{t} + d_j}{1 + d_j}\right) dt.
$$

If we let $|t| = s$ and $|z| = r$ and insert absolute values on both sides of (5.4), we obtain

$$
|\sigma_{n+1}^{(\nu)}(z) - \sigma_n^{(\nu)}(z)| \leqslant \frac{1}{2\pi} \left(\frac{r}{s}\right)^{\nu+1} |1 + d_{n+1}|^{-1} \prod_{j=1}^{n} \left(\frac{\frac{r}{s} + \rho_j}{|1 + d_j|}\right) \int_0^{2\pi} |f(t)|\, d\theta.
$$

Since $f(t)$ is regular on $|t| = s$, there exists a constant Q such that $|f(t)| \leqslant Q$ for all t such that $|t| = s$. Hence

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$$(5.5) \qquad |\sigma_{n+1}^{(\nu)}(z) - \sigma_n^{(\nu)}(z)| \leqslant \frac{Q}{|1+d_{n+1}|}\left(\frac{r}{s}\right)^{\nu+1} \prod_{j=1}^{n} \left(\frac{\frac{r}{s}+\rho_j}{|1+d_j|}\right).$$

Employing the same type of argument that was used to obtain (4.5), we have

$$(5.6) \qquad \frac{\frac{r}{s}+\rho_n}{|1+d_n|} \leqslant \exp\left\{\frac{\frac{r^2}{s^2} - 1 + 2\rho_n\left(\frac{r}{s} - \cos\theta_n\right)}{2(1+2\rho_n\cos\theta_n+\rho_n^2)}\right\}.$$

Since r/s is a constant and since $\sum_{n=1}^{\infty}\rho_n^{-1} < +\infty$, it follows from (5.6) that there exists a constant $L(r)$ such that

$$(5.7) \qquad \prod_{j=1}^{n} \frac{\frac{r}{s}+\rho_j}{|1+d_j|} \leqslant L(r)$$

for all n. We write $L(r)$ to indicate that although $L(r)$ is a constant for fixed z, $L(r)$ is a function of r. Using the inequality (5.7) and summing (5.5) over n, we get

$$(5.8) \qquad \sum_{n=0}^{\infty} |\sigma_{n+1}^{(\nu)}(z) - \sigma_n^{(\nu)}(z)| \leqslant QL(r)(r/s)^{\nu+1}\sum_{n=0}^{\infty} |1+d_{n+1}|^{-1},$$

if $r \neq 0$ for $\nu < -1$. Since $\sum_{n=0}^{\infty}|1+d_{n+1}|^{-1}$ is convergent by hypothesis, it follows that the left-hand series in (5.8) is uniformly convergent for z in any bounded domain except that z must be outside some neighbourhood of the origin when $\nu < -1$.

Since $\sigma_{n+1}{}^{(\nu)}(z) - \sigma_0{}^{(\nu)}(z) = \sum_{k=0}^{n}[\sigma_{k+1}{}^{(\nu)}(z) - \sigma_k{}^{(\nu)}(z)]$, it follows that the sequence $\{\sigma_n{}^{(\nu)}(z)\}$ is uniformly convergent for all z in any bounded domain except that whenever $\nu < -1$ we must also have $|z| \geqslant \alpha$ for some $\alpha > 0$. This completes the proof of the theorem.

From (5.1) it follows that $S_n(z)$ is a polynomial of degree n in z. Hence from (5.3) it also follows that $\sigma_n{}^{(\nu)}(z)$ is also a polynomial of degree n in z. Define $\sigma^{(\nu)}(z)$ by

$$\sigma^{(\nu)}(z) = \lim_{n\to\infty}\sigma_n{}^{(\nu)}(z)$$

whenever the limit exists. Note that $\sigma^{(\nu)}(0)$ is undefined for $\nu < -1$. Since $\{\sigma_n{}^{(\nu)}(z)\}$ is uniformly convergent in any bounded domain for $\nu \geqslant -1$, it follows that $\sigma^{(\nu)}(z)$ is an entire function of z for $\nu \geqslant -1$. Therefore, if the analytic continuation of $f(z)$ has singular points, then $f(z) \neq \sigma^{(\nu)}(z)$ for $\nu \geqslant -1$. Now suppose $\nu < -1$. By assumption $f(z)$ is analytic in a circle about the origin, and from Theorem 5.1 $\sigma^{(\nu)}(z)$, $\nu < -1$, is analytic outside every circle about the origin. Therefore, $\sigma^{(\nu)}(z) \neq f(z)$ unless $f(z)$ is entire. Thus the $[F, d_n]$ transform of Theorem 5.1 does not give the analytic continuation of a single function defined by a power series whose analytic continuation possesses a singularity. If $f(z)$ is entire we cannot say that $\sigma^{(\nu)}(z) \neq f(z)$ except in case

$\nu = -1$ where it is easy to prove that $\sigma^{(-1)}(0) \neq f(0)$. However, we have no problem of analytic continuation if $f(z)$ is entire as the power series itself converges for all z in this case. Let us say that the $[F, d_n]$ matrix provides an effective method of analytic continuation for a function of $f(z)$ defined by its power series if $f(z)$ is not entire and if $\sigma^{(\nu)}(z) = f(z)$ for some ν. Then we may conclude that the $[F, d_n]$ matrix does not form an effective analytic continuation procedure if $\sum_{n=1}^{\infty} \rho_n^{-1}$ is convergent.

References

1. R. P. Agnew, *Euler transformations*, Amer. J. Math., *66* (1944), 318–338.
2. ——— *The Lototsky method for evaluation of series*, Mich. Math. J., *4* (1957), 105–128.
3. A. Jakimovski, *A generalization of the Lototsky method*, Mich. Math. J., *6* (1959), 277–290.
4. J. Karamata, *Theorèmes sur la sommabilité exponentielle et d'autres sommabilités s'y rattachant*, Mathematica, Cluj, *9* (1953), 164–178.
5. A. V. Lototsky, *On a linear transformation of sequences and series*, Ivanor. Gos. Ped. Inst. Uc. Zap. Fig.-Math. Nauki, *4* (1953), 61–91 (Russian).
6. A. Meir, *On a theorem of A. Jakimovski on linear transformations*, Mich. Math. J., *6* (1959), 359–361.
7. Y. Okada, *Ueber die Annäherung analytischer Funktionen*, Math. Zeit., *23* (1925), 62–71.
8. V. Vuckovic, *The mutual inclusion of Karamata-Stirling methods of summation*, Mich. Math. J., *6* (1959), 291–297.

University of Kentucky
University of Minnesota

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