BULL. AUSTRAL. MATH. SOC. 40A25, 33C10, 65B10
VOL. 59 (1999) [315–322]

A GENERALISATION OF THE SUMMATION FORMULA OF PLANA

CLÉMENT FRAPPIER

An extension of the classical summation formula of Plana is obtained. The extension is obtained by using the zeros of a Bessel function of the first kind.

1. INTRODUCTION

Let $f(z)$ be analytic in the half-plane $Re(z) \geqslant 0$ and let either the series $\sum\limits_{k=0}^{\infty} f(k)$ or the integral $\int_0^{\infty} f(x)\,dx$ be convergent. Suppose, in addition, that

(A) $\lim\limits_{N\to\infty} \left|f(x \pm iN)\right| e^{-2\pi N} = 0$ uniformly in x on every finite interval, and

(B) $\int_0^{\infty} \left|f(M \pm iy)\right| e^{-2\pi y}\,dy$ exists for every $M \geqslant 0$ and tends to zero as $M \to \infty$.

Under these assumptions, we have

$$\sum_{k=0}^{\infty} f(k) = \frac{1}{2} f(0) + \int_0^{\infty} f(x)\,dx + i\int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1}\,dy. \tag{1}$$

Equation (1) is known as the summation formula of Plana. It has applications to the Γ function, the Riemann ζ function and the discrete Laplace transform [**4**].

The aim of this paper is to present a generalisation of (1). Our results involve a parameter $\alpha > -1$, the order of a Bessel function of the first kind. We obtain (1) as the special case $\alpha = 1/2$. For $\alpha = -1/2$ we shall obtain a formula similar to (1), namely

$$\sum_{k=1}^{\infty} f\left(k - \frac{1}{2}\right) = \int_0^{\infty} f(x)\,dx - i\int_0^{\infty} \frac{f(iy) - f(-iy)}{e^{2\pi y} + 1}\,dy. \tag{2}$$

We examine some examples, one of which leads us to consider, in a natural way, an extension of the Riemann ζ function. Some fragmentary properties of this function are given, including a functional equation.

Received 28th September, 1998
This research was supported by the Natural Sciences and Engineering Research Council of Canada Grant No. OGP0009331.

2. The summation formula

Before we state our results, it is convenient to recall some standard notation. Let $\alpha > -1$ be a real number. The function J_α defined by the relation

$$\frac{J_\alpha(z)}{z^\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k+\alpha} k! \Gamma(\alpha + k + 1)} \tag{3}$$

is known as the Bessel function of the first kind, of order α. The function $(J_\alpha(z)/z^\alpha)$ is an even entire function of exponential type 1, whose zeros $j_k = j_k(\alpha)$ are real and simple [**5**, Chapter XV]. We arrange these zeros such that $0 < j_1 < j_2 < \dots$ and $j_{-k} = -j_k$. The Bessel function of the second kind Y_α (Weber's function) is defined, whenever α is not an integer, by

$$Y_\alpha(z) = \frac{\cos(\alpha\pi) J_\alpha(z) - J_{-\alpha}(z)}{\sin(\alpha\pi)}. \tag{4}$$

This function is often denoted N_α (Neumann's function). The Bessel functions of the third kind $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ (Hankel's functions) are defined by

$$H_\alpha^{(1)}(z) = J_\alpha(z) + iY_\alpha(z) = i\frac{\left(e^{-\alpha\pi i} J_\alpha(z) - J_{-\alpha}(z)\right)}{\sin(\alpha\pi)} \tag{5}$$

and

$$H_\alpha^{(2)}(z) = J_\alpha(z) - iY_\alpha(z) = i\frac{\left(J_{-\alpha}(z) - e^{\alpha\pi i} J_\alpha(z)\right)}{\sin(\alpha\pi)}. \tag{6}$$

Now let $f(z)$ be analytic for $Re(z) \geqslant 0$. We define a real number $\omega = \omega(\alpha)$ by

$$\omega = \begin{cases} 2\alpha - 1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha < 0. \end{cases} \tag{7}$$

Suppose that

(a) $\lim_{N\to\infty} (x \pm iN)^\omega f(x \pm iN) e^{-2N} = 0$ uniformly in x on every finite interval, and

(b) $\int_0^\infty \left|(M \pm iy)^\omega f(M \pm iy)\right| e^{-2y}\, dy$ exists for every $M \geqslant 0$ and tends to zero as $M \to \infty$.

Theorem 1. *Let $\alpha > 0$ be a real number, not an integer, and let $f(z)$ be analytic for $Re(z) \geqslant 0$. Suppose that the series $\sum_{k=1}^{\infty} \left(j_k^{2\alpha-2} f(j_k)/\left(J_\alpha'(j_k)\right)^2\right)$ or the integral $\int_0^\infty x^{2\alpha-1} f(x)\, dx$ is convergent. If conditions (a) and (b) are satisfied then*

$$\begin{aligned} 2\sum_{k=1}^{\infty} \frac{j_k^{2\alpha-2} f(j_k)}{(J_\alpha'(j_k))^2} &= \int_0^\infty x^{2\alpha-1} f(x)\, dx - 2^{2\alpha-1}\Gamma(\alpha+1)\Gamma(\alpha) f(0) \\ &\quad - \frac{i}{2\sin(\alpha\pi)} \int_0^\infty y^{2\alpha-1} \frac{\left(J_\alpha(iy) - e^{\alpha\pi i} J_{-\alpha}(iy)\right)}{J_\alpha(iy)} \left(f(iy) - f(-iy)\right) dy. \end{aligned} \tag{8}$$

https://doi.org/10.1017/S0004972700032925 Published online by Cambridge University Press

In the special case $\alpha = 1/2$ we have $J_{1/2}(z) = \sqrt{2/(\pi z)}\sin(z)$, $j_k = k\pi$, and (8), where $f(z)$ is replaced by $f(z/\pi)$, reduces to (1). For $\alpha = 3/2$ we have $J_{3/2}(z) = \sqrt{2/(\pi z)}\Big((\sin(z)/z) - \cos(z)\Big)$, $\tan(j_k) = j_k$, and (8) becomes

$$(9)\quad \sum_{k=1}^{\infty}(1+j_k^2)f(j_k) = \int_0^{\infty} x^2 f(x)\,dx - \frac{3\pi}{2}f(0) + i\int_0^{\infty}\frac{y^2(y+1)(f(iy)-f(-iy))}{(y-1)e^{2y}+(y+1)}\,dy.$$

Theorem 2. *Let $-1 < \alpha < 0$ be a real number and let $f(z)$ be analytic for $Re(z) \geqslant 0$, with $f(0) = 0$. Suppose that the series $\sum_{k=1}^{\infty}\Big(f(j_k)\Big/ j_k^2(J'_\alpha(j_k))^2\Big)$ or the integral $\int_0^{\infty}(f(x)/x)\,dx$ is convergent. If conditions (a) and (b) are satisfied then*

$$(10)\quad \begin{aligned} 2\sum_{k=1}^{\infty}\frac{f(j_k)}{j_k^2(J'_\alpha(j_k))^2} &= \int_0^{\infty}\frac{f(x)}{x}\,dx \\ &+ \frac{1}{2\sin(\alpha\pi)}\int_0^{\infty}\frac{(J_\alpha(iy)-e^{\alpha\pi i}J_{-\alpha}(iy))}{J_\alpha(iy)}\frac{(e^{-\alpha\pi i}f(iy)-e^{\alpha\pi i}f(-iy))}{iy}\,dy. \end{aligned}$$

The particular case $\alpha = -1/2$ of (10) gives, since $J_{-1/2}(z) = \sqrt{2/(\pi z)}\cos(z)$ and $j_k = (2k-1)(\pi/2)$, the summation formula

$$(11)\quad \sum_{k=1}^{\infty}\frac{f\Big(\Big(k-\frac{1}{2}\Big)\pi\Big)}{\Big(k-\frac{1}{2}\Big)} = \int_0^{\infty}\frac{f(x)}{x}\,dx - \int_0^{\infty}\frac{(f(iy)+f(-iy))}{y(e^{2y}+1)}\,dy,$$

from which (2) follows if we replace $f(z)$ by $zf(z/\pi)$. Of course, a small modification to the hypothesis of Theorem 2 must then be made.

3. Proofs of the theorems

For the various properties of Bessel functions used in the proofs, we refer the reader to [5].

Let ε be a small positive number and let M, N be large positive numbers with $M := m\pi + \alpha(\pi/2) + (\pi/4)$ for some positive integer m. We consider, in the complex plane, the closed curve $\Gamma = \Gamma(M, N, \varepsilon)$ which is the union of the intervals $[M - iN, M + iN]$, $[M + iN, iN]$, $[iN, i\varepsilon]$, $[-i\varepsilon, -iN]$, $[-iN, M - iN]$ and the semicircle $C(\varepsilon) = \{z : |z| = \varepsilon, Re(z) \geqslant 0\}$. Thus, Γ is a rectangular curve, in the right half-plane, with corners $M \pm iN, \pm iN$, except that the interval $[i\varepsilon, -i\varepsilon]$ is replaced by $C(\varepsilon)$. The curve Γ is oriented positively. Let also $\Gamma_1 := \{z \in \Gamma : Im(z) \geqslant 0\}$ and $\Gamma_2 := \{z \in \Gamma : Im(z) \leqslant 0\}$, so that $\Gamma = \Gamma_1 \cup \Gamma_2$.

The function $z^\omega f(z)(Y_\alpha(z)/J_\alpha(z))$ is analytic on and inside Γ, except for simple poles at $z = j_k$, $1 \leqslant k \leqslant m$. By the residue theorem, we have

$$(12)\quad \frac{1}{2\pi i}\int_\Gamma z^\omega f(z)\frac{Y_\alpha(z)}{J_\alpha(z)}\,dz = -\frac{2}{\pi}\sum_{k=1}^{m}\frac{j_k^{\omega-1}f(j_k)}{(J'_\alpha(j_k))^2}.$$

https://doi.org/10.1017/S0004972700032925 Published online by Cambridge University Press

The relation $Y_\alpha(j_k) = -J_{-\alpha}(j_k)/\sin(\alpha\pi) = -2/\big(\pi j_k J'_\alpha(j_k)\big)$ has been used to compute the residue at $z = j_k$. It follows from (12) that

$$\begin{aligned} -\frac{2}{\pi}\sum_{k=1}^{m}\frac{j_k^{\omega-1}f(j_k)}{(J'_\alpha(j_k))^2} &= \frac{1}{2\pi i}\int_{\Gamma_1} z^\omega f(z)\frac{Y_\alpha(z)}{J_\alpha(z)}\,dz + \frac{1}{2\pi i}\int_{\Gamma_2} z^\omega f(z)\frac{Y_\alpha(z)}{J_\alpha(z)}\,dz \\ &= -\frac{1}{2\pi}\int_{\Gamma_1} z^\omega f(z)\frac{H_\alpha^{(1)}(z)}{J_\alpha(z)}\,dz + \frac{1}{2\pi}\int_{\Gamma_1} z^\omega f(z)\,dz \\ &\quad +\frac{1}{2\pi}\int_{\Gamma_2} z^\omega f(z)\frac{H_\alpha^{(2)}(z)}{J_\alpha(z)}\,dz - \frac{1}{2\pi}\int_{\Gamma_2} z^\omega f(z)\,dz. \end{aligned} \tag{13}$$

Cauchy's theorem, applied to the closed curves $\Gamma_1 \cup [\varepsilon, M]$ and $\Gamma_2 \cup [M, \varepsilon]$, gives

$$\int_{\Gamma_1} z^\omega f(z)\,dz = -\int_\varepsilon^M x^\omega f(x)\,dx. \tag{14}$$

and

$$\int_{\Gamma_2} z^\omega f(z)\,dz = \int_\varepsilon^M x^\omega f(x)\,dx. \tag{15}$$

Substituting in (13), we obtain

$$\begin{aligned} -\frac{2}{\pi}\sum_{k=1}^{m}\frac{j_k^{\omega-1}f(j_k)}{\big(J'_\alpha(j_k)\big)^2} &= -\frac{1}{\pi}\int_\varepsilon^M x^\omega f(x)\,dx - \frac{1}{2\pi}\int_{\Gamma_1} z^\omega f(z)\frac{H_\alpha^{(1)}(z)}{J_\alpha(z)}\,dz \\ &\quad +\frac{1}{2\pi}\int_{\Gamma_2} z^\omega f(z)\frac{H_\alpha^{(2)}(z)}{J_\alpha(z)}\,dz. \end{aligned} \tag{16}$$

In (16), the integral along Γ_1 may be written explicitly as

$$\begin{aligned} \int_{\Gamma_1} z^\omega f(z)\frac{H_\alpha^{(1)}(z)}{J_\alpha(z)}\,dz &= i\int_0^N (M+iy)^\omega f(M+iy)\frac{H_\alpha^{(1)}(M+iy)}{J_\alpha(M+iy)}\,dy \\ &\quad + \int_M^0 (x+iN)^\omega f(x+iN)\frac{H_\alpha^{(1)}(x+iN)}{J_\alpha(x+iN)}\,dx \\ &\quad + i\int_N^\varepsilon (iy)^\omega f(iy)\frac{H_\alpha^{(1)}(iy)}{J_\alpha(iy)}\,dy \\ &\quad + i\int_{\pi/2}^0 (\varepsilon e^{i\theta})^{\omega+1} f(\varepsilon e^{i\theta})\frac{H_\alpha^{(1)}(\varepsilon e^{i\theta})}{J_\alpha(\varepsilon e^{i\theta})}\,d\theta. \end{aligned} \tag{17}$$

The asymptotic expansions

$$H_\alpha^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{i(z-\alpha(\pi/2)-(\pi/4))}, \qquad |z| \to \infty, \big|\arg(z)\big| < \pi, \tag{18}$$

$$J_\alpha(z) \sim \sqrt{\frac{2}{\pi z}}\cos\Big(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\Big), \qquad |z| \to \infty, \big|\arg(z)\big| < \pi, \tag{19}$$

and the hypothesis (a) show that

$$\lim_{N\to\infty}\int_0^M (x+iN)^\omega f(x+iN)\frac{H_\alpha^{(1)}(x+iN)}{J_\alpha(x+iN)}\,dx = 0. \tag{20}$$

The same asymptotic expansions and the hypothesis (b) imply then that

$$\lim_{M\to\infty}\int_0^\infty (M+iy)^\omega f(M+iy)\frac{H_\alpha^{(1)}(M+iy)}{J_\alpha(M+iy)}\,dy = 0. \tag{21}$$

Thus, (17) becomes

$$\begin{aligned}\lim_{\substack{M\to\infty\\N\to\infty}}\int_{\Gamma_1} z^\omega f(z)\frac{H_\alpha^{(1)}(z)}{J_\alpha(z)}\,dz &= i\int_\varepsilon^\infty (iy)^\omega f(iy)\frac{H_\alpha^{(1)}(iy)}{J_\alpha(iy)}\,dy\\ &\quad -i\int_0^{\pi/2}(\varepsilon e^{i\theta})^{\omega+1} f(\varepsilon e^{i\theta})\frac{H_\alpha^{(1)}(\varepsilon e^{i\theta})}{J_\alpha(\varepsilon e^{i\theta})}\,d\theta.\end{aligned} \tag{22}$$

In a similar manner, using the asymptotic expansion

$$H_\alpha^{(2)}(z)\sim\sqrt{\frac{2}{\pi z}}e^{-i(z-\alpha(\pi/2)-(\pi/4))},\qquad |z|\to\infty,\,\left|\arg(z)\right|<\pi, \tag{23}$$

instead of (18), we obtain

$$\begin{aligned}\lim_{\substack{M\to\infty\\N\to\infty}}\int_{\Gamma_2} z^\omega f(z)\frac{H_\alpha^{(2)}(z)}{J_\alpha(z)}\,dz &= -i\int_{-\infty}^{-\varepsilon} (iy)^\omega f(iy)\frac{H_\alpha^{(2)}(iy)}{J_\alpha(iy)}\,dy\\ &\quad -i\int_{-\pi/2}^{0}(\varepsilon e^{i\theta})^{\omega+1} f(\varepsilon e^{i\theta})\frac{H_\alpha^{(2)}(\varepsilon e^{i\theta})}{J_\alpha(\varepsilon e^{i\theta})}\,d\theta.\end{aligned} \tag{24}$$

We infer from (16) that

$$\begin{aligned}2\sum_{k=1}^{\infty}\frac{j_k^{\omega-1}f(j_k)}{(J_\alpha'(j_k))^2} &= \int_0^\infty x^\omega f(x)\,dx + \lim_{\varepsilon\to 0}\left\{-\frac{i}{2}\int_\varepsilon^\infty (iy)^\omega f(iy)\frac{H_\alpha^{(1)}(iy)}{J_\alpha(iy)}\,dy\right.\\ &\quad +\frac{i}{2}\int_{-\infty}^{-\varepsilon}(iy)^\omega f(iy)\frac{H_\alpha^{(2)}(iy)}{J_\alpha(iy)}\,dy\\ &\quad -\frac{i}{2}\int_0^{\pi/2}(\varepsilon e^{i\theta})^{\omega+1} f(\varepsilon e^{i\theta})\frac{H_\alpha^{(1)}(\varepsilon e^{i\theta})}{J_\alpha(\varepsilon e^{i\theta})}\,d\theta\\ &\quad \left.+\frac{i}{2}\int_{-\frac{\pi}{2}}^{0}(\varepsilon e^{i\theta})^{\omega+1} f(\varepsilon e^{i\theta})\frac{H_\alpha^{(2)}(\varepsilon e^{i\theta})}{J_\alpha(\varepsilon e^{i\theta})}\,d\theta\right\}.\end{aligned} \tag{25}$$

Using now the asymptotic expansion

$$J_\alpha(z)\sim\frac{z^\alpha}{2^\alpha\Gamma(\alpha+1)},\qquad z\to 0, \tag{26}$$

https://doi.org/10.1017/S0004972700032925 Published online by Cambridge University Press

we obtain

$$\lim_{\varepsilon\to 0}\frac{i}{2}\int_0^{\pi/2}(\varepsilon e^{i\theta})^{\omega+1}f(\varepsilon e^{i\theta})\frac{H_\alpha^{(1)}(\varepsilon e^{i\theta})}{J_\alpha(\varepsilon e^{i\theta})}\,d\theta \tag{27}$$
$$= \begin{cases} 2^{2\alpha-2}\Gamma(\alpha+1)\Gamma(\alpha)f(0) & \text{if } \alpha>0 \\ -\dfrac{\pi e^{-\alpha\pi i}f(0)}{4\sin(\alpha\pi)} & \text{if } \alpha<0, \end{cases}$$

and

$$\lim_{\varepsilon\to 0}\frac{i}{2}\int_{-\pi/2}^{0}(\varepsilon e^{i\theta})^{\omega+1}f(\varepsilon e^{i\theta})\frac{H_\alpha^{(2)}(\varepsilon e^{i\theta})}{J_\alpha(\varepsilon e^{i\theta})}\,d\theta \tag{28}$$
$$= \begin{cases} -2^{2\alpha-2}\Gamma(\alpha+1)\Gamma(\alpha)f(0) & \text{if } \alpha>0 \\ \dfrac{\pi e^{\alpha\pi i}f(0)}{4\sin(\alpha\pi)} & \text{if } \alpha<0. \end{cases}$$

Substituting in (25) we get, for $\alpha > 0$,

$$2\sum_{k=1}^{\infty}\frac{j_k^{2\alpha-2}f(j_k)}{(J'_\alpha(j_k))^2} = \int_0^\infty x^{2\alpha-1}f(x)\,dx - 2^{2\alpha-1}\Gamma(\alpha+1)\Gamma(\alpha)f(0) \tag{29}$$
$$-\frac{1}{2}\int_0^\infty y^{2\alpha-1}\left(e^{\alpha\pi i}f(iy)\frac{H_\alpha^{(1)}(iy)}{J_\alpha(iy)} + e^{-\alpha\pi i}f(-iy)\frac{H_\alpha^{(2)}(-iy)}{J_\alpha(-iy)}\right)dy,$$

from which Theorem 1 follows since $J_\alpha(e^{-\pi i}z) = e^{-\alpha\pi i}J_\alpha(z)$ for $Im(z) > 0$. □

Theorem 2 is obtained similarly. In that case ($\alpha < 0$) we have necessarily $f(0) = 0$.

4. Examples and remarks

4.1. A generalised Riemann ζ function. An interesting application of Theorem 1 arises if we apply it, in appropriate circumstances, to a function of the form $f(z) = 1/(z+\varepsilon)^\nu$, $\varepsilon > 0$, $\nu := s + 2\alpha - 1$. We obtain the relation

$$2\sum_{k=1}^{\infty}\frac{j_k^{2\alpha-2}}{(J'_\alpha(j_k))^2(j_k+\varepsilon)^\nu} = \frac{\Gamma(2\alpha)\Gamma(s-1)}{\Gamma(s+2\alpha-1)\varepsilon^{s-1}} - 2^{2\alpha-1}\frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\varepsilon^{s+2\alpha-1}} \tag{30}$$

$$+\frac{e^{-\alpha\pi i}}{2\sin(\alpha\pi)\varepsilon^{s+2\alpha-1}}\int_0^\infty\frac{(i\varepsilon y)^{2\alpha}}{iy}\frac{(J_\alpha(i\varepsilon y) - e^{\alpha\pi i}J_{-\alpha}(i\varepsilon y))}{J_\alpha(i\varepsilon y)}\left(\frac{1}{(1+iy)^\nu} - \frac{1}{(1-iy)^\nu}\right)dy,$$

where $\alpha > 0$ and $Re(s) > 1$. In the left hand side of (30), we can easily let $\varepsilon \to 0$. We are led to consider the function ζ_α defined (for all α) by

$$\zeta_\alpha(s) = 2\pi^{s-1}\sum_{k=1}^{\infty}\frac{1}{j_k^{s+1}(J'_\alpha(j_k))^2}, \qquad Re(s) > 1. \tag{31}$$

https://doi.org/10.1017/S0004972700032925 Published online by Cambridge University Press

We readily see that $\zeta_{1/2}(s) = \zeta(s)$ and $\zeta_{-1/2}(s) = (2^s - 1)\zeta(s)$. Also,

$$\zeta_{3/2}(s) = \pi^s \sum_{k=1}^{\infty} \frac{(1 + j_k^2)}{j_k^{s+2}}$$

where $\tan(j_k) = j_k$. If we were able to represent suitably a corresponding Γ_α function then ζ_α would satisfy a practical functional equation analogous to [**1**, p.259]

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s). \tag{32}$$

More precisely, we have

$$\sin\left(\pi\alpha - \frac{\pi s}{2}\right)\zeta_\alpha(s) = -ie^{\alpha\pi i}2^{\alpha-(1/2)}(2\pi)^{s-1}\sin(\pi s)\Gamma_\alpha(1-s)\zeta_\alpha(1-s), \tag{33}$$

where Γ_α is an extension of the classical Γ function. The relation $\Gamma(s)\Gamma(1-s) = (\pi/\sin(\pi s))$ becomes the particular case $\alpha = 1/2$ of

$$\Gamma_\alpha(s)\Gamma_\alpha(1-s) = \frac{\pi e^{-2\alpha\pi i}\big(\sin(2\pi\alpha) - \sin(\pi s)\big)}{2^{2\alpha-1}\sin^2(\pi s)}, \tag{34}$$

which follows from (33).

We don't give details concerning the function ζ_α because there is a difficulty in representing Γ_α in explicit form. We have only the representation

$$\Gamma_\alpha(s) = \frac{-1}{2^{\alpha+(1/2)}\zeta_\alpha(s)}\int_0^\infty r^{s-1}\frac{H_\alpha^{(1)}\left(\frac{ir}{2}\right)}{J_\alpha\left(\frac{ir}{2}\right)}\,dr, \tag{35}$$

with $Re(s) > 2\alpha$ if $\alpha > 0$ and $Re(s) > 0$ if $\alpha < 0$. We have $\Gamma_{1/2}(s) = \Gamma(s)$ and $\Gamma_{-(1/2)}(s) = 2\big((2^{1-s} - 1)/(2^s - 1)\big)\Gamma(s)$.

The analytic continuation form of (31) is

$$\zeta_\alpha(s) = \frac{i\pi^{s-1}}{4\sin(\alpha\pi)\sin\big((2\alpha - s)\pi/2\big)}\int_C \frac{1}{z^s}\frac{\big(J_\alpha(-iz) - e^{\alpha\pi i}J_{-\alpha}(-iz)\big)}{J_\alpha(-iz)}\,dz, \tag{36}$$

where C is a curve starting at $-\infty$ on the negative real axis, surrounding the origin in such a way that the numbers ij_k, $k = \pm 1, \pm 2, \dots$, are excluded from its interior, and returning at $-\infty$ along the negative real axis. The zeros of ζ_α remain to be studied.

4.2. Let $f(z) = (z + \varepsilon)^\nu e^{-\delta z}$, $\varepsilon > 0$, with $Re(\nu) > 0$, $Re(\delta) > 0$ and $|Im(\delta)| < 2$. By letting $\varepsilon \to 0$ we obtain, from Theorem 1,

$$\begin{aligned} 2\sum_{k=1}^{\infty}\frac{j_k^{2\alpha+\nu-2}e^{-\delta j_k}}{(J_\alpha'(j_k))^2} &= \frac{\Gamma(2\alpha+\nu)}{\delta^{2\alpha+\nu}} \\ &- \frac{1}{\sin(\alpha\pi)}\int_0^\infty y^{2\alpha+\nu-1}\frac{\big(J_\alpha(iy) - e^{\alpha\pi i}J_{-\alpha}(iy)\big)}{J_\alpha(iy)}\sin\left(\delta y - \frac{\nu\pi}{2}\right)dy. \end{aligned} \tag{37}$$

The same example, with $\nu = 0$, gives

$$(38)\qquad 2\sum_{k=1}^{\infty} \frac{j_k^{2\alpha-2} e^{-\delta j_k}}{\left(J'_\alpha(j_k)\right)^2} = \frac{\Gamma(2\alpha)}{\delta^{2\alpha}} - 2^{2\alpha-1}\Gamma(\alpha+1)\Gamma(\alpha) - \frac{1}{\sin(\alpha\pi)} \int_0^\infty y^{2\alpha-1} \frac{(J_\alpha(iy) - e^{\alpha\pi i} J_{-\alpha}(iy))}{J_\alpha(iy)} \sin(\delta y)\, dy.$$

4.3. As a numerical example related to (2), we take $f(z) = 1/(1+z)^2$. The neat result is

$$(39)\qquad \int_0^\infty \frac{y\, dy}{(1+y^2)^2(e^{2\pi y}+1)} = \frac{5}{4} - \frac{\pi^2}{8}.$$

4.4. Exact quadrature formula. If $f(z)$ is even then, according to our assumptions, it is necessarily analytic in the whole complex plane. In that case, the last integral appearing in (8) is zero, so that

$$(40)\qquad \int_0^\infty x^{2\alpha-1} f(x)\, dx = 2\sum_{k=1}^{\infty} \frac{j_k^{2\alpha-2} f(j_k)}{\left(J'_\alpha(j_k)\right)^2} + 2^{2\alpha-1}\Gamma(\alpha+1)\Gamma(\alpha) f(0),$$

which is a consequence of [**3**, Theorem 1]

$$(41)\qquad \int_0^\infty x^{2\alpha+1}\left(f(x) + f(-x)\right) dx = \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{j_k^{2\alpha}}{\left(J'_\alpha(j_k)\right)^2} \left(f\left(\frac{j_k}{\tau}\right) + f\left(-\frac{j_k}{\tau}\right)\right),$$

with $\tau = 1$. In fact, (40) follows from (41) where $f(x)$ is replaced by $(1/x^2)\ \left(f(x) - (2^\alpha\Gamma(\alpha+1)(J_\alpha(x)/x^\alpha))^2 f(0)\right)$. Note that (41) is valid for $Re(\alpha) > -1$. See also [**2**, Theorem 2] for a more general result of the form (40).

References

[**1**] T.M. Apostol, *Introduction to analytic number theory* (Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[**2**] R. Ben Ghanem and C. Frappier, 'Explicit quadrature formulae for entire functions of exponential type', *J. Approx. Theory* **92** (1998), 267–279.

[**3**] C. Frappier and P. Olivier, 'A quadrature formula involving zeros of Bessel functions', *Math. Comp.* **60** (1993), 303–316.

[**4**] P. Henrici, *Applied and computational complex analysis Vol. 1 and 2* (John Wiley and Sons, New York, 1988 and 1991).

[**5**] G.N. Watson, *A treatise on the theory of Bessel functions*, (2nd edition) (Cambridge University Press, Cambridge, 1952).

Département de Mathématiques et de génie industriel
École Polytechnique
C.P. 6079 Succ. Centre-Ville
Montréal (P.Q.)
Canada H3C 3A7
e-mail: clement.frappier@courrier.polymtl.ca