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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF THE p-LAPLACE EQUATIONS

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Abstract

We are concerned with the existence of solutions of

$$-\Delta_p u = f(x,u) + h(x) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Δ_p is the p-Laplacian, $p \in (1,\infty)$, and Ω is a bounded smooth domain in $\mathbb{R}^n$.

For $h(x) \equiv 0$ and $f(x,u)$ satisfying proper asymptotic spectral conditions, existence of a unique positive solution is obtained by invoking the sub-supersolution technique and the spectral method. For $h(x) \not\equiv 0$, with assumptions on asymptotic behavior of $f(x,u)$ as $u \to \pm\infty$, an existence result is also proved.

1. Introduction

In this paper we investigate the existence of solutions for the Dirichlet problem of the following quasilinear elliptic equation:

$$-\Delta_p u = f(x,u) + h(x) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $p \in (1,\infty)$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $h(x) \in L^\infty$ and f is to be specified below.

For $p \neq 2$, operator Δ_p has appeared in a variety of physical fields. For example, applications of Δ_p have been seen in fluid dynamics. The shear stress $\vec{\tau}$ and the velocity gradient ∇u of the fluid are related in the manner that $\vec{\tau}(x) = r(x)|\nabla u|^{p-2}\nabla u$, where $p = 2$ (respectively, $p < 2$, $p > 2$) if the fluid is Newtonian (respectively, pseudoplastic, dilatant). Consequently the equation governing the motion of the fluid involves the p-Laplacian. The p-Laplacian also appears in the study of flow through porous media ($p = 3/2$), nonlinear elasticity ($p \geq 2$) and glaciology ($p \in (1, 4/3]$). We refer to Diaz [8] and Lions [14] for more background material and applications. For

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f homogeneous in u, that is, $f(x,u) = f(x)u^\sigma$ ($\sigma > 0$), existence and nonexistence results were obtained by many authors, see for example Azorero and Alonso [2], Diaz [8], Guedda and Veron [10], Lions [14], Otani [15] and Szulkin [19]. Variational methods were employed extensively when trying to find positive solutions. If $f^+(x) = \max(f(x), 0) \not\equiv 0$, eigenvalue problem $-\Delta_p u = \lambda f(x)|u|^{p-2}u$ with homogeneous Dirichlet condition has been proved to possess a unique simple positive eigenvalue with positive eigenfunction. See Anane [1], Azorero and Alonso [2], and Otani and Teshima [16]. For the case where $f(x,\cdot)$ is of order u^{p-1} but nonhomogeneous in u, Del Pino, Elgueta and Manasevich [6] studied the existence of solutions of (1.1) with $h \equiv 0$ for the one dimensional case, Boccardo, Drátbek and Kučera [3] considered the higher dimensional case with $h \not\equiv 0$. While for $p = 2$, problem (1.1) has been studied extensively, see for example Brezis and Oswald [4], Costa and Gonçalves [5], Kazdan and Kramer [12] and references therein.

The objective of this paper is to extend some of the known results for the special case $p = 2$ to the general quasilinear case. In particular, we will obtain the following result:

With $h(x) \equiv 0$, if $f(x,t) = f(t)$, $\liminf_{t\to 0^+} f(t)/t^{p-1} = a_0$, $\limsup_{t\to\infty} f(t)/t^{p-1} = a_\infty$, and

$$0 < a_\infty < \lambda_1 < a_0, \tag{$*$}$$

where λ_1 is the smallest positive eigenvalue of $-\Delta_p u = \lambda|u|^{p-2}u$ with homogeneous Dirichlet condition, then problem (1.1) has a positive solution. Moreover, the solution is unique and condition ($*$) is necessary if $f(t)/t^{p-1}$ is strictly decreasing. The method we employ follows the sub-supersolution procedure utilized in [5]. The construction of a sub-supersolution pair is made feasible by establishing some properties for indefinite eigenvalue problem for the p-Laplacian, which are of independent interest and extend the known results of Hess and Kato [11] for second order elliptic operators. For $h(x) \not\equiv 0$ we decompose $f(x,u)$ into

$$f(x,u) = a_+(x)|u|^{p-2}u^+ - a_-(x)|u|^{p-2}u^- + f_1(x,u),$$

with $f_1(x,u)$ as the lower order term in u, and use topological degree theory to establish an existence result. We note that, due to the set-up of our problem, more general existence results are actually obtained.

The rest of the paper is organized as follows. In Section 2, we introduce the needed notations and prove several lemmas which provide the foundation for the construction of a sub-supersolution pair as well as for utilization of topological degree theory. In Section 3, the sub-supersolution pair is constructed and the existence of positive solution of (1.1) is obtained. Finally, Section 4 studies the case $h(x) \not\equiv 0$.

2. Preliminaries

In this section we present some notations and results needed in the sequel. For the special case $p = 2$, these results are well known, *cf.* Hess and Kato [11].

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. We denote $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for u in $W^{1,p}(\Omega)$ with $p \in (1, \infty)$. By Theorems 2.2.1 and 2.2.3 of Lions [14], the problem

$$-\Delta_p u = f(x) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.1}$$

has a unique solution $u \in W_0^{1,p}(\Omega)$ for any $f \in L^q$ $(1/p + 1/q = 1)$. Regularity theorems (see Tolksdorf [20, page 806] and Di Benedetto [7]) further show that if $f \in L^\infty(\Omega)$, then $u \in C^{1+\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. By uniqueness of the solution of (2.1), the inverse of $-\Delta_p$ exists and is henceforth denoted by $R : L^\infty \to L^\infty$. We first have

LEMMA 2.1. *R is compact, continuous and strictly positive, that is, for $f \geq 0$, $f \not\equiv 0$, $Rf > 0$.*

PROOF. Compactness follows from the fact that $C^{1+\alpha}(\overline{\Omega})$ is embedded in $L^\infty(\Omega)$ compactly and densely.

Let $u_k = R(f_k)$, $f_k \to f_0$ in L^∞. By compactness, for some $u_0 \in L^\infty \cap W_0^{1,p}(\Omega)$, $u_k \to u_0$ in L^∞, and $u_k \to u_0$ in $W_0^{1,p}(\Omega)$ weakly. For this f_0, there exists a $\tilde{u} \in L^\infty$ such that $\tilde{u} = R(f_0)$. It then follows from $u_k = R(f_k)$ and $\tilde{u} = R(f_0)$ that

$$\begin{aligned} J(u_k, \tilde{u}) &:= \int (|\nabla u_k|^{p-2}\nabla u_k - |\nabla \tilde{u}|^{p-2}\nabla \tilde{u})\nabla(u_k - \tilde{u}) \\ &= \int (f_k - f_0)(u_k - \tilde{u}). \end{aligned}$$

Note that for any vectors a and b,

$$|a - b|^p \leq c \cdot \left\{(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b)\right\}^{t/2} \cdot (|a|^p + |b|^p)^{1-t/2},$$

where $t = \min\{p, 2\}$ (see Lindqvist [13], Appendix). Thus by Hölder's inequality we conclude

$$\|\nabla u_k - \nabla \tilde{u}\|_p^p \leq c \cdot (J(u_k, \tilde{u}))^{t/2} \cdot (\|\nabla u_k\|_p^p + \|\nabla \tilde{u}\|_p^p)^{1-t/2}.$$

Consequently $u_k \to \tilde{u}$ in $W_0^{1,p}(\Omega)$. Thus $\tilde{u} = u_0$, and this implies continuity of R.

Let $u = Rf$, then $-\Delta_p u = f$. Multiplying this equation by $u^- := u^+ - u$ and integrating show that $u^- = 0$ on the set $(f(x) \neq 0)$ and $u^- = const.$ on the set $(f(x) = 0)$. We conclude from continuity of u^- that $u^- = 0$, that is, $u \geq 0$. It then follows from Lemma 2 of Otani and Teshima [16] that $u > 0$. This concludes the proof.

REMARK 2.1. The above proof actually implies that, if $f_k \to f_0$ in L^q $(\frac{1}{p} + \frac{1}{q} = 1)$, and $u_k := R(f_k) \to$ (some) $u_0 \in W_0^{1,p}(\Omega)$ weakly in $W_0^{1,p}(\Omega)$, then $u_0 = R(f_0)$.

From now on we denote $r = n/p$ if $p \le n$, and $r = 1$ if $p > n$. Let $m(x) \in L^{r+\varepsilon}(\Omega)$ for some $\varepsilon > 0$, $m^+(x) = \max(m(x), 0) \not\equiv 0$. Then the following is well known (see Anane [1], Azorero and Alonso [2], and Otani and Teshima [16]):

PROPOSITION 2.2. *The eigenvalue problem*

$$-\Delta_p u = \lambda m(x)|u|^{p-2} \quad in\ \Omega, \quad u = 0\ on\ \partial\Omega \tag{2.2}$$

has a unique positive eigenvalue $\lambda_1(m)$ which is simple and has an associated positive eigenfunction $\varphi \in W_0^{1,p}(\Omega)$. $\lambda_1(m)$ is characterized by the following variational formula

$$\lambda_1(m) = \inf\left\{\int |\nabla u|^p : \int m|u|^p = 1, u \in W_0^{1,p}(\Omega)\right\}. \tag{2.3}$$

Moreover, if $m \in L^\infty$, then $\partial\varphi/\partial\nu < 0$ on $\partial\Omega$, where ν denotes the outward normal of $\partial\Omega$.

Next we use the variational characteristic of $\lambda_1(m)$ to establish

LEMMA 2.3. *Let $\lambda_1(m)$ be as given in Proposition 2.2 and $m(x) \in L^\infty(\Omega)$. Then*

(i) *λ_1 is decreasing in m, that is, $m_1 > m_2$ with $m_2^+ \not\equiv 0$ implies that $\lambda_1(m_1) < \lambda_1(m_2)$.*

(ii) *If $m_k \to m_0$ in L^∞ with $m_0^+ \not\equiv 0$, then $\lambda_1(m_k) \to \lambda_1(m_0)$.*

PROOF. (i). For $u \in W_0^{1,p}(\Omega)$ with $\int m_2|u|^p = 1$, we have $a := \int m_1|u|^p \ge 1$. We then derive that $\int |\nabla u|^p \ge a\lambda_1(m_1)$, thus $\lambda_1(m_1) \le \lambda_1(m_2)$.

Suppose $\lambda_1(m_1) = \lambda_1(m_2)$. Let u_1, u_2 be the positive eigenfunctions associated with $\lambda_1(m_1)$, $\lambda_1(m_2)$ respectively and set

$$I(u_1, u_2) := \int (-\Delta_p u_1)\frac{u_1^p - u_2^p}{u_1^{p-1}} - \int (-\Delta_p u_2)\frac{u_1^p - u_2^p}{u_2^{p-1}}.$$

We note that such $I(u_1, u_2)$ is well defined since both $\partial u_1/\partial\nu$, $\partial u_2/\partial\nu < 0$ on $\partial\Omega$ (*cf.* Anane [1]). Calculation shows

$$I(u_1, u_2) = \int \lambda_1(m_1)(m_1 - m_2)(u_1^p - u_2^p).$$

By choosing $u_1 - u_2 < 0$ in Ω we have $I(u_1, u_2) \le 0$. It then follows from Proposition 1 of Anane [1] that $u_1 = ku_2$ for some constant k, a contradiction. Hence $\lambda_1(m_1) < \lambda_1(m_2)$.

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(ii). We first claim that $\lambda_1(m_k)$ is bounded. Indeed, let u_0 be the positive eigenfunction associated with $\lambda_1(m_0)$ satisfying $\int m_0|u_0|^p = 1$. For k sufficiently large, we have $1/2 \leq \int m_k|u_0|^p \leq 3/2$. We thus conclude that $\lambda_1(m_k) \leq (\int |\nabla u_0|^p)/(\int m_k|u_0|^p)$ is bounded.

Now, since $\int |\nabla u_k|^p$ is bounded, where u_k is the positive eigenfunction associated with $\lambda_1(m_k)$ satisfying $\int m_k|u_k|^p = 1$, we may assume, taking a subsequence if necessary, that for some $\tilde{u} \in W_0^{1,p}(\Omega)$, $u_k \to \tilde{u}$ weakly in $W_0^{1,p}(\Omega)$, $u_k \to \tilde{u}$ in L^p if $p < n$ and in C if $p \geq n$, and $\lambda_1(m_k) \to \tilde{\lambda}$. It then follows that $\int m_k|u_k|^p \to \int m_0|\tilde{u}|^p$. Since u_k satisfies

$$-\Delta_p u_k = \lambda_1(m_k) m_k u_k^{p-1} \quad \text{in } \Omega, \qquad u_k = 0 \text{ on } \partial\Omega,$$

we let

$$J(u_k, u_l) := \int (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_l|^{p-2}\nabla u_l)\nabla(u_k - u_l),$$

then, as in the proof of Lemma 2.1, we have

$$\|\nabla u_k - \nabla u_l\|_p^p \leq c \cdot (J(u_k, u_l))^{t/2} \cdot (\|\nabla u_k\|_p^p + \|\nabla u_l\|_p^p)^{1-t/2},$$

where $t = \min\{p, 2\}$. We then derive that u_k is a Cauchy sequence in $W_0^{1,p}(\Omega)$ and hence $u_k \to \tilde{u}$ in $W_0^{1,p}(\Omega)$.

We now claim that $\tilde{\lambda} = \lambda_1(m_0)$. Assume, on the contrary, that $\tilde{\lambda} - \lambda_1(m_0) = \varepsilon > 0$. Since $m_k \to m_0$ in L^∞, we have, for k sufficiently large,

$$\int |\nabla u_0|^p \Big/ \int m_k|u_o|^p - \lambda_1(m_0) < \varepsilon/4$$

and

$$|\lambda_1(m_k) - \tilde{\lambda}| < \varepsilon/4.$$

We then obtain that

$$\begin{aligned} 0 < \int |\nabla u_0|^p \Big/ \int m_k|u_o|^p &< \lambda_1(m_0) + \varepsilon/4 \\ &= \tilde{\lambda} - 3\varepsilon/4 < \lambda_1(m_k) - \varepsilon/2, \end{aligned}$$

which contradicts the variational characteristic of $\lambda_1(m_k)$. Consequently we have $\tilde{\lambda} = \lambda_1(m_0)$. The proof is complete.

Remark 2.2. It follows from the above proof that $\tilde{u} = u_0$, which implies that the positive eigenfunction u is continuous in $W_0^{1,p}(\Omega)$ as a function of m.

To establish next lemma some definitions are needed.

DEFINITION 2.1. Let K be a closed convex subset of a Banach space X. Suppose $A : K \to X^*$ is bounded.

(i) A is called a pseudomonotone operator if it satisfies the following condition: if $x_k \to x$ weakly in X and $\limsup_{k\to\infty} \langle A(x_k), x_k - x\rangle \le 0$, then

$$\langle A(x), x - y\rangle \le \liminf_{k\to\infty} \langle A(x_k), x_k - y\rangle$$

for all $y \in X$.

(ii) A is coercive if $\langle A(x), x\rangle/\|x\|_X \to \infty$ as $\|x\|_X \to \infty$.

The following lemma is standard and can be found in, for example, Lions [14].

LEMMA 2.4. *Assume that A is pseudomonotone and coercive, X is a reflexive Banach space and K is a closed convex subset of X. Then, for any $f \in X^*$, there exists an $x \in K$ such that for all $y \in K$,*

$$\langle A(x) - f, y - x\rangle \ge 0.$$

Now we are ready to introduce

LEMMA 2.5. *Let $\lambda \in (0, \lambda_1(m))$. Then for any $h \in W^{1,r}$, $h \ge 0$ in Ω and $h \not\equiv 0$, there exists a nonnegative solution $u \in W_0^{1,p}(\Omega)$ of*

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u + h(x) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2.4}$$

If, in addition, $m, h \in L^\infty$, then $u > 0$ in Ω, $\partial u/\partial \nu < 0$ on $\partial\Omega$ and u is unique.

PROOF. We first prove the existence part. We introduce an operator $A = A_1 + A_2$ from $W_0^{1,p}(\Omega)$ to $(W_0^{1,p}(\Omega))'$ by

$$\begin{aligned}(Au, v) &= (A_1u, v) + (A_2u, v)\\ &= \int |\nabla u|^{p-2}\nabla u\nabla v - \lambda\int m|u|^{p-2}uv.\end{aligned}$$

Obviously A is bounded. Next we verify that A is pseudomonotone and coercive.

Indeed, for some $\alpha \in (0,1)$, $\alpha\lambda_1(m) = \lambda$, we have, for $u \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int |\nabla u|^p &= (1-\alpha)\int |\nabla u|^p + \alpha\int |\nabla u|^p\\ &\ge (1-\alpha)\int |\nabla u|^p + \alpha\lambda_1(m)\int m|u|^p\\ &= (1-\alpha)\int |\nabla u|^p + \lambda\int m|u|^p.\end{aligned}$$

Hence we deduce that

$$(Au, u) \geq (1-\alpha)\|u\|^p_{W^{1,p}},$$

which implies coerciveness of A.

Note that A_1 is pseudomonotone since it is monotone (see Lions [14]) while A_2 is continuous. Thus $A = A_1 + A_2$ is pseudomonotone.

Now we apply Lemma 2.4 to the sets $X = K = W_0^{1,p}(\Omega)$ to conclude the existence of a solution u to (2.4).

To prove nonnegativity of the solution u, we proceed as follows. Since $u \in W_0^{1,p}(\Omega)$, we have $u^- \in W_0^{1,p}(\Omega)$ (see Gilbarg and Trudinger [9]). Multiplying (2.4) by u^- and integrating by parts, we obtain

$$-\int |\nabla u^-|^p + \lambda \int m|u^-|^p = \int hu^-. \tag{2.5}$$

Since $\lambda < \lambda_1(m)$, the left-hand side of (2.5) is negative if $u^- \not\equiv 0$ while the right-hand side is nonnegative, a contradiction. We thus conclude that $u^- \equiv 0$, that is, $u \geq 0$.

Now, assume $m, h \in L^\infty$. We write (2.4) as

$$-\Delta_p u + \lambda m^-|u|^{p-2}u = \lambda m^+|u|^{p-2}u + h(x).$$

$u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$ then follow from Lemma 4 of Otani and Teshima [16].

Next, suppose $v > 0$ is another solution. As in the proof of Lemma 2.3, we let

$$I(u, v) = \int (-\Delta_p u)\frac{u^p - v^p}{u^{p-1}} - \int (-\Delta_p v)\frac{u^p - v^p}{v^{p-1}}.$$

$I(u, v)$ is well defined since $\partial u/\partial \nu, \partial v/\partial \nu < 0$ on $\partial\Omega$. If $u \not\equiv v$, then

$$I(u, v) = \int \frac{h(u^p - v^p)(v^{p-1} - u^{p-1})}{u^{p-1}v^{p-1}} \leq 0.$$

Again Proposition 1 of Anane [1] implies that $u = kv$ for some constant k. We thus conclude that $u \equiv v$. This completes the proof.

For general problem

$$-\Delta_p u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.6}$$

we introduce

DEFINITION 2.2. A function $\bar{u} \in W^{1,p}(\Omega)$ is said to be a supersolution of (2.6) if

$$\int |\nabla \bar{u}|^{p-2}\nabla \bar{u}\nabla \varphi \geq \int f(x, \bar{u})\varphi$$

for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Subsolution is defined accordingly.

3. Existence results

To obtain the existence of a positive solution of

$$-\Delta_p u = f(x,u) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.1}$$

we employ sub-supersolution technique, following Costa and Gonçalves [5] in principle. We postulate the following conditions on $f(x,u)$ throughout this section.

(H1) f satisfies the Caratheodory condition, that is, f is measurable with respect to x for all $u \in \mathbb{R}$ and continuous with respect to u for almost all $x \in \Omega$. Assume further that for some $\varepsilon > 0$, and for any $K > 0$,

$$\sup_{0\le u\le K} |f(x,u)| \in L^{r+\varepsilon}. \tag{3.2}$$

(H2) The asymptotic limits

$$a_0(x) = \liminf_{u\to 0^+} \frac{f(x,u)}{u^{p-1}}, \quad a_\infty(x) = \limsup_{u\to\infty} \frac{f(x,u)}{u^{p-1}} \tag{3.3}$$

exist uniformly for a.e. $x \in \Omega$. Moreover, they satisfy that $a_\infty \in L^\infty$, $a_\infty^+ \not\equiv 0$, and either $\infty > a_0(x) > c_0 > 0$ or $a_0(x) \equiv \infty$, and

$$\lambda_1(a_0) < 1 < \lambda_1(a_\infty). \tag{3.4}$$

REMARK 3.1. For the case $a_0(x) \equiv +\infty$, we take $\lambda_1(a_0) = 0$ in (3.4). If a_0, a_∞ are constants, then (3.4) is equivalent to $0 < a_\infty < \lambda_1(1) < a_0$, a condition used frequently for the case $p = 2$, see for example Brezis and Oswald [4] and references therein.

We first construct a pair of sub-supersolution.

THEOREM 3.1. *Assume (H1) and (H2) hold. Then there exists a positive supersolution $\bar{u} \in W_0^{1,p}(\Omega)$ of (3.1) such that $\partial\bar{u}/\partial\nu < 0$ on $\partial\Omega$.*

PROOF. We choose $\varepsilon > 0$ small such that $\lambda_1(a_\infty + \varepsilon) > 1$. This is possible by Lemma 2.3. By virtue of the definition of a_∞, we can find a function $b_\varepsilon(x) \in W^{1,r} \cap L^\infty$ with $b_\varepsilon \ge 1$ in $\overline{\Omega}$ such that

$$f(x,u) \le (a_\infty + \varepsilon)u^{p-1} + b_\varepsilon$$

for $u \ge 0$. Now we consider the following problem

$$-\Delta_p u = (a_\infty + \varepsilon)|u|^{p-2}u + b_\varepsilon \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{3.5}$$

Lemma 2.5 implies that (3.5) has a positive solution $u_\varepsilon \in W_0^{1,p}(\Omega)$ with $\partial u_\varepsilon/\partial\nu < 0$ on $\partial\Omega$. We then choose $\bar{u} = u_\varepsilon$ to conclude the proof.

THEOREM 3.2. *Under the same conditions, there exists a positive subsolution* $\underline{u} \in W_0^{1,p}(\Omega)$ *of (3.1) such that* $\underline{u} \leq \bar{u}$.

PROOF. (i). For the case $a_0(x) \in L^\infty$, we again choose an $\varepsilon > 0$ so small that $a_0 - \varepsilon > 0$ and $\lambda_1(a_0 - \varepsilon) < 1$. We further take $K > 0$ small such that

$$f(x, u) \geq (a_0 - \varepsilon)u^{p-1}$$

for all $0 \leq u \leq K$. Let v_1 be a positive eigenfunction of the following eigenvalue problem

$$-\Delta_p u = \lambda_1(a_0 - \varepsilon)(a_0 - \varepsilon)|u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

We can further assume that $v_1 \leq K$. It then follows that

$$\begin{aligned} -\Delta_p v_1 &= \lambda_1(a_0 - \varepsilon)(a_0 - \varepsilon)v_1^{p-1} \\ &\leq \lambda_1(a_0 - \varepsilon) f(x, v_1) \leq f(x, v_1) \end{aligned}$$

since $\lambda_1(a_0 - \varepsilon) < 1$. We thus conclude that v_1 is a subsolution of (3.1). Since $\partial\bar{u}/\partial\nu < 0$ and $\partial v_1/\partial\nu < 0$ on $\partial\Omega$, we can choose $\alpha \in (0, 1)$ small such that $\alpha v_1 \leq \bar{u}$ in Ω. Then $\underline{u} = \alpha v_1$ is as required.

(ii). For the case $a_0(x) \equiv +\infty$, replace in the above proof $a_0 - \varepsilon$ by a large $M > 0$. Then the same conclusion can be deduced. The proof is complete.

The next theorem, originally due to Kazdan and Kramer [12] for $p = 2$, guarantees the existence of solutions for our problem.

THEOREM 3.3. *Let* $\underline{u}, \bar{u} \in W_0^{1,p} \cap L^\infty$ *be a pair of sub-supersolution of (3.1). If* $\underline{u} \leq \bar{u}$, *then (3.1) has a solution* $u \in W_0^{1,p}(\Omega)$ *such that* $\underline{u} \leq u \leq \bar{u}$ *if (H1) is assumed.*

REMARK 3.2. The proof we give below follows the same arguments as that of Kazdan and Kramer [12] and Costa and Gonçalves [5] for $p = 2$, but we require and obtain weaker regularity. In fact, as indicated by Tolksdorf [20], one cannot expect C^2 regularity if $p < n$ and $p \neq 2$. A different proof for the case $p = 2$ was given in Sattinger [17], where it was assumed that $f(x, u) + ku$ is nondecreasing in u for some $k \geq 0$.

PROOF. For the given sub-supersolution pair $\underline{u} \leq \bar{u}$ in L^∞, we define an operator T by

$$Tu = \begin{cases} \underline{u}, & \text{if } u(x) \leq \underline{u}(x); \\ u, & \text{if } \underline{u}(x) \leq u(x) \leq \bar{u}(x); \\ \bar{u}, & \text{if } u(x) \geq \bar{u}(x), \end{cases}$$

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and consider the problem

$$-\Delta_p u + |u|^{p-2}u = f(x, Tu) + |Tu|^{p-2}Tu \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{3.6}$$

We first claim that problem (3.1) together with condition $\underline{u} \le u \le \bar{u}$ is equivalent to problem (3.6).

Obviously, solution u of (3.1) satisfying $\underline{u} \le u \le \bar{u}$ also solves (3.6). On the other hand, let u be a solution of (3.6) and denote $\tilde{u} = u - \bar{u}$. We will show that $\tilde{u} \le 0$. Since $\bar{u}$ is a supersolution of (3.1), we have

$$\int \left(|\nabla\bar{u}|^{p-2}\nabla\bar{u} - |\nabla u|^{p-2}\nabla u\right)\nabla\varphi \\ \ge \int \left(f(x,\bar{u}) - f(x,Tu) - |Tu|^{p-2}Tu + |u|^{p-2}u\right)\varphi$$

for all $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \ge 0$. If $\tilde{u}^+ \not\equiv 0$, taking $\varphi = \tilde{u}^+$ in the above inequality yields

$$0 > \int \left(|\nabla\bar{u}|^{p-2}\nabla\bar{u} - |\nabla u|^{p-2}\nabla u\right)\nabla\tilde{u}^+ \ge 0,$$

a contradiction. Hence $u \le \bar{u}$. Analogously we can prove $\underline{u} \le u$. It then follows that $\underline{u} \le u \le \bar{u}$ and consequently u solves (3.1).

Now we consider the solvability of a family of problems

$$-\Delta_p u + |u|^{p-2}u = t(f(x, Tu) + |Tu|^{p-2}Tu) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.7}$$

for $t \in [0, 1]$. It is easy to verify that solution of (3.7) is a fixed point of the following operator

$$RK_t(u) := R\left(t(f(x, Tu) + |Tu|^{p-2}Tu) - |u|^{p-2}u\right),$$

where R is the inverse of $-\Delta_p$ defined in Section 2. We claim that RK_t is compact from L^∞ to L^∞ for all $t \in [0, 1]$. Indeed we need only to verify that RK_t is bounded since R is compact (see Lemma 2.1). Let $v = RK_t(u)$, that is,

$$-\Delta_p v = t(f(x, Tu) + |Tu|^{p-2}Tu) - |u|^{p-2}u.$$

For $p > n$, L^∞ boundedness follows from the Sobolev embedding theorem since $v \in W_0^{1,p}(\Omega)$. For $p < n$, since for some $M > 0$, $M \ge \|\underline{u}\|_\infty, \|\bar{u}\|_\infty$, then by (H1), the norm of $f(x, Tu)$ in $L^{r+\varepsilon}$ is independent of u, thus Theorem 1 of Serrin [18] implies that $\|v\|_\infty \le M_1$, where M_1 is independent of u and $t \in [0, 1]$. For $p = n$, Theorem 2 of Serrin [18] implies the same.

The above analysis also shows that there exists a constant $M_0 > 0$, such that for all $t \in [0, 1]$, RK_t has no fixed points outside the ball $B := B_{M_0}$ in L^∞, where B_{M_0} is centered at the origin with radius M_0. Thus we conclude

$$\deg(I - RK_1, B, 0) = \deg(I - RK_0, B, 0).$$

If $I - tRK_0$ has a zero point u_0, then $-\Delta_p u_0 = -t^{p-1}|u_0|^{p-2}u_0$. Multiplying both sides of the equation by u_0 and integrating by parts we obtain

$$0 \le \int |\nabla u_0|^p = -t^{p-1} \int |u_0|^p \le 0,$$

which implies $u_0 = 0$ if $t \neq 0$. This shows that $I - tRK_0$ is homotopic to I. It then follows that

$$\deg(I - RK_1, B, 0) = \deg(I - RK_0, B, 0) = 1,$$

that is, there exists a solution to $u \in B$ of (3.7) with $t = 1$. Consequently we obtain a solution u of (3.1) with $\underline{u} \le u \le \bar{u}$. This ends the proof.

Now, combining Theorems 3.1, 3.2 and 3.3 we have

THEOREM 3.4. *Assume that (H1) and (H2) hold. Then there exists a positive solution* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of (3.1). If, in addition,* $f(x,t)/t^{p-1}$ *is decreasing for* $t \in (0,\infty)$, *then the solution is unique in* $L^\infty(\Omega)$.

PROOF. We need only verify the uniqueness part.

Let $v > 0$ be another solution. Proposition 1 of [1] is not readily applicable here since we do not know whether $\partial v/\partial n < 0$ on $\partial\Omega$ or not. Instead we adapt a device due to Lindqvist [13]. Let

$$u_\varepsilon = u + \varepsilon, \qquad v_\varepsilon = v + \varepsilon,$$

for $\varepsilon > 0$, and choose test function $(u_\varepsilon^p - v_\varepsilon^p)/u_\varepsilon^{p-1}$ for equation (3.1), and $(u_\varepsilon^p - v_\varepsilon^p)/v_\varepsilon^{p-1}$ for (3.1) with u replaced by v, respectively. Multiplying (3.1) and the corresponding equation for v by the test functions and subtracting the resultant equations, we obtain

$$\int \left((-\Delta_p u) \frac{u_\varepsilon^p - v_\varepsilon^p}{u_\varepsilon^{p-1}} - (-\Delta_p v) \frac{u_\varepsilon^p - v_\varepsilon^p}{v_\varepsilon^{p-1}} \right) = \int \left(\frac{f(x,u)}{u_\varepsilon^{p-1}} - \frac{f(x,v)}{v_\varepsilon^{p-1}} \right) (u_\varepsilon^p - v_\varepsilon^p).$$

It is apparent that, by our assumption,

$$\lim_{\varepsilon\to 0} \int \left(\frac{f(x,u)}{u_\varepsilon^{p-1}} - \frac{f(x,v)}{v_\varepsilon^{p-1}} \right) (u_\varepsilon^p - v_\varepsilon^p) = \int \left(\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v^{p-1}} \right) (u^p - v^p) \le 0.$$

Then the same arguments as that in the proof of Lemma 1 of [13] imply that $u = kv$ for some $k > 0$. It is easy to see that $k = 1$, that is, $u = v$. The proof is complete.

Finally we state a theorem which says that, in essence, (H2) is necessary.

THEOREM 3.5. *Assume that, for all* $x \in \overline{\Omega}$, $f(x,t)/t^{p-1}$ *is strictly decreasing for* $t \in (0,\infty)$ *and* $a_0 \in L^\infty$. *If (3.1) has a positive solution* u, *then*

$$\lambda_1(a_0) < 1 < \lambda_1(a_\infty).$$

PROOF. By monotonicity of $f(x,t)/t^{p-1}$, we have

$$a_\infty(x) < f(x,t)/t^{p-1} < a_0(x)$$

for all $t > 0$, $x \in \overline{\Omega}$. Define $m(x)$ by

$$m(x) = \begin{cases} f(x,u(x))/u(x)^{p-1}, & \text{if } x \in \Omega; \\ a_0(x), & \text{if } x \in \partial\Omega, \end{cases}$$

where $u(x)$ is a positive solution of (3.1). Then $a_\infty(x) < m(x) < a_0(x)$, hence $m(x) \in L^\infty$, $m^+ \not\equiv 0$. Since a positive solution u of (3.1) also solves

$$-\Delta_p u = m(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

we derive that $\lambda_1(m) = 1$ from simplicity of positive eigenfunction. Lemma 2.3 further shows that

$$\lambda_1(a_0) < \lambda_1(m) = 1 < \lambda_1(a_\infty).$$

This completes the proof.

4. Nonhomogeneous case

In this section we will deal with the nonhomogeneous problem, that is, $h(x) \not\equiv 0$ in (1.1). We assume that f satisfies (H1) and

$$\lim_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} = a_\pm(x) \in L^\infty. \tag{4.1}$$

Thus we can decompose $f(x,u)$ into

$$f(x,u) = a_+(x)|u|^{p-2}u^+ - a_-(x)|u|^{p-2}u^- + f_1(x,u),$$

where

$$\lim_{s\to\infty} \frac{\max_{|u|\le s} f_1(x,u)}{s^{p-1}} = 0.$$

We also assume $a_\pm^+(x) \not\equiv 0$ so that $\lambda_1(a_\pm)$ are well defined. We further restrict ourselves to the case where

$$\lambda_1(a_\pm) > 1. \tag{4.2}$$

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REMARK 4.1. We note that in the case where $a_{\pm}$ are positive constants, (4.2) is equivalent to $a_{\pm} < \lambda_1(1)$. Moreover, in the case $a_{\pm}(x) \leq \lambda_1(1)$ with strict inequalities on subregions in Ω with positive measure, a condition routinely used in the literature, (4.2) holds. Indeed, (4.2) is slightly more general than that. It is conceivable that $a_{\pm} > \lambda_1(1)$ in some subregions of Ω but (4.2) remains true.

We first investigate the homogeneous case.

LEMMA 4.1. *The problem*

$$-\Delta_p u = a_+(x)|u|^{p-2}u^+ - a_-(x)|u|^{p-2}u^- \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{4.3}$$

has only trivial solution.

PROOF. Let u be a solution. Multiplying both sides of (4.3) by u^+ and integrating yield that

$$\int |\nabla u^+|^p = \int a_+(x)|u^+|^p.$$

On the other hand, for any $u \in W_0^{1,p}(\Omega)$ with $\int a_+(x)|u|^p \neq 0$,

$$\int |\nabla u|^p \geq \lambda_1(a_+) \int a_+(x)|u|^p > \int a_+(x)|u|^p,$$

by (4.2). It then follows that $u^+ \equiv 0$. Similarly we derive that $u^- \equiv 0$. Thus $u \equiv 0$. The lemma is proved.

Now we define, for $\eta > 0$,

$$\begin{aligned} F(\eta, v) &= f(x, \eta v)/\eta^{p-1} \\ &= a_+|v|^{p-2}v^+ - a_-|v|^{p-2}v^- + f(x, \eta v)/\eta^{p-1}, \end{aligned}$$

and

$$\Phi(\eta, v) = v - R\left(F(\eta, v) + h/\eta^{p-1}\right),$$

which maps L^∞ into itself.

REMARK 4.2. We note that if $\Phi(\eta, v) = 0$ for some $\eta > 0$, then ηv is a solution of (1.1).

LEMMA 4.2. *There exist $\eta_0 > 0$, $M > 0$ such that for any $\eta \geq \eta_0$ and $\|v\|_\infty \geq M$, we have $\Phi(\eta, v) \neq 0$.*

PROOF. Assume, on the contrary, that there exist two sequences $\{\eta_k\}$ and $\{v_k\}$ such that $\eta_k \to \infty$, $\|v_k\|_\infty \to \infty$, and $\Phi(\eta_k, v_k) = 0$, that is,

$$-\Delta_p v_k = F(\eta_k, v_k) + h/\eta_k^{p-1}.$$

Denote $u_k = v_k/\|v_k\|_\infty$. We then obtain

$$\begin{aligned}-\Delta_p u_k &= \left(F(\eta_k, v_k) + h/\eta_k^{p-1}\right)\|v_k\|_\infty^{-(p-1)}\\ &= a_+|u_k|^{p-2}u_k^+ - a_-|u_k|^{p-2}u_k^- + \frac{f_1(x, \eta_k v_k) + h}{\eta_k^{p-1}\|v_k\|_\infty^{p-1}}.\end{aligned}$$

We thus derive that $\int |\nabla u_k|^p$ is bounded. Hence for some $u_0 \in W_0^{1,p}(\Omega)$, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that $u_k \to u_0$ weakly in $W_0^{1,p}(\Omega)$ and strongly in L^p. It follows from compactness of R that $u_k \to u_0$ in L^∞ and $\|u_0\|_\infty = 1$. Combining with continuity of R we conclude that

$$-\Delta_p u_0 = a_+(x)|u_0|^{p-2}u_0^+ - a_-(x)|u_0|^{p-2}u_0^-.$$

Lemma 4.1 then implies that $u_0 \equiv 0$, which is impossible. Thus $\Phi(\eta, v) \neq 0$ for η and $\|v\|_\infty$ sufficiently large. The proof of this lemma is complete.

Let B_M denote the ball in L^∞ with radius M and centered at the origin. It follows from Lemma 4.2 that $\deg(\Phi(\eta, v), B_M, 0)$ is well defined for $\eta \geq \eta_0$. Moreover we have

LEMMA 4.3. $\deg(\Phi(\eta, v), B_M, 0) \neq 0$, *for* $\eta \geq \eta_0$.

PROOF. We observe that, by Lemma 4.2

$$\Psi(t, \eta, v) := v - R\left(a_+|v|^{p-2}v^+ - a_-|v|^{p-2}v^- + \frac{f_1(x, \eta v) + h}{\eta^{p-1}}t^{p-1}\right)$$

defines a homotopy for $t \in [0, 1]$. We then have

$$\deg(\Psi(1, \eta, v), B_M, 0) = \deg(\Psi(0, \eta, v), B_M, 0).$$

On the other hand, since

$$R(a_+|v|^{p-2}v^+ - a_-|v|^{p-2}v^-)$$

is an odd operator, the Borsuk Theorem implies that

$$\deg(\Phi(\eta, v), B_M, 0) = \deg(\Psi(0, \eta, v), B_M, 0) \neq 0.$$

This ends the proof of the lemma.

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We now state our existence result

THEOREM 4.1. *Assume that (4.1) and (4.2) hold with* $f_1(x, 0) = 0$. *Then for any* $h \in L^\infty$, $h \not\equiv 0$, *problem (1.1) has a nontrivial solution.*

PROOF. The existence follows from Lemma 4.3, while $h \not\equiv 0$ implies that the solution cannot be trivial.

REMARK 4.3. In the paper by Boccardo, Drábek and Kučera [3], existence results are obtained for the case (in essence) $a_\pm = \lambda_1(1)$.

REMARK 4.4. We note that, if $-\Delta_p u$ is replaced by $-\Delta_p u + b(x)|u|^{p-2}u$ with $b(x) \in L^\infty$, $b(x) \geq 0$ a.e. in Ω, the same proofs can be carried through and the same results of this paper remain valid.

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