SEMIGROUPS OF LINEAR TRANSFORMATIONS WITH RESTRICTED RANGE

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Abstract

In 1975, Symons described the automorphisms of the semigroup $T(X, Y)$ consisting of all total transformations from a set $X$ into a fixed subset $Y$ of $X$. Recently Sanwong, Singha and Sullivan determined all maximal (and all minimal) congruences on $T(X, Y)$, and Sommanee studied Green’s relations in $T(X, Y)$. Here, we describe Green’s relations and ideals for the semigroup $T(V, W)$ consisting of all linear transformations from a vector space $V$ into a fixed subspace $W$ of $V$.

Keywords and phrases: linear transformation, Green’s relations, ideals.

1. Introduction

If $X$ is a set, we let $T(X)$ denote the semigroup (under composition) of all total transformations of $X$. In addition, if $Y \subseteq X$, we let $X\alpha = \text{ran } \alpha$ denote the range of $\alpha$ and write

$$T(X, Y) = \{ \alpha \in T(X) \mid X\alpha \subseteq Y \}.$$

This is a subsemigroup of $T(X)$. In fact, if $|Y| = 1$ then $T(X, Y)$ contains exactly one element (namely, the constant mapping with range $Y$).

In 1975, Symons [6] described all the automorphisms of $T(X, Y)$: this is an elegant and significant result and, surprisingly, it depends on whether $Y$ contains exactly two, or more than two, elements (the former case is the much harder one to resolve). In [3] the authors characterized the regular elements in $T(X, Y)$, and all maximal (and all minimal) congruences on $T(X, Y)$ were described in [4]. Also, in [5] Sommanee studied Green’s relations in $T(X, Y)$. Here we describe Green’s relations and ideals for the semigroup $T(V, W)$ consisting of all linear transformations from a vector space $V$ into a fixed subspace $W$ of $V$. As a consequence, we show that $T(V, W)$ is almost never isomorphic to $T(U)$ for any vector space $U$, and thus it is worth studying the algebraic properties of the semigroup $T(V, W)$ in its own right.
2. Green’s relations on $T(V, W)$

Suppose that $W$ is a nonzero proper subspace of a vector space $V$, and let $T(V)$ denote the semigroup (under composition) of all linear mappings from $V$ into itself. Our aim in this section is to consider properties of the subsemigroup of $T(V)$ defined by

$$T(V, W) = \{ \alpha \in T(V) \mid V \alpha \subseteq W \}.$$ 

To do this, we need some notation. For each $\alpha \in T(V)$, we write $\ker \alpha$ and $V \alpha = \text{ran} \alpha$ for the kernel and the range of $\alpha$, respectively, and we write

$$n(\alpha) = \dim(\ker \alpha) \quad \text{and} \quad r(\alpha) = \dim(\text{ran} \alpha).$$

As an abbreviation, we write a subset $\{ e_i \mid i \in I \}$ of $V$ as $\{ e_i \}$, letting the subscript denote an (unspecified) index set $I$ (this is comparable with [1, Volume 2, p. 241]). We write the subspace of $V$ generated by a linearly independent subset $\{ e_i \}$ of $V$ as $\langle e_i \rangle$; and, when we write $U = \{ e_i \}$, we tacitly assume that the set $\{ e_i \}$ is a basis for the subspace $U$.

Often it is necessary to construct some $\alpha \in T(V)$ by first choosing a basis $\{ e_i \}$ for $V$ and some subset $\{ u_i \}$ of $V$, and then letting $e_i \alpha = u_i$ for each $i \in I$ and extending this action by linearity to the whole of $V$. To abbreviate matters, we simply say, given $\{ e_i \}$ and $\{ u_i \}$ within context, that $\alpha \in T(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ u_i \end{pmatrix}.$$

To characterize Green’s relations on $T(V, W)$, we need to describe all of its regular elements. This was done in [2, Theorem 2.1], but we include a proof for completeness.

**Lemma 1.** The set $Q$ of all regular elements in $T(V, W)$ forms a semigroup and is given by

$$Q = \{ \alpha \in T(V, W) \mid V \alpha \subseteq W \alpha \}.$$

**Proof.** Clearly, if $\alpha \in Q$ and $\beta \in T(V)$, then $V \alpha \subseteq W \alpha$ implies that $V \alpha \beta \subseteq W \alpha \beta$, so $Q$ is a right ideal of $T(V)$ and, in particular, it is a subsemigroup of $T(V, W)$. Suppose that $\alpha = \alpha \beta \alpha$ for some $\beta \in T(V, W)$. Then $u \alpha = (u \alpha \beta) \alpha \in W \alpha$ for all $u \in V$, so $V \alpha \subseteq W \alpha$ and hence $\alpha \in Q$.

Conversely, suppose that $\alpha \in T(V, W)$ and $V \alpha \subseteq W \alpha = \langle w_j \alpha \rangle$, where $w_j \in W$ for each $j$. Then $\{ w_j \}$ is linearly independent. Also, if $v \in V$ then $v \alpha = (\sum x_j w_j) \alpha$ for some scalars $x_j$, and so $V = \ker \alpha \oplus \langle w_j \rangle$. If $\ker \alpha = \langle u_i \rangle$ and $V = W \alpha \oplus \langle v_k \rangle$, we can write

$$\alpha = \begin{pmatrix} u_i & w_j \\ 0 & w_j \alpha \end{pmatrix}, \quad \beta = \begin{pmatrix} v_k & w_j \alpha \\ 0 & w_j \end{pmatrix},$$

and observe that $V \beta = \langle w_j \rangle \subseteq W$, so $\beta \in T(V, W)$ and $\alpha = \alpha \beta \alpha$. \qed
Note that $Q$ is always nonempty: if $W = V$ then $Q = T(V)$ which is regular for all vector spaces $V$ (see [1, Volume I, Exercise 2.2.6]); if $W = [0]$ then $Q$ contains only the zero mapping in $T(V)$, and hence it is trivially regular; and if $W = \langle w_j \rangle$ and $V = \langle u_i \rangle \oplus W$ then clearly

$$\alpha = \begin{pmatrix} u_i & w_j \\ 0 & w_j \end{pmatrix} \in Q.$$  

In addition, although $Q$ is always a right ideal of $T(V, W)$, it is almost never a left ideal. For example, if $W = \langle w \rangle$ and $V = \langle v, w \rangle$ and

$$\alpha = \begin{pmatrix} v & w \\ 0 & w \end{pmatrix}, \quad \lambda = \begin{pmatrix} w & v \\ 0 & w \end{pmatrix},$$

then $\alpha \in Q$ but $\lambda \alpha = \lambda \notin Q$.

**Lemma 2.** Let $\gamma \in Q$ and $\beta \in T(V, W)$. Then $\beta = \lambda \gamma$ for some $\lambda \in T(V, W)$ if and only if $\operatorname{ran} \beta \subseteq \operatorname{ran} \gamma$. Consequently, if $\alpha, \beta \in T(V, W)$ then $\alpha \mathcal{L} \beta$ in $T(V, W)$ if and only if $\alpha = \beta$ or $(\operatorname{ran} \alpha = \operatorname{ran} \beta$ and $\alpha, \beta \in Q)$.

**Proof.** Clearly, if $\beta = \lambda \gamma$ for some $\lambda \in T(V, W)$ then $\operatorname{ran} \beta \subseteq \operatorname{ran} \gamma$. Conversely, suppose that $\operatorname{ran} \beta \subseteq \operatorname{ran} \gamma \subseteq W \gamma$ and write $\beta = \langle v_i \beta \rangle$. Then, for each $i$, there exists $w_i \in W$ such that $v_i \beta = w_i \gamma$ and we let $\gamma = \langle w_i \gamma \rangle \oplus \langle w_j \gamma \rangle$. Note that, if $\ker \beta = \langle u_r \rangle$ and $\ker \gamma = \langle u_s \rangle$, then both $\{u_r \} \cup \{v_i \}$ and $\{u_s \} \cup \{w_i \} \cup \{w_j \}$ are linearly independent. Also, $V = \ker \beta \oplus \langle v_i \rangle$ and $V = \ker \gamma \oplus \langle w_i \rangle \oplus \langle w_j \rangle$. Thus, we can write

$$\beta = \begin{pmatrix} u_r & v_i \\ 0 & w_i \gamma \end{pmatrix}, \quad \gamma = \begin{pmatrix} u_s & w_i \gamma \\ 0 & w_j \gamma \end{pmatrix},$$

and define $\lambda \in T(V, W)$ by

$$\lambda = \begin{pmatrix} u_r & v_i \\ 0 & w_i \end{pmatrix}.$$  

Then $\beta = \lambda \gamma$, as required. Now suppose that $\alpha \mathcal{L} \beta$ in $T(V, W)$, so $\alpha = \lambda \beta$ and $\beta = \lambda' \alpha$ for some $\lambda, \lambda' \in T(V, W)^1$. If $\lambda = 1$ or $\lambda' = 1$ then $\alpha = \beta$. On the other hand, if $\lambda, \lambda' \neq 1$ then $\lambda, \lambda' \in T(V, W)$ and

$$\alpha = \lambda \lambda' \alpha \quad \text{and} \quad \beta = \lambda' \lambda \beta.$$  

Hence, $V \alpha = (V \lambda \lambda') \alpha \subseteq W \alpha$, and similarly $V \beta \subseteq W \beta$, so $\alpha, \beta \in Q$, and clearly $\operatorname{ran} \alpha = \operatorname{ran} \beta$. The converse is clear by the first part of the lemma. \qed

**Lemma 3.** If $\alpha, \beta \in T(V, W)$, then $\beta = \alpha \mu$ for some $\mu \in T(V, W)$ if and only if $\ker \alpha \subseteq \ker \beta$. Consequently, $\alpha \mathcal{R} \beta$ in $T(V, W)$ if and only if $\ker \alpha = \ker \beta$.

**Proof.** Clearly, if $\beta = \alpha \mu$ for some $\mu \in T(V, W)$, then $\ker \alpha \subseteq \ker \beta$. Conversely, suppose that $\ker \alpha \subseteq \ker \beta$. Write $\ker \alpha = \langle u_i \rangle$, $\ker \beta = \langle u_i, u_j \rangle$ and $V = \ker \beta \oplus \langle v_k \rangle$. Then

$$\alpha = \begin{pmatrix} u_i & u_j & v_k \\ 0 & w_j & w_k \end{pmatrix}, \quad \beta = \begin{pmatrix} u_i & u_j & v_k \\ 0 & 0 & w_k \end{pmatrix},$$

\[3\] Linear transformations with restricted range 443

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for some $w_j, w_k, w'_k \in W$. Let $V = \text{ran } \alpha \oplus \langle \upsilon \rangle$ and define $\mu \in T(V, W)$ by

$$\mu = \begin{pmatrix} \upsilon & w_j & w_k \\ 0 & 0 & w'_k \end{pmatrix}.$$  

Then $\beta = \alpha \mu$, as required, and the remaining assertion is clear.  

Note that, if $\alpha \in T(V, W)$, then $\dim(W \alpha) \leq r(\alpha)$. Also, recall that the rank-nullity theorem for arbitrary vector spaces can be proved by showing that, for each $\pi \in T(V)$, the mapping

$$V \pi \to V/ \ker \pi, \quad v \pi \to v + \ker \pi,$$

is a well-defined (vector space) isomorphism. Hence, if $\ker \beta \subseteq \ker \alpha$, then $r(\beta) \geq r(\alpha)$.

We also need to observe that if $\alpha \in T(V, W)$, and we write $W \cap \ker \alpha = \langle u_i \rangle$ and $W = \langle u_i \rangle \oplus \langle u_j \rangle$, then $\dim(W \alpha) = |J|$. This follows from the fact that $\{u_j \alpha \}$ is a basis for $W \alpha$, and the restriction $\alpha|\{u_j \}$ is a (vector space) isomorphism from $\langle u_j \rangle$ onto $W \alpha$.

**Lemma 4.** If $\alpha, \beta \in T(V, W)$, then $\beta = \lambda \alpha \mu$ for some $\lambda, \mu \in T(V, W)$ if and only if $r(\beta) \leq \dim(W \alpha)$. Consequently, $\alpha \not\sim \beta$ in $T(V, W)$ if and only if one of the following equalities occurs:

1. $\ker \alpha = \ker \beta$;
2. $r(\alpha) = \dim(W \alpha) = \dim(W \beta) = r(\beta)$.

**Proof.** If $\beta = \lambda \alpha \mu$ for some $\lambda, \mu \in T(V, W)$, then $V \beta = (V \lambda) \alpha \mu \subseteq (W \alpha) \mu$, so $\dim(V \beta) \leq \dim(W \alpha)$. Conversely, suppose that the condition holds and write

$$\beta = \begin{pmatrix} u_i & v_k \\ 0 & w_k \end{pmatrix}, \quad \alpha = \begin{pmatrix} u_j & w'_k \\ 0 & w_k \end{pmatrix},$$

where $V \beta = \langle w_k \rangle, \langle w'_k \rangle \subseteq W$ and $V \alpha = \langle w'_k \alpha \rangle \oplus \langle \upsilon \rangle \subseteq W$. Let $V = \langle w'_k \alpha \rangle \oplus \langle \upsilon_m \rangle$ and define $\lambda, \mu \in T(V, W)$ by

$$\lambda = \begin{pmatrix} u_i & v_k \\ 0 & w'_k \end{pmatrix}, \quad \mu = \begin{pmatrix} \upsilon_m & w'_k \alpha \\ 0 & w_k \end{pmatrix}.$$  

Then $\beta = \lambda \alpha \mu$, as required.

Now, suppose that $\beta = \lambda \alpha \mu$ and $\alpha = \lambda' \beta \mu'$ for some $\lambda, \lambda', \mu, \mu' \in T(V, W)^1$. If $\lambda = 1$ then $\ker \alpha \subseteq \ker \beta$; and if $\lambda \neq 1$ then $r(\beta) = \dim(V \lambda \alpha \mu) \leq \dim(W \alpha)$. In other words, the supposition implies that

$$\ker \alpha \subseteq \ker \beta \quad \text{or} \quad r(\beta) \leq \dim(W \alpha) \quad \text{and}$$

$$\ker \beta \subseteq \ker \alpha \quad \text{or} \quad r(\alpha) \leq \dim(W \beta),$$

and the different combinations give the following possibilities:
Clearly, the given condition implies that $\ker \alpha = \ker \beta$;

(J2) $r(\alpha) = \dim(W\alpha) = \dim(W\beta) = r(\beta)$;

(J3) $\ker \alpha \subseteq \ker \beta$ and $r(\alpha) \leq \dim(W\beta)$;

(J4) $\ker \beta \subseteq \ker \alpha$ and $r(\beta) \leq \dim(W\alpha)$.

However, if (J3) occurs then $W \cap \ker \alpha \subseteq W \cap \ker \beta$ and $r(\alpha) \geq r(\beta)$. Hence, if

$$W \cap \ker \beta = (W \cap \ker \alpha) \oplus U_1 \quad \text{and} \quad W = (W \cap \ker \beta) \oplus U_2,$$

then $W\alpha = U_1\alpha \oplus U_2\alpha$ and $W\beta = U_2\beta$. Consequently,

$$\dim(W\beta) = \dim U_2 = \dim(U_2\alpha) \leq \dim(W\alpha),$$

$$\dim(W\alpha) \leq r(\alpha) \leq \dim(W\beta) \quad \text{and} \quad r(\beta) \leq r(\alpha) \leq \dim(W\beta) \leq r(\beta).$$

It follows that (J2) holds, and similarly, (J4) also implies (J2). For the converse, recall that $R \subseteq J$. Hence, if either (J1) or (J2) occurs then Lemma 3, and the first part of this lemma, imply that $\alpha J \beta$. \hfill \Box

From Lemma 4, we see that, if $\alpha J \beta$, then $r(\alpha) = r(\beta)$. However, the converse is false, even if $V$ has finite dimension. This differs from the situation for $T(V)$ and arbitrary $V$, since it is well known that $\alpha J \beta$ in $T(V)$ if and only if $r(\alpha) = r(\beta)$; see [1, Volume 1, Exercise 2.2.6].

**Example 1.** Let $V = \langle e_1, e_2, e_3 \rangle$ and $W = \langle e_1, e_2 \rangle$, and define $\alpha, \beta \in T(V, W)$ by

$$\alpha = \begin{pmatrix} e_1 & e_2 & e_3 \\ 0 & e_2 & e_1 \end{pmatrix}, \quad \beta = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_1 & 0 & e_2 \end{pmatrix}. $$

Then $\ker \alpha \neq \ker \beta$, so (J1) does not hold. Also, $r(\alpha) = r(\beta) = 2$, but $\dim(W\alpha) = \dim(W\beta) = 1$, so (J2) does not hold. Hence, $\alpha$ and $\beta$ are not $J$-related in $T(V, W)$. Furthermore, $V\alpha \not\subseteq W\alpha$ and $V\beta \not\subseteq W\beta$, so $\alpha, \beta \not\in Q$. Hence, $\alpha, \beta$ are also not $L$-related in $T(V, W)$, even though $\ran \alpha = \ran \beta$.

In fact, this example shows more: namely, even though $r(\beta) = r(\alpha)$ and $\dim(W\beta) = \dim(W\alpha)$ for the given $\alpha$ and $\beta$, nonetheless $\beta \neq \lambda \alpha \mu$ for all $\lambda, \mu \in T(V, W)$. This is unlike the situation in $T(V)$, where $\beta = \lambda \alpha \mu$ for some $\lambda, \mu \in T(V)$ if and only if $\dim(V\beta) \leq \dim(V\alpha)$. However, by restricting our attention to $Q$, we regain the normal situation.

**Lemma 5.** If $\alpha, \beta \in Q$, then $\beta = \lambda \alpha \mu$ for some $\lambda, \mu \in Q$ if and only if $r(\beta) \leq r(\alpha)$. Consequently, $\alpha J \beta$ in $Q$ if and only if $r(\alpha) = r(\beta)$, and hence $J = D$ on $Q$.

**Proof.** Clearly, the given condition implies that $r(\beta) \leq r(\alpha)$. Conversely, suppose that $\alpha, \beta \in Q$ and $r(\beta) \leq r(\alpha)$. Then $V\alpha = W\alpha$ and $V\beta = W\beta$, so we can write

$$\beta = \begin{pmatrix} u_r & w_i \\ 0 & w_i \beta \end{pmatrix}, \quad \alpha = \begin{pmatrix} u_s & w_i' \\ 0 & w_i' \alpha \end{pmatrix}. $$

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Now let $V = \langle u_i \rangle \oplus \langle w'_i \alpha \rangle$ and define $\lambda, \mu \in T(V, W)$ by

$$
\lambda = \begin{pmatrix} u_r & w_i \\ 0 & w'_i \alpha \end{pmatrix}, \quad \mu = \begin{pmatrix} v_t & w_i' \alpha \\ 0 & w_i \beta \end{pmatrix}.
$$

Then $\beta = \lambda \alpha \mu$. Moreover, since $V = \langle u_r \rangle \oplus \langle w_i \rangle$, we know that $V \lambda = \langle w_i \lambda \rangle \subseteq W \lambda$, and so $\lambda \in Q$. Similarly, $V \mu = \langle (w'_i \alpha) \mu \rangle \subseteq W \mu$ (since $w'_i \alpha \in W$ for each $i$) and hence $\mu \in Q$.

Finally, if $\beta = \lambda \alpha \mu$ and $\alpha = \lambda' \beta' \mu'$ for some $\lambda, \lambda', \mu, \mu' \in Q$ then, regardless of whether $\lambda = 1$ or $\mu = 1$,

$$
\dim(V \beta) = \dim(V \lambda) \alpha \mu \leq \dim(V \alpha) \mu \leq \dim(V \alpha).
$$

That is, $r(\beta) \leq r(\alpha)$, and similarly $r(\alpha) \leq r(\beta)$. The converse is clear from the first part of this lemma. Finally, since $Q$ is a regular subsemigroup of $T(V, W)$, T. E. Hall’s theorem allows us to deduce that the $\mathcal{L}$ and $\mathcal{R}$ relations on $Q$ are the restrictions of those on $T(V, W)$ to $Q$. Thus, by Lemmas 2 and 3, if $\alpha, \beta \in Q$ then $\alpha \mathcal{L} \beta$ in $Q$ if and only if $\ker \alpha = \ker \beta$, and $\alpha \mathcal{R} \beta$ in $Q$ if and only if $\ker \alpha = \ker \beta$. Consequently, a standard argument shows that, if $r(\alpha) = r(\beta)$, then $\alpha \mathcal{D} \beta$, and we conclude that $\mathcal{J} = \mathcal{D}$ on $Q$. \hfill $\Box$

**Lemma 6.** If $\alpha, \beta \in T(V, W)$, then $\alpha \mathcal{D} \beta$ in $T(V, W)$ if and only if either $\ker \alpha = \ker \beta$ or $r(\alpha) = r(\beta)$ and $\alpha, \beta \in Q$.

**Proof.** If $\alpha \mathcal{D} \beta$ in $T(V, W)$, then $\alpha \mathcal{R} \gamma \mathcal{L} \beta$ for some $\gamma \in T(V, W)$. Hence, $\ker \alpha = \ker \gamma$, and either $\gamma = \beta$ or ($\ker \gamma \neq \beta$) $\ker \gamma \neq \beta$. If $\ker \alpha = \ker \gamma$ and $\gamma = \beta$ then $\ker \alpha = \ker \beta$, as required. On the other hand, suppose that $\ker \alpha = \ker \gamma$, $\ker \gamma \neq \beta$, and $\gamma, \beta \in Q$. Then $\alpha = \gamma \mu$ for some $\mu \in T(V, W)$, so $V \gamma \leq W \gamma$ implies that $V \alpha \leq W \alpha$, and hence $\alpha \in Q$. Similarly, $\beta \in Q$. Also,

$$
r(\beta) = r(\gamma) = \dim(V / \ker \gamma) = \dim(V / \ker \alpha) = r(\alpha).
$$

Conversely, if $\ker \alpha = \ker \beta$ then $\alpha \mathcal{R} \beta$, and so $\alpha \mathcal{D} \beta$ (since $\mathcal{R} \subseteq \mathcal{D}$). On the other hand, if $\alpha, \beta \in Q$ and $r(\alpha) = r(\beta)$, then $V \alpha = W \alpha$ and $V \beta = W \beta$, so we can write

$$
\alpha = \begin{pmatrix} u_r & w_j \\ 0 & w_j \alpha \end{pmatrix}, \quad \beta = \begin{pmatrix} u_s & w'_j \alpha \\ 0 & w'_j \beta \end{pmatrix},
$$

where $\langle w_j \rangle \subseteq W$ and $\langle w'_j \rangle \subseteq W$. If $\gamma \in T(V, W)$ is defined by

$$
\gamma = \begin{pmatrix} u_r & w_j \\ 0 & w'_j \beta \end{pmatrix},
$$

then $\ker \gamma = \ker \alpha$, $\gamma \neq \ker \beta$ and $\gamma \in Q$, so $\alpha \mathcal{R} \gamma \mathcal{L} \beta$. \hfill $\Box$
Recall that \( \mathcal{D} \subseteq \mathcal{J} \) on any semigroup, and it is well known that \( \mathcal{D} = \mathcal{J} \) on any \( T(V) \) (see [1, Volume 1, Exercise 2.2.6]). However, this fails for \( T(V, W) \), as we now show.

**Example 2.** If \( \alpha \mathcal{D} \beta \) in \( T(V, W) \) then either \( \ker \alpha = \ker \beta \) (so (J1) holds) or \( r(\alpha) = r(\beta) \) and \( \alpha, \beta \in Q \) (hence \( \dim(\alpha) = \dim(\beta) \) and (J2) holds). However, \( \mathcal{J} \setminus \mathcal{D} \) can be nonempty. For example, suppose that \( V = \langle u_0, u_1, u_2, w_1, w_k \rangle \) and \( W = \langle w_1, w_k \rangle \), where \( K \) is infinite. In this event, we can define \( \alpha, \beta \in T(V, W) \) by

\[
\alpha = \begin{pmatrix} u_0 & u_1 & \{u_2, w_1, w_k\} \\ 0 & w_1 & w_k \end{pmatrix}, \quad \beta = \begin{pmatrix} u_0 & u_1 & u_2 & \{w_1, w_k\} \\ 0 & 0 & w_1 & w_k \end{pmatrix}.
\]

Then \( u_2 - w_1 \in \ker \alpha \), so \( \ker \alpha \subsetneq \ker \beta \); and \( r(\alpha) = |K| = \dim(W\beta) \), so \( \alpha, \beta \) satisfy (J2). But, although \( r(\alpha) = r(\beta) \), we observe that \( W\alpha = \langle w_k \rangle = W\beta \), so \( W\alpha \subsetneq V\alpha \) and \( W\beta \subsetneq V\beta \). Therefore, \( \alpha, \beta \notin Q \), and hence \( \alpha, \beta \) are not \( \mathcal{D} \)-related in \( T(V, W) \).

**Corollary 7.** If \( \dim W < \aleph_0 \) then \( \mathcal{D} = \mathcal{J} \) on \( T(V, W) \).

**Proof.** Suppose that \( \alpha, \beta \in T(V, W) \) and \( \alpha \mathcal{J} \beta \). By Lemma 3, if \( \ker \alpha = \ker \beta \) then \( \alpha \mathcal{D} \beta \); and, by Lemma 4, if \( \ker \alpha \neq \ker \beta \) then \( r(\alpha) = \dim(W\alpha) = \dim(W\beta) = r(\beta) \). Consequently, in this case, if \( \dim W < \aleph_0 \) then \( r(\alpha), r(\beta) < \aleph_0 \), and it follows that \( V\alpha = W\alpha \) and \( V\beta = W\beta \). Thus, \( \alpha, \beta \in Q \) and \( r(\alpha) = r(\beta) \), so \( \alpha \mathcal{D} \beta \). \( \square \)

### 3. Ideals in \( T(V, W) \)

In what follows, \( Y = A \cup B \) means that \( Y \) is a *disjoint* union of \( A \) and \( B \), and \( r' \) denotes the *successor* of a cardinal \( r \). Also, as an abbreviation, we sometimes write \( T = T(V, W) \).

As might be expected, the ideals of \( Q \) are easy to describe.

**Theorem 8.** The ideals of \( Q \) are precisely the sets

\[ Q_r = \{ \alpha \in Q \mid r(\alpha) < r \}, \]

where \( 1 \leq r \leq \dim W \). In addition, \( Q_r \) is principal if and only if \( r = s' \), where \( 1 \leq s \leq \dim W \).

**Proof.** If \( \alpha \in Q_r \) and \( \beta \in Q \), then \( \dim(V\alpha)\beta \leq \dim(V\alpha) \) and \( V(\beta\alpha) \subseteq V\alpha \), so \( \alpha\beta \in Q_r \) and \( \beta\alpha \in Q_r \), and hence \( Q_r \) is an ideal of \( Q \). Conversely, suppose that \( I \) is an ideal of \( Q \) and let \( r \) be the least cardinal greater than \( r(\alpha) \) for all \( \alpha \in I \). Then \( I \subseteq Q_r \). Let \( \beta \in Q_r \) and suppose that \( r(\beta) = s < r \). Then there exists \( \alpha \in I \) with \( r(\alpha) \geq s \); otherwise, \( r(\alpha) < s \) for all \( \alpha \in I \), contradicting the choice of \( r \). That is, \( r(\beta) \leq r(\alpha) \), and Lemma 5 implies that \( \beta = \lambda\alpha\mu \) for some \( \lambda, \mu \in Q \). That is, \( Q_r \subseteq I \), and equality follows.

Finally, if \( r = s' \) for some \( s \) such that \( 1 \leq s \leq \dim W \), then Lemma 5 implies that \( Q_r \subseteq Q^1\alpha Q^1 \) for each \( \alpha \in Q_r \) with rank \( s \), and it follows that \( Q_r \) is principal.
Conversely, suppose that \( Q_r = Q^1 \alpha Q^1 \) for some \( \alpha \in Q_r \). Let \( r(\alpha) = s \) and assume there is a cardinal \( t \) such that \( s < t < r \). Since \( r \leq \dim W \), there exists \( \beta \in Q \) with \( r(\beta) = t \). For example, we can write \( W = \langle e_i, e_j \rangle \) and \( V = \langle e_i, e_j, e_k \rangle \) where \( |I| = t \), and let

\[
\beta = \begin{pmatrix}
  \{e_j, e_k\} & e_i \\
  0 & e_i
\end{pmatrix}.
\]

Now \( \beta \in Q_r \), so \( \beta = \lambda \alpha \mu \) for some \( \lambda, \mu \in Q^1 \). But this implies that \( r(\beta) \leq r(\alpha) \), which is a contradiction. Therefore, \( t \) does not exist and thus \( r = s' \). \( \Box \)

To determine the ideals of \( T = T(V, W) \), we let \( 1 \leq r \leq \dim W \) and write

\[
T_r = \{ \alpha \in T \mid r(\alpha) < r \}.
\]

If \( \alpha \in T_r \) and \( \lambda, \mu \in T \) then \( V \lambda \alpha \subseteq V \alpha \) and \( \dim(V \alpha \mu) \leq \dim V \alpha \), so \( T_r \) is an ideal of \( T \).

Let \( Y \subseteq T(V, W) \) be nonempty and let

\[
\begin{align*}
r(Y) &= \min \{ r \mid r > \dim(W \alpha) \text{ for all } \alpha \in Y \}, \\
K(Y) &= \{ \beta \in T(V, W) \mid \ker \beta \supseteq \ker \alpha \text{ for some } \alpha \in Y \}.
\end{align*}
\]

Note that \( r(Y) \) always exists since the cardinals are well ordered.

**Lemma 9.** With the above notation, both \( T_{r(Y)} \cup K(Y) \) and \( T_{r(Y)'} \cup K(Y) \) are ideals of \( T(V, W) \).

**Proof.** Since \( \ker \beta \subseteq \ker \beta \mu \) for each \( \mu \in T(V, W) \), \( K(Y) \) is a right ideal of \( T(V, W) \). On the other hand, if \( \lambda \in T(V, W) \) and \( \beta \in K(Y) \) then Lemma 3 implies that \( \beta = \alpha \mu \) for some \( \alpha \in Y \) and \( \mu \in T(V, W) \), hence

\[
\dim(V \lambda \beta) \leq \dim(W \beta) = \dim(W \alpha \mu) \leq \dim(W \alpha) < r(Y).
\]

Therefore \( \lambda \beta \in T_{r(Y)} \). The result now follows since \( T_{r(Y)} \) and \( T_{r(Y)'} \) are themselves ideals of \( T(V, W) \). \( \Box \)

**Example 3.** Let \( V = \langle u_1, u_2, u_3, w_1, w_2, w_3 \rangle \), \( W = \langle w_1, w_2, w_3 \rangle \) and

\[
\alpha = \begin{pmatrix}
  u_1 & \{w_1, w_2, w_3\} & u_2 & u_3 \\
  0 & 0 & w_2 & w_3
\end{pmatrix}.
\]

If \( T = T(V, W) \) then \( \alpha T \) is not only a right ideal but also a left ideal of \( T \) since \( \lambda \alpha = 0 \) for each \( \lambda \in T \). Let \( Y = \{ \alpha \} \). Then \( r(Y) = 1 \) and \( T_{r(Y)} = \{0\} \), and clearly \( \alpha T = K(\{\alpha\}) \).
Example 4. Let \( V = \langle u_1, u_2, u_3, w_1, w_2, w_3 \rangle \), \( W = \langle w_1, w_2, w_3 \rangle \) and
\[
\alpha = \begin{pmatrix} u_1 & w_2 & w_3 \\ 0 & w_1 & w_2 & w_3 \end{pmatrix},
\lambda_1 = \begin{pmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{pmatrix}.
\]
Now \( \alpha T \) is not a left ideal of \( T \) since
\[
\lambda_1 \alpha = \begin{pmatrix} w_1 & w_2 & w_3 \\ u_1 & 0 & 0 \end{pmatrix} \not\in \alpha T.
\]
Let \( Y = \{ \alpha \} \). Then \( r(Y) = 2 \) and \( T^1 \alpha T^1 \subseteq T_2 \cup K(\{ \alpha \}) \): for example, if \( \lambda, \mu \in T \) then \( \dim(V \lambda \alpha) \leq \dim(W \lambda \alpha) < 2 \) and \( \alpha \mu \in K(\{ \alpha \}) \). In fact, if \( \beta \in T_2 \) then \( \dim(V \beta) \leq 1 = \dim(W \alpha) \), so \( \beta \in T \alpha T \) by Lemma 4. And, if \( \beta \in K(\{ \alpha \}) \) then \( \beta = \alpha \mu \) for some \( \mu \in T \), so \( \beta \in T^1 \alpha T^1 \). Hence \( T^1 \alpha T^1 = T_2 \cup K(\{ \alpha \}) \). On the other hand, \( T \alpha T \subseteq T_2 \) since \( \dim(V \lambda \alpha \mu) \leq \dim(W \alpha) \mu \leq \dim(W \alpha) \) for all \( \lambda, \mu \in T \), and \( T_2 \subseteq T \alpha T \) by Lemma 4.

For our main result, we need a technical lemma.

Lemma 10. If \( \beta \in T \) and \( r < \dim(W \beta) = \dim(V \beta) = s \), then there exists \( \lambda \in T \) such that \( \dim(W \lambda \beta) = r \) and \( \dim(V \lambda \beta) = s \).

Proof. If \( s \) is finite then \( W \beta = V \beta \), so we write
\[
\beta = \begin{pmatrix} u_p & w_1 & \ldots & w_r & w_{r+1} & \ldots & w_s \\ 0 & w'_1 & \ldots & w'_r & w'_{r+1} & \ldots & w'_s \end{pmatrix}.
\]
Choose \( u \in V \setminus W \) and note that \( u + w_j \not\in W \) for each \( j = r + 1, \ldots, s \). Also, the set \( \{u_1, \ldots, w_r, u + w_{r+1}, \ldots, u + w_s\} \) is linearly independent: for example, if there are scalars such that
\[
\sum_{i=1}^r x_i w_i + \sum_{j=r+1}^s y_j (u + w_j) = 0,
\]
then \( \sum_{i=1}^r x_i w_i + \sum_{j=r+1}^s y_j w_j \in \langle u \rangle \) and this implies that \( x_i = y_j = 0 \) for each \( i \) and \( j \). Write \( V = \langle u_\ell \rangle \oplus \langle w_1, \ldots, w_r, u + w_{r+1}, \ldots, u + w_s \rangle \) and let
\[
\lambda = \begin{pmatrix} u_\ell & w_1 & \ldots & w_r & u + w_{r+1} & \ldots & u + w_s \\ 0 & w_1 & \ldots & w_r & w_{r+1} & \ldots & w_s \end{pmatrix}.
\]
Then \( \dim(W \lambda \beta) = r \) and \( \dim(V \lambda \beta) = s \), as required.

If \( s \) is infinite, write
\[
\beta = \begin{pmatrix} u_p & w_j & v_k \\ 0 & w_j & w_k \end{pmatrix},
\]
where $|J| + |K| = |J| = s \geq \aleph_0$. This implies that $|K| \leq |J|$, and clearly there exist $\lambda_1 \in T$ and $u_d \in V$ such that

$$\lambda_1 \beta = \begin{pmatrix} u_d & w_j \\ 0 & w'_j \end{pmatrix} \in Q.$$ 

Since $r < |J|$, we can write $J = M \cup N$ where $|M| = r$ and $|N| = |J|$. Then, as before, if $u \in V \setminus W$ then $\{w_m\} \cup \{u + w_n\}$ is linearly independent and we let

$$\lambda_2 = \begin{pmatrix} u_\ell & w_m & u + w_n \\ 0 & w_m & w_n \end{pmatrix}.$$ 

Then $\dim(W\lambda_2 \lambda_1 \beta) = r$ and $\dim(V\lambda_2 \lambda_1 \beta) = s$, as required. \hfill $\Box$

The proper ideals of $T(W)$ are well known: in fact, they are in one-to-one correspondence with the cardinals $r$ such that $1 \leq r \leq \dim W$ (see [1, Volume 1, Exercise 2.2.6]). However, the result for $T(V, W)$ is very different.

**Theorem 11.** The ideals of $T(V, W)$ are precisely the sets $T_0 \cup K(Y)$ and $T_0 \cup K(Y)$, where $r = r(Y)$ and $Y$ is a nonempty subset of $T(V, W)$.

**Proof.** Let $I$ be an ideal of $T$. If $I = \{0\}$, we let $Y = I$, so $r(Y) = 1$, $T_1 = \{0\}$; and, if $\beta \in K(\{0\})$ then $\ker \beta = V$, so $\beta = 0$ and thus $K(\{0\}) = \{0\}$. That is, $\{0\} = T_1 \cup K(\{0\})$.

Suppose that $\alpha \in I$ is nonzero and write

$$\alpha = \begin{pmatrix} u_p & w_j & v_k \\ 0 & w'_j & w_k \end{pmatrix},$$

where $v_k \notin W$ for each $k$. If $J = \emptyset$ then $K \neq \emptyset$ and $\dim(W\alpha) < \dim(V\alpha)$. On the other hand, if $J \neq \emptyset$, choose $1 \in J$ and $u \in V \setminus W$, write $V = \langle u \rangle \oplus \langle \ell \rangle$ where $W \subseteq \langle \ell \rangle$, and let

$$\lambda = \begin{pmatrix} v_\ell & u \\ 0 & w_1 \end{pmatrix}.$$ 

Then $\lambda \alpha \in I$ and $\dim(W\lambda \alpha) = 0 < 1 = \dim(V\lambda \alpha)$. That is, in each case, if

$$Y = \{\alpha \in I \mid \dim(W\alpha) < \dim(V\alpha)\},$$

then $Y \neq \emptyset$. We assert that $I$ equals $T_r \cup K(Y)$ or $T_r \cup K(Y)$, where $r = r(Y)$.

First suppose that $\dim(W\beta) < r$ for all $\beta \in I$. In this case, if $\beta \in I$ and $r(\beta) < r$ then $\beta \in T_r$ and, if $\dim(W\beta) < r \leq r(\beta)$, then $\beta \in Y$ and so $\beta \in K(Y)$. Thus, in this case, $I \subseteq T_r \cup K(Y)$. Conversely, suppose that $\beta \in T_r$. If $\dim(W\alpha) < r(\beta) < r$ for all $\alpha \in Y$, we contradict the choice of $r = r(Y)$. Hence, $r(\beta) \leq \dim(W\alpha)$ for some $\alpha \in Y \subseteq I$, hence $\beta \in I$ by Lemma 4. Clearly, $K(Y) \subseteq I$, so we conclude that $I = T_r \cup K(Y)$. 

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Next suppose that \( r \leq \dim(W\pi) \) for some \( \pi \in I \). In this case, if \( \dim(W\pi) < \dim(V\pi) \) then \( \pi \in Y \) and we contradict the choice of \( r \). Hence \( \dim(W\pi) = \dim(V\pi) \). Now, if \( r < \dim(W\pi) = \dim(V\pi) = s \), then Lemma 10 implies that \( \dim(W\lambda\pi) = r < s = \dim(V\lambda\pi) \) for some \( \lambda \in T \), which contradicts the choice of \( r \) (since \( \lambda\pi \in I \)). Hence, in this case, \( r = \dim(W\pi) = \dim(V\pi) \) and thus \( \pi \in T_r \).

Clearly this conclusion holds for any \( \beta \in I \) such that \( r \leq \dim(W\beta) \). On the other hand, if \( \beta \in I \) and \( \dim(W\beta) < r \), then we have already seen that \( \beta \in T_r \cup K(Y) \). So, in this case, \( I \subseteq T_r \cup K(Y) \). Conversely, if \( \beta \in T_r \) then \( r(\beta) \leq r = \dim(W\pi) = \dim(V\pi) \) for some \( \pi \in I \), so \( \beta \in I \) by Lemma 4. As before, \( K(Y) \subseteq I \), and now we conclude that \( I = T_r \cup K(Y) \). \( \square \)

**Example 5.** Let \( 1 \leq r \leq \dim W \) and write

\[
J_r = \{ \alpha \in T \mid \dim(W\alpha) < r \}.
\]

If \( \alpha \in J_r \) and \( \lambda, \mu \in T \), then \( W\lambda\alpha \subseteq W\alpha \) and \( \dim(W\alpha\mu) \leq \dim(W\alpha) \), so \( J_r \) is an ideal of \( T \). Clearly \( T_r \subseteq J_r \) if \( r \leq s \leq \dim W \), and the containment can be proper. For example, suppose that \( s \) is finite and \( u \in V \setminus W \). Write \( W = \langle w_i \rangle \) with \( |I| = s \) and \( V = \langle v_p \rangle \oplus \langle u, w_i \rangle \). Let \( 1 \in I \) and \( J = I \setminus \{1\} \), and note that

\[
\alpha = \begin{pmatrix} v_p & w_1 & v \\ w_j & w_1 \end{pmatrix} \in J_s \setminus T_s.
\]

(1)

More generally, let \( Y = \{ \alpha \in J_r : \dim(W\alpha) < \dim(V\alpha) \} \). Since \( \dim(W\alpha) < r \) for all \( \alpha \in J_r \), we know that \( r(Y) \leq r \). Suppose that \( r(Y) < r \). If \( r \) is finite then the \( \alpha \) defined in (1) with \( s = r(Y) \) satisfies \( r(Y) = \dim(W\alpha) \), hence it belongs to \( J_r \). However, it also satisfies \( \dim(W\alpha) < \dim(V\alpha) \), so it contradicts the choice of \( r(Y) \), and we conclude that \( r(Y) = r \). Likewise, if \( r \) is infinite, we write \( W = \langle w_i, w_j \rangle \) where \( |I| = r(Y) < r \leq |J| \) and let \( V = \langle v_q \rangle \oplus \langle w_i, v + w_j \rangle \). Now consider

\[
\alpha = \begin{pmatrix} v_q & w_i & v + w_j \\ 0 & w_i & w_j \end{pmatrix} \in J_r.
\]

Since this also contradicts the choice of \( r(Y) \), we again conclude that \( r(Y) = r \). Therefore \( J_r = T_r \cup K(Y) \) by Theorem 11. \( \square \)

Recall that, for any vector space \( U \), the ideals of \( T(U) \) form a chain under containment. The next result shows that \( T(V, W) \) is almost never isomorphic to any \( T(U) \).

**Corollary 12.** If \( \dim V \geq 3 \), then \( T(V, W) \) is not isomorphic to \( T(U) \) for any vector space \( U \).

**Proof.** By our assumption at the start, \( \dim W \geq 1 \) and \( W \neq V \).

Suppose that \( \dim W = 1 \). In this case, \( \text{codim } W \geq 2 \) and we can write \( V = \langle v_1, v_2, v_m \rangle \oplus \langle w_1 \rangle \) where \( W = \langle w_1 \rangle \). Define nonzero \( \pi_1, \pi_2 \in T(V, W) \) by

\[
\pi_1 = \begin{pmatrix} \{v_m, v_2, w_1\} & v_1 \\ 0 & w_1 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} \{v_m, v_1, w_1\} & v_2 \\ 0 & w_1 \end{pmatrix}.
\]
Let $Y_1 = \{\pi_1\}$ and $Y_2 = \{\pi_2\}$. If $\beta \in K(Y_1)$ then $\ker \beta \supseteq \ker \pi_1 \supseteq W$. Hence, if \(\lambda \in T(V, W)\) then $V\lambda \beta \subseteq W\beta = \{0\}$, so $\lambda \beta = 0 \in K(Y_1)$. That is, $K(Y_1)$ and $K(Y_2)$ are ideals of $T(V, W)$, where $\pi_1 \in K(Y_1) \setminus K(Y_2)$ and $\pi_2 \in K(Y_2) \setminus K(Y_1)$. In other words, $K(Y_1)$ and $K(Y_2)$ are ideals of $T(V, W)$ which are not comparable under containment, so the ideals of $T(V, W)$ do not form a chain in this case.

Now suppose that $\dim W \geq 2$. If $w_1, w_2 \in W$ are linearly independent, and $u \in V \setminus W$, then $v_1 = u + w_1$ and $v_2 = u + w_2$ are linearly independent in a complement of $W$ in $V$. Write $V = \langle v_m \rangle \oplus \langle w_n \rangle$, where $\{v_m\} = \{v_1, v_2\} \cup \{v_p\}$ and $\{w_n\} = \{w_1, w_2\} \cup \{w_q\}$. Define nonzero $\alpha, \beta \in T(V, W)$ by

$$
\alpha = \begin{pmatrix}
\{v_p, w_n\} & v_1 & v_2 \\
0 & w_1 & w_2 
\end{pmatrix}, \quad 
\beta = \begin{pmatrix}
\{v_m, \{w_n\} \setminus \{w_1\}\} & w_1 \\
0 & w_1
\end{pmatrix}.
$$

Clearly, $\alpha \in J_1 \setminus T_2$ and $\beta \in T_2 \setminus J_1$. That is, $J_1$ and $T_2$ are ideals which are not comparable under containment, so the ideals of $T(V, W)$ do not form a chain, and the result follows.

It is not hard to see that part (b) of the next result also holds if $Y = \alpha T^1$, which clearly also equals $\beta T^1$ for some $\beta \neq \alpha$. So, it is unlikely that there are conditions which determine precisely when $T_r \cup K(Y)$ is principal.

**Theorem 13.** Let $\alpha \in T(V, W) = T$, say. Then:

(a) $T\alpha T = T_r \cup K(Y)$, where $Y = T\alpha T$, $r = r(Y) = s'$ and $s = \dim(W\alpha)$;

(b) $T^1\alpha T^1 = T_r \cup K(Y)$, where $Y = \{\alpha\}$, $r = r(Y) = s'$ and $s = \dim(W\alpha)$.

**Proof.** (a) Let $s = \dim(W\alpha)$ and, with our usual choice of bases, write

$$
\alpha = \begin{pmatrix}
\{u_i, v_k\} & w_j & v_k \\
0 & w_j & w_k
\end{pmatrix}.
$$

Let $V = \langle u'_i \rangle \oplus \langle w'_j \rangle \oplus \langle w'_k \rangle$ and define $\delta, \varepsilon \in T$ by

$$
\delta = \begin{pmatrix}
\{u_i, v_k\} & w_j \\
0 & w_j
\end{pmatrix}, \quad \varepsilon = \begin{pmatrix}
\{u'_\ell, w_k\} & w'_j \\
0 & w'_j
\end{pmatrix}.
$$

Clearly, if $\pi = \delta \alpha \varepsilon$, then $\dim(W\pi) = s$. Now, if $\lambda, \mu \in T$ then $\dim(W\lambda \alpha \mu) \leq \dim(W\alpha) \mu \leq \dim(W\alpha) = s$ (note that possibly $\alpha \notin T\alpha T$). Hence, if $Y = T\alpha T$ and $r = r(Y)$ then $r \geq s'$. Suppose that $r > s'$. Then, by the definition of $r(Y)$, there exists $\lambda, \mu \in T$ such that $s' \leq \dim(W\lambda \alpha \mu) \leq \dim(W\alpha) = s$, which is a contradiction (regardless of whether $s$ is finite or infinite). Hence, $r \leq s'$, and equality follows. Next, if $\beta \in K(Y)$ then $\beta = \gamma \mu'$ for some $\gamma \in Y$ and $\mu' \in T$. That is, $\beta = \lambda \alpha \mu. \mu'$ for some $\lambda, \mu \in T$ and so $\beta \in T\alpha T$. Moreover, for each $\lambda, \mu \in T$, $\dim(V\lambda \alpha \mu) \leq \dim(W\alpha) \mu \leq s$. In other words, $K(Y) \subseteq T\alpha T \subseteq T_{s'}$. In fact, if $\beta \in T_{s'}$ then $r(\beta) \leq s = \dim(W\alpha)$, so $\beta \in T\alpha T$ by Lemma 4. Thus, we obtain $T\alpha T = T_{s'} \cup K(Y)$, as required.
(b) If $\lambda, \mu \in T$ and $\lambda \neq 1$ then $r(\lambda \alpha \mu) \leq \dim (W \alpha) \mu \leq s$, so $\lambda \alpha \mu \in \cal T_{s'}$, and clearly $\alpha \mu \in K(Y)$ when $Y = \{\alpha\}$. Thus, $T^1 \alpha T^1 \subseteq \cal T_{s'} \cup K(Y)$. Conversely, if $\beta \in \cal T_{s'}$ then $r(\beta) \leq s = \dim (W \alpha)$, so $\beta \in T \alpha T$ by Lemma 4; and, if $\beta \in K(Y)$ then $\beta \in \alpha T^1$ by Lemma 3. Therefore, $T^1 \alpha T^1 = \cal T_{s'} \cup K(Y)$ where $r(Y) = s'$ (since $Y = \{\alpha\}$). \hfill \Box

In passing, we note that if $1 \leq r \leq \dim W$, $Y = T_r$ and $\beta \in K(Y)$, then $r(\beta) \leq r(\alpha) < r$, so $\beta \in T_r$ and thus $K(Y) \subseteq T_r$. Also, $r(Y) = r$ since $Q_r \subseteq T_r$. That is, the ideal $T_r$ takes the form $T_r(Y) \cup K(Y)$, when $Y = T_r$.

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References


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