81.35 Does it give the right angle?

In spring last year I received quite unexpectedly the following letter from a schoolboy in Oman who was then aged 12 years.

Dear Sir

I have developed 70° angle using a compass. I would be grateful if you could examine the enclosed and let me know whether it is correct or not.

Thank you.

Yours faithfully

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Gigar’s construction of 70° angle

Steps for the construction of 70° angle.

1. Construct \( \overline{AR} \).
2. Construct \( \angle ZAR = 90° \) (Note: while constructing \( \angle ZAR = 90° \), the radius \( \overline{AB} \) is kept same, even while constructing intersecting arcs at \( Z \)).
3. With \( A \) as the centre and \( AZ \) as the radius, construct an arc \( a \), intersecting \( \overline{AR} \) at \( C \).
4. Taking \( BC \) as the radius and \( C \) as the centre, construct an arc to intersect arc \( a \) at \( D \). Then taking \( D \) as the centre and with the same radius, construct an arc intersecting arc \( a \) at \( X \).
5. Bisect \( \angle ZAX \) by taking \( Z \) and \( X \) as the centres of two arcs of equal radii which intersect at \( Q \).
6. Join \( Q \) and \( A \). Hence \( \angle QAR = 70° \).

The reader might like to construct this angle using ruler and compasses and measure it before reading on. Is this indeed a true construction for 70°? Furthermore, the reader might attempt a trigonometrical investigation ...

My reply to this enterprising student now follows ...

Dear Gigar,

Thank you for sending me your interesting construction for an angle of
70°. I knew at once that it could not be completely accurate because mathematicians have proved that 60° cannot be trisected using compass and straightedge (which would give 20°). If 70° could be constructed then so could 20° because 20° + 70° = 90°, so there would be a contradiction.

I have made your construction accurately using a computer package called *Cabri géomètre* (see Figure 1), and found the angle to be (approximately) 69.40°.

![Figure 1 Gigar's construction](image)

I have also analysed the diagram using trigonometry which shows that, taking \( AB = 1, \ AZ = AC = AD = AX = \sqrt{3}, \ CD = DX = \sqrt{3} - 1 \), from which it follows that \( \angle DAC = \angle XAD = 24.40000808° \), from which I deduce that \( \angle QAC = 69.40000808° \) (to the accuracy of my calculator).

The mathematics behind the proof concerning which angles can and which cannot be constructed is not simple. Accounts can be found in the following two books and other similar books. Note that they are *algebra* books rather than *geometry* books!


The essence of the proof is based upon the following ideas.

A real number \( r \) is said to be *constructible* if, starting from a segment of length 1, we can construct a line segment of length \( |r| \) in a finite number of steps using compass and straightedge only.

It has been proved that a real number is constructible if and only if it can be obtained from the number 1 by successive applications of the algebraic operations of addition, subtraction, multiplication, division and square rooting. (These are also the geometric operations which can be performed by construction using compass and straightedge.)

For example \( 3 + \sqrt{4 + 5\sqrt{6}} \) is constructible but \( \sqrt{2} \) is not.
It has been shown that $60^\circ$ can be trisected only if $\cos 20^\circ$ is a constructible number. However, $\cos 20^\circ$ is a root of the cubic equation $8x^3 - 6x - 1 = 0$ which comes from the trigonometric identity that $\cos 3x = 4 \cos^3 x - 3 \cos x$ with $x = 20^\circ$ and using $\cos 60^\circ = 1/2$. The cubic $8x^3 - 6x - 1 = 0$ does not factorise over the rational numbers so $x$ is not a constructible number so $\cos 20^\circ$ is not a constructible number and so $60^\circ$ cannot be trisected!

I hope that this is of use to you. Your construction is very interesting, being simple and yet obtaining a value close to $70^\circ$. (One can get as close as one wishes if the constructions are very complex.) When I first studied your construction I wondered if it was the simplest possible construction to get that close. However, a student of mine (Simon) has discovered a simple construction very similar to yours which yields a much closer approximation. I describe it below, and leave you to investigate it (see Figure 2).

Best wishes

David Green

Simon's construction of $70^\circ$ angle

Steps for the construction of $70^\circ$ angle.

1. Construct $\overline{AR}$.
2. Construct $\angle ZAR = 90^\circ$ (Note: while constructing $\angle ZAR = 90^\circ$, the radius $AB$ is kept same, even while constructing intersecting arcs at $Z$).
3. With $A$ as the centre and $AZ$ as the radius, construct an arc $a$, intersecting $\overline{AHatC}$.
4. Taking $BC$ as the radius and $B$ as the centre, construct an arc to intersect arc $\overline{AR}$ at $D$. Then taking $DC$ as the radius and $C$ as centre, construct an arc intersecting arc $a$ at $X$.
5. Bisect $\angle ZAX$ by taking $Z$ and $X$ as the centres of two arcs of equal radii which intersect at $Q$.
6. Join $Q$ and $A$. Hence $\angle QAR = 70^\circ$.

FIGURE 2 Simon's construction

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