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$\mathbb{A}^1$-CONNECTEDNESS OF MODULI OF VECTOR BUNDLES ON A CURVE

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Abstract In this note, we prove that the moduli stack of vector bundles on a curve with a fixed determinant is $\mathbb{A}^1$-connected. We obtain this result by classifying vector bundles on a curve up to $\mathbb{A}^1$-concordance. Consequently, we classify $\mathbb{P}^n$-bundles on a curve up to $\mathbb{A}^1$-weak equivalence, extending a result in [3] of Asok-Morel. We also give an explicit example of a variety which is $\mathbb{A}^1$-h-cobordant to a projective bundle over $\mathbb{P}^2$ but does not have the structure of a projective bundle over $\mathbb{P}^2$, thus answering a question of Asok-Kebekus-Wendt [2].

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1. Introduction

Let C be a smooth projective curve of genus g over a field k. Fix a line bundle $\mathcal{L} \in \mathrm{Pic}(C)$. Consider the following moduli stack $Bun_{n,\mathcal{L}}$.

$Bun_{n,\mathcal{L}}(Y) = \{$ category of rank n vector bundles on $C \times Y$, such that for any object $\mathcal{V}$, $\det\mathcal{V} \cong p^*(\mathcal{L})$, where $p : C \times Y \to C\}$

This is a smooth algebraic stack [1, Prop 1.3]. Any stack can be regarded as a simplicial sheaf via the nerve construction (see [6]) and thus it defines an object in the $\mathbb{A}^1$-homotopy category. Therefore, it makes sense to talk about $\mathbb{A}^1$-connectedness of $Bun_{n,\mathcal{L}}$. Following is the main theorem of this note:

Theorem 1.1. *$Bun_{n,\mathcal{L}}$ is $\mathbb{A}^1$-connected for any curve C over an infinite field k and $\mathcal{L} \in \mathrm{Pic}(C)$.*

The proof of Theorem 1.1 relies on finding an explicit $\mathbb{A}^1$-concordance (see [2, Definition 5.1] or Definition 2.1) between a vector bundle $\mathcal{E}$ of rank n and determinant $\mathcal{L}$ to the vector bundle $\mathcal{O}_C^{n-1} \oplus \mathcal{L}$. This is achieved by induction on n. In the course of this proof, we also achieve the classification of vector bundles of a given rank on the curve C

up to $\mathbb{A}^1$-concordance (Theorem 2.5). Once the question of $\mathbb{A}^1$-connectedness of $Bun_{n,\mathcal{L}}$ is settled, it is natural to wonder the same about its open substack $Bun^s_{n,\mathcal{L}}$, the moduli of stable vector bundles for a curve of genus greater than 1. Assuming that n and degree of $\mathcal{L}$ are co-prime, k algebraically closed, the coarse moduli space is known to be rational ([8, Theorem 1.2]) and one may be tempted to conclude that $\mathbb{A}^1$-connectedness of an algebraic stack is dictated by that of its coarse moduli space (if it exists). However, in Example 2.9, we show that this simply is not the case. We give an example of "stacky" $\mathbb{P}^1$, an orbifold with $\mathbb{P}^1$ as a coarse moduli space, which is not $\mathbb{A}^1$-connected. It should also be noted that the $\mathbb{A}^1$-connectedness is not preserved under rationality of a morphism of schemes as illustrated by $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$.

Related to $\mathbb{A}^1$-concordance is the notion of an $\mathbb{A}^1$-h-cobordism ([3, Definition 3.1.1] or Definition 3.1). Furthermore, projectivizations of two $\mathbb{A}^1$-concordant vector bundles are $\mathbb{A}^1$-h-cobordant. As an application of $\mathbb{A}^1$-connectedness of $Bun_{n,\mathcal{L}}$, we obtain the following theorem which classifies $\mathbb{P}^n$-bundles over any curve of genus g up to $\mathbb{A}^1$-weak equivalence. This extends the result on classification of $\mathbb{P}^n$-bundles over $\mathbb{P}^1$ given in [3].

Theorem 1.2. *Let $X = \mathbb{P}_C(\mathcal{E})$ and $Y = \mathbb{P}_C(\mathcal{F})$ be $\mathbb{P}^n$-bundles over C, where C lies over an infinite field. Then the following are equivalent:*

(1) *X and Y are $\mathbb{A}^1$- weakly equivalent.*

(2) *X and Y are $\mathbb{A}^1$-h-cobordant.*

(3) $\det(\mathcal{E}) \otimes \det(\mathcal{F})^{-1} = \mathcal{L}^{\otimes n+1}$*, for some* $\mathcal{L} \in \mathrm{Pic}(C)$.

In another application of our theorem, we answer a question raised in [2]: whether a variety which is $\mathbb{A}^1$-h-cobordant to a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ has a structure of $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. The answer is no and we prove in the following theorem that the suggested example in op. cit. indeed works.

Theorem 1.3. *Let $X := \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, where $\mathcal{E} := \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1_k$ (where k is an infinite field). Then X is $\mathbb{A}^1$-h-cobordant to $\mathbb{P}^1_k \times \mathbb{P}^2_k$ but doesn't have the structure of a $\mathbb{P}^1_k$-bundle over $\mathbb{P}^2_k$.*

2. Classification of vector bundles on a curve up to $\mathbb{A}^1$-concordance

In this section, we classify vector bundles on a curve up to $\mathbb{A}^1$-concordance (Theorem 2.5) and obtain the proof of Theorem 1.1 as a consequence of that. Recall the following definition from [2].

Definition 2.1. [2, Definition 5.1] Let X be a scheme over a field k. Then two given vector bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ on X are said to be directly $\mathbb{A}^1$-concordant if there exists a vector bundle $\mathcal{E}$ on $X \times \mathbb{A}^1$ such that $i_0^*\mathcal{E} \cong \mathcal{E}_0$ and $i_1^*\mathcal{E} \cong \mathcal{E}_1$, where $i_k : X \times \{k\} \hookrightarrow X \times \mathbb{A}^1$, for $k = 0,1$. $\mathcal{E}_0$ and $\mathcal{E}_1$ on X are $\mathbb{A}^1$-concordant if they are equivalent under the equivalence relation generated by direct $\mathbb{A}^1$-concordance.

Lemma 2.2. *Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be $\mathbb{A}^1$-concordant vector bundles on a normal variety X, and let $\mathcal{V}$ be a vector bundle on $X \times \mathbb{A}^1$. Then $(i_0^*(\mathcal{V}) \otimes \mathcal{L}) \oplus \mathcal{E}_0$ and $(i_1^*(\mathcal{V}) \otimes \mathcal{L}) \oplus \mathcal{E}_1$ are $\mathbb{A}^1$-concordant, for any $\mathcal{L} \in \mathrm{Pic}(X)$.*

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Proof. It is enough to prove the lemma in the case when $\mathcal{E}_0$ and $\mathcal{E}_1$ are directly $\mathbb{A}^1$-concordant. Let the direct $\mathbb{A}^1$-concordance be given by a vector bundle $\mathcal{E}$ on $X \times \mathbb{A}^1$. Note that $p^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X \times \mathbb{A}^1)$ (where $p : X \times \mathbb{A}^1 \to X$) is an isomorphism (see [5, II, Prop. 6.6]) with the inverse given by $i_0^* = i_1^*$. Then the lemma immediately follows from the definition by considering the vector bundle $(\mathcal{V} \otimes p^*\mathcal{L}) \oplus \mathcal{E}$ and the fact that the pullback functor commutes with the direct sums. □

In light of the previous lemma, the following corollary is rather obvious, but we state it nevertheless, keeping in mind its direct application in the proof of Theorem 1.1.

Corollary 2.3. *Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be $\mathbb{A}^1$-concordant vector bundles on a normal variety X Then the following statements hold:*

(1) *$\mathcal{O}_X^n \oplus \mathcal{E}_0$ and $\mathcal{O}_X^n \oplus \mathcal{E}_1$ are $\mathbb{A}^1$-concordant for any $n \geq 0$.*
(2) *$\mathcal{O}_X(m) \oplus \mathcal{E}_0$ and $\mathcal{O}_X(m) \oplus \mathcal{E}_1$ are $\mathbb{A}^1$-concordant for any m.*

Proof. For the first statement, take $\mathcal{V} = \mathcal{O}^n_{X \times \mathbb{A}^1}$, keeping the notation of the previous lemma in mind.

For the second statement, take $\mathcal{V} = \mathcal{O}_{X \times \mathbb{A}^1}$ and $\mathcal{L} = p^*\mathcal{O}_X(m)$. □

2.1. $\mathbb{A}^1$-concordance via Ext classes

Now we look at a way of constructing $\mathbb{A}^1$-concordance between vector bundles.

Proposition 2.4. *Let $0 \to \mathcal{E}_0 \to \mathcal{E} \to \mathcal{E}_1 \to 0$ be any short exact sequence of vector bundles on a projective scheme X over a field k. Then $\mathcal{E}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{E}_0 \oplus \mathcal{E}_1$.*

Proof. Consider $\mathcal{E}$ as an element in $\mathrm{Ext}^1(\mathcal{E}_1,\mathcal{E}_0)$. If $\mathcal{E}$ is trivial, then our claim is obvious, so assume to the contrary. Consider the moduli functor $\mathbf{Ext^1}(\mathcal{E}_1,\mathcal{E}_0)$ given by $Y \mapsto \mathrm{Ext}^1(p^*\mathcal{E}_1,p^*\mathcal{E}_0)$, where $p : X \times Y \to X$. It is well-known ([7, Proposition 3.1]) that this functor is representable by $\mathbb{A}^n_k$, where $n = \dim(\mathrm{Ext}^1(\mathcal{E}_1,\mathcal{E}_0))$ as a vector space over k and $n > 0$ by the assumption that $\mathcal{E}$ is nontrivial. Therefore, by representability, there is a universal class $\mathcal{V}$ (of vector bundle) on $X \times \mathbb{A}^n_k$ whose pullback to $X \times t_i \hookrightarrow X \times \mathbb{A}^n_k$, $i = 0,1$ is $\mathcal{E}$ and $\mathcal{E}_0 \oplus \mathcal{E}_1$ respectively for some k-rational points $t_i \in \mathbb{A}^n_k$. For any two given k-rational points in $\mathbb{A}^n_k$ (in our case t_0 and t_1), there is a closed embedding $i : \mathbb{A}^1_k \hookrightarrow \mathbb{A}^n_k$ such that the composition $\mathrm{Spec}\, k \xrightarrow{0} \mathbb{A}^1_k \xrightarrow{i} \mathbb{A}^n_k$ is t_0 and the composition $\mathrm{Spec}\, k \xrightarrow{1} \mathbb{A}^1_k \xrightarrow{i} \mathbb{A}^n_k$ is t_1. Now consider the pullback of the universal class, $(id_X \times i)^*\mathcal{V}$, via the map $id_X \times i : X \times \mathbb{A}^1_k \to X \times \mathbb{A}^n_k$. By construction, the vector bundle $(id_X \times i)^*\mathcal{V}$ on $X \times \mathbb{A}^1_k$ gives a direct $\mathbb{A}^1$-concordance between $\mathcal{E}$ and $\mathcal{E}_0 \oplus \mathcal{E}_1$ □

2.2. Classification result and proof of Theorem 1.1

Theorem 2.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be rank n vector bundles on the curve C. Then the following hold:*

(1) *$\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}_C^{n-1} \oplus \det(\mathcal{E})$.*
(2) *$\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{F}$ iff $\det(\mathcal{E}) \cong \det(\mathcal{F})$.*

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Proof. We first prove (1) for the case when $n=2$. For the general case we will use induction.

Case 1: $n=2$. First, assume $\mathcal{E}$ is globally generated and denote $\det(\mathcal{E})$ by $\mathcal{L}$. Then, by [5, II, Exercise 8.2], we have the following short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{E}' \to 0 \tag{2.1}$$

where $\mathcal{E}'$ is a line bundle. By the Whitney sum formula of Chern classes ([4, Theorem 5.3(c)])

$$c_1(\mathcal{L}) = c_1(\mathcal{E}) = c_1(\mathcal{O}_C) + c_1(\mathcal{E}') = c_1(\mathcal{E}').$$

Therefore, by Proposition 2.4, $\mathcal{E}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C \oplus \mathcal{L}$. For a general $\mathcal{E}$, choose $m >> 0$ such that $\mathcal{E}(m)$, $\mathcal{L}(m)$ are globally generated. Then again, by applying [5, II, Exercise 8.2], we get a short exact sequence for $\mathcal{E}(m)$ which we tensor by $\mathcal{O}(-m)$ to obtain the following short exact sequence.

$$0 \to \mathcal{O}_C(-m) \to \mathcal{E} \to \mathcal{L}(m) \to 0 \tag{2.2}$$

This proves $\mathcal{E}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C(-m) \oplus \mathcal{L}(m)$. As the final step, we now prove that $\mathcal{O}_C(-m) \oplus \mathcal{L}(m)$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C \oplus \mathcal{L}$. Note that m is chosen such that $\mathcal{L}(m)$ is globally generated, therefore $\mathcal{O}_C(m) \oplus \mathcal{L}(m)$ is globally generated. Hence, we have a short exact sequence which shows $\mathcal{O}_C(-m) \oplus \mathcal{L}(m)$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C \oplus \mathcal{L}$.

$$0 \to \mathcal{O}_C(-m) \to \mathcal{O}_C \oplus \mathcal{L} \to \mathcal{L}(m) \to 0 \tag{2.3}$$

Therefore, $\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}_C \oplus \mathcal{L}$.

Case 2: Now we handle the general case. So assume $n > 2$ and choose m such that $\mathcal{E}(m)$ and $\mathcal{L}(m)$ are globally generated. Then we have a short exact sequence giving a direct $\mathbb{A}^1$-concordance between $\mathcal{E}$ and $\mathcal{O}_C(-m) \oplus \mathcal{E}'$, where $\mathcal{E}'$ is a vector bundle of rank $n-1$ with determinant $\mathcal{L}(m)$. By induction, $\mathcal{E}'$ is $\mathbb{A}^1$-concordant to $\mathcal{O}_C^{n-2} \oplus \mathcal{L}(m)$. Therefore, by the second statement of Corollary 2.3, we have an $\mathbb{A}^1$-concordance between $\mathcal{O}_C(-m) \oplus \mathcal{O}_C^{n-2} \oplus \mathcal{L}(m)$ and $\mathcal{O}_C(-m) \oplus \mathcal{E}'$. Hence, $\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}(-m) \oplus \mathcal{O}_C^{n-2} \oplus \mathcal{L}(m)$. Now by the short exact sequence 2.3, $\mathcal{O}_C \oplus \mathcal{L}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C(-m) \oplus \mathcal{L}(m)$, which implies, by the first statement of Corollary 2.3, that $\mathcal{O}(-m) \oplus \mathcal{O}_C^{n-2} \oplus \mathcal{L}(m)$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C^{n-1} \oplus \mathcal{L}$, thus finishing the proof of (1).

For proving (2), we first observe that if $\det(\mathcal{E}) \cong \det(\mathcal{F})$, then (1) implies that $\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{F}$. Hence, it remains to show that if $\mathcal{E}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{F}$, then $\det(\mathcal{E}) \cong \det(\mathcal{F})$.

So assume that $\mathcal{E}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{F}$, which by definition gives us a vector bundle $\mathcal{E}'$ on $C \times \mathbb{A}^1$ such that $i_0^*\mathcal{E}' \cong \mathcal{E}$ and $i_1^*\mathcal{E}' \cong \mathcal{F}$. We have $c_1(\det\mathcal{E}) = c_1(i_0^*\mathcal{E}') = i_0^*(c_1(\mathcal{E}'))$, where the first equality follows from the isomorphism $i_0^*\mathcal{E}' \cong \mathcal{E}$ and the fact that for any vector bundle $\mathcal{V}$, $c_1(\mathcal{V}) = c_1(\det\mathcal{V})$, while the second equality is the functoriality of Chern classes ([4, Theorem 5.3(d)]). Similarly, we have $c_1(\det\mathcal{F}) = c_1(i_1^*\mathcal{E}') = i_1^*(c_1(\mathcal{E}'))$. Moreover, $(i_0)^* = (i_1)^* : \mathrm{CH}^1(C \times \mathbb{A}^1) \to \mathrm{CH}^1(C)$, which implies $c_1(\det\mathcal{E}) = c_1(\det\mathcal{F})$ Therefore, $\det(\mathcal{E}) \cong \det(\mathcal{F})$. $\square$

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Before we proceed with the proof of Theorem 1.1, we recall some standard definitions.

Definition 2.6. Let $\mathcal{X}$ be a simplicial sheaf and U a scheme. Then x and y in $\mathcal{X}(U)$ are said to be naively $\mathbb{A}^1$-homotopic if there exists $f : \mathbb{A}^1_U \to \mathcal{X}$ such that $f_0 = x$ and $f_1 = y$, where f_i is the composition $U \xrightarrow{i} \mathbb{A}^1_U \xrightarrow{f} \mathcal{X}$, for $i = 0,1$.

Definition 2.7. For a given simplicial sheaf $\mathcal{X}$, we define $\mathcal{S}(\mathcal{X})$ to be the Nisnevich sheafification of the presheaf $U \mapsto \mathcal{X}(U)/\sim$, where $\sim$ is the equivalence relation generated by naive $\mathbb{A}^1$-homotopies.

The following standard lemma will be required in our proof. It is essentially [11, Section 2, Corollary 3.22] combined with [9, Lemma 6.1.3.],

Lemma 2.8. *A simplicial sheaf $\mathcal{X}$ is $\mathbb{A}^1$-connected if $\mathcal{S}(\mathcal{X})(F) = *$ for every finitely generated field extension F over k.*

Now we have all the ingredients in place to prove Theorem 1.1.

Proof of Theorem 1.1. We regard $Bun_{n,\mathcal{L}}$ as a simplicial sheaf. By definition, any two F-valued points of $Bun_{n,\mathcal{L}}$ are two rank n (with determinant condition) vector bundles, say $\mathcal{E}_0$ and $\mathcal{E}_1$ on C. A morphism $\mathbb{A}^1_F \to Bun_{n,\mathcal{L}}$ is a vector bundle $\mathcal{E}$ on $C \times \mathbb{A}^1$. Then $\mathcal{E}_0$ and $\mathcal{E}_1$ are naively $\mathbb{A}^1$-homotopic if and only if they are $\mathbb{A}^1$-concordant. By Theorem 2.5, both $\mathcal{E}_0$ and $\mathcal{E}_1$ are $\mathbb{A}^1$-concordant to $\mathcal{O}_C^{n-1} \oplus \mathcal{L}$. Hence, they are $\mathbb{A}^1$-concordant to each other. Therefore, by Lemma 2.8, $Bun_{n,\mathcal{L}}$ is $\mathbb{A}^1$-connected. $\square$

Motivated by the question of $\mathbb{A}^1$-connectedness of moduli stack of stable vector bundles, we observe in the example below that there does not seem to be an immediate way of concluding $\mathbb{A}^1$-connectedness of a stack by looking at its coarse moduli space.

Example 2.9. Let C be a curve of genus 2 over a field with characteristics not equal to 2. In particular, it is a hyperelliptic curve (See [5, IV, Exercise 1.7(a)]). Therefore, there is a finite morphism $f : C \to \mathbb{P}^1$ of degree 2, and we have an action of the finite group $G := \mathbb{Z}/2\mathbb{Z}$ on C. By [5, IV, Exercise 2.2(a)], such a morphism is unramified at all but 6 points (denoted as closed subscheme Z') of C. So the action of G is free on $C \setminus Z'$. Let Z denote the closed subset in $\mathbb{P}^1$ corresponding to the 6 branched points. The quotient stack $[C/G]$ has coarse moduli space $\mathbb{P}^1$, and the morphism $\pi : [C/G] \to \mathbb{P}^1$ gives an isomorphism of an open subscheme of $[C/G]$ with $\mathbb{P}^1 \setminus Z$. See [12, Example 8.1.12] for more details on quotient stacks.

Let $E(G)$ denote the simplicially contractible, simplicial sheaf with $E(G)_n = G^{n+1}$ (See [11, Example 1.11, page 128]). The morphism $C \to [C/G]$ is a G-torsor. Moreover, G acts freely on the space $E(G) \times C$ and $(E(G) \times C)/G \simeq [C/G]$. $\pi_0^{\mathbb{A}^1}(G) \cong \mathbb{Z}/2\mathbb{Z}$, being a finite abelian group, is a strictly $\mathbb{A}^1$-invariant sheaf. So all the hypotheses of the statement of [10, Theorem 6.50] are satisfied, as a consequence of which we obtain the following long exact sequence of $\mathbb{A}^1$-homotopy groups/pointed sets, where $*$ is a chosen basepoint.

$$\cdots \to \pi_0^{\mathbb{A}^1}(G,*) \to \pi_0^{\mathbb{A}^1}(E(G) \times C,*) \to \pi_0^{\mathbb{A}^1}([C/G],*) \to *$$

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But on the account of $E(G)$ being simplicially contractible and C being $\mathbb{A}^1$-rigid (as all curves of genus $g > 0$ are), $\pi_0^{\mathbb{A}^1}(E(G)\times C) \cong \pi_0^{\mathbb{A}^1}(C) \cong C$. So by long exact sequence, $[C/G]$ being $\mathbb{A}^1$-connected would imply surjection of finite group $\mathbb{Z}/2\mathbb{Z}$ on $C \cong \pi_0^{\mathbb{A}^1}(C)$, which cannot happen.

Remark 2.10. By definition, any hyperelliptic curve of genus g admits a finite map of degree 2 to $\mathbb{P}^1$. By Hurwitz's theorem, such a morphism has $2g+2$ ramified points. Therefore, Example 2.9 can be generalised to an hyperelliptic curve of any genus (which is necessarily greater than 1).

3. Applications

As applications of the results in the previous section, we give a proof of Theorem 1.2 and Theorem 1.3. We first recall the following definition from [3].

Definition 3.1. [3, Definition 3.1.1] Let X_0 and X_1 be smooth and proper varieties over k. They are directly $\mathbb{A}^1$-h- cobordant if there exists a smooth scheme X with $f: X \to \mathbb{A}^1$ a proper surjective morphism such that

(1) the fibers of f over 0 and 1 are X_0 and X_1 respectively
(2) the natural maps $X_i \hookrightarrow X$ for $i = 0,1$ are $\mathbb{A}^1$-weak equivalences.

X_0 and X_1 are $\mathbb{A}^1$-h-cobordant if they are equivalent under the equivalence relation generated by direct $\mathbb{A}^1$-h-cobordance.

While $\mathbb{A}^1$-concordance is a relation between vector bundles, $\mathbb{A}^1$-h-cobordism a relation between proper schemes. Note that by [2, Lemma 6.4], projectivizations of $\mathbb{A}^1$-concordant vector bundles are $\mathbb{A}^1$-h-cobordant.

Recall that, given a locally free sheaf $\mathcal{E}$ of rank $n+1$ on a scheme X, the associated $\mathbb{P}^n$-bundle, denoted $\mathbb{P}_X(\mathcal{E})$, is the scheme $\mathbf{Proj}_X(\mathrm{Sym}(\mathcal{E}))$. Here, $\mathbf{Proj}$ is the relative proj construction and $\mathrm{Sym}(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ as an $\mathcal{O}_X$-module. See [5, II, page 162] for more details.

We now paraphrase the classification of $\mathbb{P}^n$-bundles on $\mathbb{P}^1$ up to $\mathbb{A}^1$-weak equivalence proved in [3] to highlight that Theorem 1.2 is its direct generalisation to an arbitrary smooth projective curve.

Proposition 3.2. [3, Proposition 3.2.10] *Let $X := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^n \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ and $Y := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^n \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ be two $\mathbb{P}^n$-bundles over $\mathbb{P}^1$. Then the following statements are equivalent:*

(1) *X and Y are $\mathbb{A}^1$- weakly equivalent.*
(2) *X and Y are $\mathbb{A}^1$-h-cobordant.*
(3) *$n+1$ divides $a-b$.*

Note that in case of $C = \mathbb{P}^1$, the condition $\det(\mathcal{E}) \otimes \det(\mathcal{F})^{-1} = \mathcal{L}^{\otimes n+1}$ in Theorem 1.2 exactly translates to the fact that $n+1$ divides $a-b$ as stated in the Proposition 3.2. This is due to $\mathrm{Pic}(\mathbb{P}^1)$ being isomorphic to $\mathbb{Z}$. For a general curve, Picard group is much more complicated and humongous (think of the Jacobian variety of a curve), so one doesn't get any further simplification. We now prove Theorem 1.2, which is an extension of the previous proposition.

Proof of Theorem 1.2. (3) $\implies$ (2): By Theorem 2.5, $\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}_C^n \oplus \mathcal{L}_1$, where $\mathcal{L}_1 = \det(\mathcal{E})$ and $\mathcal{F}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}_C^n \oplus \mathcal{L}_2$, where $\mathcal{L}_2 = \det(\mathcal{F})$. Hence, X and $\mathbb{P}_C(\mathcal{O}_C^n \oplus \mathcal{L}_1)$ are $\mathbb{A}^1$-h-cobordant. In the exact same manner, Y and $\mathbb{P}_C(\mathcal{O}_C^n \oplus \mathcal{L}_2)$ are $\mathbb{A}^1$-h-cobordant. Suppose $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} = \mathcal{L}^{\otimes n+1}$. That implies $\mathcal{L}_1 = \mathcal{L}^{\otimes n+1} \otimes \mathcal{L}_2$ for some $\mathcal{L} \in \mathrm{Pic}(C)$. Let $\mathcal{E}' = (\mathcal{O}_C^n \oplus \mathcal{L}_2) \otimes \mathcal{L}$. Then $\det(\mathcal{E}') = \mathcal{L}_1$. Therefore, $\mathbb{P}(\mathcal{E}')$ is $\mathbb{A}^1$-h-cobordant to $\mathbb{P}(\mathcal{O}_C^n \oplus \mathcal{L}_1)$. Furthermore, $\mathbb{P}(\mathcal{E}')$ is isomorphic (as a scheme) to $\mathbb{P}(\mathcal{O}_C^n \oplus \mathcal{L}_2)$ by the general fact that tensoring a vector bundle by a line bundle gives an isomorphism of projectivization of the two vector bundles. This proves X and Y are $\mathbb{A}^1$-h-cobordant.

(2) $\implies$ (1): this is immediate from the definition of $\mathbb{A}^1$-h-cobordism.

(1) $\implies$ (3): $\mathbb{A}^1$-invariance of the Chow rings implies that it is enough to show that the Chow rings of $\mathbb{P}(\mathcal{O}_C^n \oplus \mathcal{L}_1)$ and $\mathbb{P}(\mathcal{O}_C^n \oplus \mathcal{L}_2)$ are not isomorphic if $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \neq \mathcal{L}^{\otimes n+1}$ for any $\mathcal{L} \in \mathrm{Pic}(C)$. The Chow ring of C – which is simply $\mathbb{Z} \oplus \mathrm{Pic}(C)$, with product of any two line bundles under the ring structure being zero – is denoted R. For simplicity of notation, we will denote $\mathcal{O}_C^n \oplus \mathcal{L}_i$, $i = 1,2$ by $\mathcal{E}_i$. Then by the projective bundle formula for the Chow rings [4, Theorem 9.6], the Chow ring of $\mathbb{P}(\mathcal{E}_1)$ is $R_1 := R[\zeta]/(\zeta^{n+1} + c_1(\mathcal{E}_1)\zeta^n)$. But $c_1(\mathcal{E}_1) = c_1(\mathcal{L}_1)$. In the ring R_1, ζ as well as any element $x \in \mathrm{Pic}(C)$ has grading 1 with $xy = 0$ for $x,y \in \mathrm{Pic}(C)$. Let's assume we have a graded ring isomorphism ϕ between R_1 and $R_2 := R[\sigma]/(\sigma^{n+1} + c_1(\mathcal{L}_2)\sigma^n)$. Then such an isomorphism has to respect the grading and hence $\phi(\zeta) = x + a\sigma$, where $x \in \mathrm{Pic}(C)$ and $a \in \mathbb{Z}$. Similarly, $\phi^{-1}(\sigma) = y + b\zeta$, where $b \in \mathbb{Z}$ and $y \in \mathrm{Pic}(C)$. We first prove that $a = \pm 1$. The condition $\phi^{-1} \circ \phi(\zeta) = \zeta$ implies that $x + ay + ab\zeta = \zeta$. Hence, $ab = 1$, so $a = \pm 1$.

By the graded ring structure of R_1, as discussed before, $x^i = 0$ for any $i > 1$. Moreover, $\phi(\zeta^{n+1} + c_1(\mathcal{L}_1)\zeta^n)$ has to be divisible by $\sigma^{n+1} + c_1(\mathcal{L}_2)\sigma^n$ in R_2. First assume $a = 1$. Proof for the case $a = -1$ is similar. We expand $\phi(\zeta^{n+1} + c_1(\mathcal{L}_1)\zeta^n)$ as $\sigma^{n+1} + \sigma^n((n+1)x + c_1(\mathcal{L}_1))$ and this expression is divisible by $\sigma^{n+1} + c_1(\mathcal{L}_2)\sigma^n$. Comparing coefficients, we conclude that $c_1(\mathcal{L}_1) - c_1(\mathcal{L}_2) = (n+1)x$. This implies that $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} = \mathcal{L}^{\otimes n+1}$, where $c_1(\mathcal{L}) = x$. □

Now, we answer a question raised in [2] negatively.

Question 3.3. [2, Question 6.9.1] *If X is any smooth projective variety that is $\mathbb{A}^1$-h-cobordant to a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, does X have the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$?*

The authors further add the answer is possibly no, and nontrivial rank three vector bundles over $\mathbb{P}^1$ deformable to the trivial one are the likely counterexamples. We now prove Theorem 1.3 which shows that the example alluded to above is indeed a correct counterexample.

Proof of Theorem 1.3. By Theorem 2.5, $X := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}^1$ is $\mathbb{A}^1$-h-cobordant to the trivial $\mathbb{P}^2$- bundle on $\mathbb{P}^1$, namely, $\mathbb{P}^1 \times \mathbb{P}^2$. However, X and $\mathbb{P}^1 \times \mathbb{P}^2$ are not isomorphic as schemes. By [5, II, Exercise 7.9(b)], an isomorphism would imply that for some line bundle on $\mathbb{P}^1$, say $\mathcal{O}(a)$ where $a \in \mathbb{Z}$, $\mathcal{O}(a) \otimes \mathcal{E} \simeq \mathcal{O}(a) \oplus \mathcal{O}(a-1) \oplus \mathcal{O}(a+1) \simeq \mathcal{O}^{\oplus 3}$ is an isomorphism of vector bundles on $\mathbb{P}^1$, which cannot happen.

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Now suppose $X \xrightarrow{\theta} \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}') := Y \xrightarrow{\phi} \mathbb{P}^2$, with θ an isomorphism of schemes, for some rank 2 vector bundle $\mathcal{E}'$ on $\mathbb{P}^2$. We thus have the following diagram

$$\begin{tikzcd} X \simeq Y \arrow[r, "\phi"] \arrow[d, "\pi"] & \mathbb{P}^2 \\ \mathbb{P}^1 & \end{tikzcd}$$

Without loss of generality we can assume (by twisting $\mathcal{E}'$ with a suitable line bundle in $\mathrm{Pic}(\mathbb{P}^2)$ as $c_1(\mathcal{E}' \otimes \mathcal{L}) = c_1(\mathcal{E}') + 2c_1(\mathcal{L})$), $c_1(\mathcal{E}') \in \{0,1\}$. Since Y is $\mathbb{A}^1$-weakly equivalent to the trivial bundle on $\mathbb{P}^2$, their Chow rings are isomorphic. By [2, Lemma 4.5], we have $c_1(\mathcal{E}')^2 - 4c_2(\mathcal{E}') = 0$. So $c_1(\mathcal{E}') = 0 = c_2(\mathcal{E}')$. It thus suffices to show that $\mathcal{E}'$ splits as a direct sum of line bundles as this will prove that $\mathcal{E}' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$. By the assumption that $X \simeq Y$, this will imply that X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$, which from the discussion in the first paragraph of this proof cannot happen.

We will prove that ϕ has a section. Such a section will give the following short exact sequence.

$$0 \to \mathcal{L}_1 \to \mathcal{E}' \to \mathcal{L}_2 \to 0 \tag{3.1}$$

As both Chern classes of $\mathcal{E}'$ vanish, by the Whitney sum formula of Chern classes, both $\mathcal{L}_1$ and $\mathcal{L}_2$ will be trivial. Therefore such a short exact sequence has to be a split one. This will prove $\mathcal{E}' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$.

Define $F \hookrightarrow Y$ to be $\theta \circ \pi^{-1}(z)$ for a point $z \in \mathbb{P}^1$. By construction, $F \simeq \mathbb{P}^2$. We claim ϕ maps F isomorphically onto $\mathbb{P}^2$. First we claim that $\phi_{|F}$ is surjective. If not, then $Z := \phi(F)$ is either a point or an irreducible curve (not necessarily smooth) in $\mathbb{P}^2$. Since $\phi : Y \to \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle map, the fiber of ϕ over each point of $\mathbb{P}^2$ is $\mathbb{P}^1$. Therefore, Z cannot be a point. So assume Z is an irreducible curve in $\mathbb{P}^2$. Consider smooth points $z_1 \neq z_2 \in Z$. Then using flatness of ϕ, we have $\phi^{-1}(z_i) \simeq \mathbb{P}^1 \subset F$ for $i = 1,2$. However, any two lines in $\mathbb{P}^2$ intersect, so $\phi^{-1}(z_1)$ and $\phi^{-1}(z_i)$ intersect in F(which is isomorphic to $\mathbb{P}^2$). This contradicts our assumption that $z_1 \neq z_2$. This establishes the surjectivity of $\phi_{|F}$. We also conclude $\phi|_F$ is a degree d morphism to $\mathbb{P}^2$ with $d \geq 1$.

We now show that $d = 1$. This is achieved by comparing the graded ring isomorphism induced on the Chow rings of X and Y. The Chow ring of X is $R_1 := \mathbb{Z}[x,y]/(x^2,y^3)$, where

(i) x is the divisor $\mathbb{P}^2$ as a fiber over a point of $\mathbb{P}^1$

(ii) y corresponds to a divisor D', such that the pushforward $\pi_*(\mathcal{O}_X(D'))$ to $\mathbb{P}^1$ is the vector bundle $\mathcal{E}$.

Similarly, the Chow ring of Y is $R_2 := \mathbb{Z}[s,t]/(s^2,t^3)$ where

(i) t corresponds to fiber of $\mathbb{P}^1$ (as a degree 1 curve in $\mathbb{P}^2$) via ϕ

(ii) s corresponds to a divisor D, such that the pushforward $\phi_*(\mathcal{O}_Y(D))$ to $\mathbb{P}^2$ is the rank two vector bundle $\mathcal{E}'$.

Let $\psi : R_1 \to R_2$ be an isomorphism of graded rings. Then $\psi(x) = as + bt$, where $a,b \in \mathbb{Z}$. We have the condition that $\psi(x^2) = \psi(x)^2 = a^2s^2 + 2abst + b^2t^2$ lies in the ideal generated

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by s^2 and t^3. This implies $b=0$. Morever, $\psi^{-1}\circ\psi(x)=x$. Therefore, $a=\pm 1$. In a similar fashion, one proves that any isomorphism between R_1 and R_2 sends y to $\pm t$. Therefore, we conclude that the graded ring isomorphism between R_1 and R_2 is given by $x\mapsto \pm s$ and $y\mapsto \pm t$. This implies s is equivalent (in the Chow ring) to the class of $F\simeq \mathbb{P}^2$. Grauert's theorem ([5, III, Corollary 12.9]) implies that the intersection multiplicity of divisor D corresponding to s (see the description of R_2 above) with any fiber of the map ϕ is 1. As s and F are equivalent in the Chow ring, the same holds for F. This cannot happen unless $d=1$ because if not, one can consider a point z' in $\mathbb{P}^2$ such that the set $\phi|_{\mathbb{P}^2}^{-1}(z')$ has more than one point. This will force $\phi^{-1}(z')=\mathbb{P}^1$ to intersect $\mathbb{P}^2$ in more than one point, meaning an intersection multiplicity of greater than 1 which, as we just proved, cannot happen. This proves $\phi|_F$ is an isomorphism onto $\mathbb{P}^2$ and hence establishes the existence of a section of ϕ. Via the short exact sequence 3.1, this proves that $\mathcal{E}'$ is a trivial rank 2 vector bundle on $\mathbb{P}^2$, and thereby finishes the proof. □

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References

[1] J. Alper, P. Belmans, D. Bragg, J. Liang and T. Tajakka, Projectivity of the moduli space of vector bundles on a curve in *Stacks Project Expository Collection* (Cambridge University Press, 2022).

[2] A. Asok, S. Kebekus and M. Wendt, Comparing A1-h-Cobordism and A1-weak equivalence, *Ann. Sc. Norm. Super. Pisa. Cl. Sci. (2)* XVII (2017), 531–572.

[3] A. Asok and F. Morel, Smooth varieties up to A1-homotopy and algebraic h-cobordisms, *Adv. Math.(5)* **227** (2011), 1990–2058.

[4] D. Eisenbud and J. Harris, *3264 and All That: A Second Course in Algebraic Geometry* (Cambridge University Press, 2016).

[5] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics no.52 (Springer-Verlag, New York, 1977).

[6] S. Hollander, A homotopy theory for stacks, *Israel Journal of Mathematics* **163** (2008), 93–124.

[7] H. Lange, Universal families of extensions, *Journal of Algebra* **83** (1983), 101–112.

[8] A. King and A Schofield, Rationality of moduli of vector bundles on curves, *Indag. Mathem. (N.S) (4)* **10** (1999), 519–535.

[9] F. Morel, The stable A1-connectivity theorems, *K-Theory* **35** (2005), 1–68.

[10] F. Morel, *A1-algebraic Topology Over a Field*, Lecture Notes in Mathematics vol. 2052 (Springer, Heidelberg, 2012).

[11] F. Morel and V. Voevodsky, A1-homotopy theory of schemes, *Inst. Hautes Etudes Sci. Publ. Math.* **90** (1999), 45–143.

[12] M. Olsson, *Algebraic Spaces and Stacks*, Colloquium Publications 62 (American Mathematical Society, 2016).