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COMPOSITION OPERATORS ON Q^p SPACES

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Abstract

A holomorphic map φ of the unit disk into itself induces an operator C_φ on holomorphic functions by composition. We characterize bounded and compact composition operators C_φ on Q^p spaces, which coincide with the $BMOA$ for $p = 1$ and Bloch spaces for $p > 1$. We also give boundedness and compactness characterizations of C_φ from analytic function space X to Q^p spaces, X = Dirichlet space $\mathscr{D}$, Bloch space B or $B^0 = \{f : f' \in H^\infty\}$.

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1. Introduction

First, we introduce some basic notation which is used in this paper. Let D and ∂D be the unit disk and the unit circle in the finite complex plane $\mathbb{C}$, respectively. Also let $dm(z)$ be the Lebesgue measure on D. Denote by $g(z, a) = \log|(1-\bar{a}z)/(a-z)|$ the Green function for D with pole at a. Also denote by H^∞ the set of bounded analytic functions on D.

Let $\varphi : D \to D$ be an analytic self-map of the unit disk D. The composition operator C_φ induced by such φ is the linear map on the space of all analytic functions on the unit disk defined by

$$C_\varphi(f) = f \circ \varphi.$$

A fundamental problem concerning composition operators is to relate function-theoretic properties of φ to operator-theoretic properties of the restrictions of C_φ to various Banach spaces of analytic functions. It is well known that C_φ preserves many

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analytic function spaces such as Hardy spaces, Bergman spaces, Bloch type spaces and *BMOA*. The compactness problem for Hardy-space composition operators was solved in 1987 by Shapiro [Sh]. The boundedness and compactness of composition operators on Bergman spaces and Bloch type spaces were solved by Smith and Yang [SmYa], Madigan and Matheson [MaMa] and Lou [Lo]. Recently, the composition operators on *BMOA* was studied by Tjani [Tj] and Bourdon, Cima and Matheson [BoCiMa]. In [BoCiMa] it was shown that the compactness of composition operators C_φ on *BMOA* is equivalent to a little-oh Carleson measure condition holding uniformly for all functions in the unit ball of *BMOA*; see [BoCiMa, Theorem 3.1]. Motivated by [BoCiMa] and [Tj], in this paper we study the composition operators C_φ on Q^p spaces, for $0 < p < \infty$. Note that $Q^1 = BMOA$ and $Q^p = B$, Bloch space, for $1 < p < \infty$. This paper is organized as follows, in Section 1 and Section 2, introduction and preliminaries are provided. Next, in Section 3, we give the compact characterization of composition operators $C_\varphi : Q^p \to Q^p$ for $0 < p < \infty$ via a Carleson measure condition. In Section 4, we give the bounded and compact characterizations of C_φ from Dirichlet space $\mathscr{D}$ to Q^p and Q_0^p $(0 < p < \infty)$ spaces. In Section 5, we study the boundedness and compactness of C_φ from B^0 to Q^p and Q_0^p $(0 < p < \infty)$, where $B^0 = \{f : f' \in H^\infty\}$. In the final section, we obtain necessary and sufficient conditions for composition operators C_φ to be bounded and compact from Bloch space to Q^p and Q_0^p.

Throughout this paper, the letter C denotes different positive constants which are not necessarily the same from line to line.

2. Preliminaries

2.1. Notations The space Q^p is defined by means of a modified Garcia norm which was introduced by Aulaskari, Xiao and Zhao in 1995 [AuXiZh]. The definition can be given in the following way. For $p \in (-1, \infty)$, we say that $f \in Q^p$ if f is analytic in D and

$$(2.1) \qquad \|f\|_{1(Q^p)} = |f(0)| + \left(\sup_{a\in D} \iint_D |f'(z)|^2 g(z,a)^p \, dm(z)\right)^{1/2} < \infty.$$

It is clear that Q^p is a Banach space relative to the above norm. From [AuStXi], $\|f\|_1$ is equivalent to the following norm on Q^p

$$(2.2) \qquad \|f\|_{2(Q^p)} = |f(0)| + \left(\sup_{a\in D} \iint_D |f'(z)|^2 \left(1 - |\sigma_a(z)|^2\right)^p dm(z)\right)^{1/2} < \infty,$$

where $\sigma_a(z) = (z-a)/(1-\bar{a}z)$. The subspace Q_0^p of Q^p consists of those functions f such that the integral in the display in (2.1) tends to 0 as $|a| \to 1$. Q_0^p is a closed subspace.

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When $p = 1$, Q^p is $BMOA$, which is the space of analytic functions on D that are of bounded mean oscillation on the unit circle ∂D (see [Ba] and [Ga] for more information on $BMOA$). When $p \in (1, \infty)$, it is well known (see [AuLa] for $p > 1$ and [Xi] for $p = 2$) that Q^p coincides with the Bloch space B of functions f analytic in D with

$$\|f\|_B = |f(0)| + \sup_{z \in D} \left(1 - |z|^2\right)|f'(z)| < \infty.$$

From discussion of [AuLa], we know that the Bloch norm $\|f\|_B$ is equivalent to $\|f\|_{1(Q^p)}$ and $\|f\|_{2(Q^p)}$ for $p > 1$. When $p = 0$, Q^p is the classical Dirichlet space $\mathscr{D}$ of functions analytic in D satisfying

$$\|f\|_{\mathscr{D}} = |f(0)| + \left(\int_D |f'(z)|^2 \, dm(z)\right)^{1/2} < \infty,$$

when $p \in (-1, 0)$, Q^p consists of complex constants ([EsXi]).

Also, $Q_0^1 = VMOA$, the subspace of $BMOA$ consisting of functions of vanishing mean oscillation on ∂D ([Ga]), and for $p > 1$, $Q_0^p = B_0$, the little Bloch space of functions f analytic on D for which (see [AuLa] and [Xi])

$$f'(z)\left(1 - |z|^2\right) \to 0, \quad |z| \to 1.$$

It is well known ([AuXiZh]) that for $0 < p_1 < p_2 < 1$,

$$\mathscr{D} \subsetneq Q^{p_1} \subset Q^{p_2} \subsetneq BMOA \subset B.$$

The spaces Q^p, $p \in (0, 1)$ are of independent interest.

2.2. Carleson measure Our characterization of compact composition operators on Q^p involves Carleson type measures.

For $p \in (0, \infty)$ we say that a positive Borel measure μ on D is a bounded p-Carleson measure provided that

$$\sup_{I \subset \partial D} \frac{\mu(S(I))}{|I|^p} < \infty, \tag{2.3}$$

where $S(I)$ means the Carleson square based on I,

$$S(I) = \left\{ z \in D : 1 - \frac{|I|}{2\pi} \le |z| < 1, \ \frac{z}{|z|} \in I \right\}.$$

If

$$\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^p} = 0, \tag{2.4}$$

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then we say that μ is a compact p-Carleson measure.

Let $0 < h < 1, 0 \le \theta \le 2\pi$, and set

$$S(h,\theta) = \left\{z \in D : \left|z - e^{i\theta}\right| < h\right\}.$$

It is easy to see that (2.3) ans (2.4) are equivalent to

$$\sup_{h\in(0,1),\ \theta\in[0,2\pi)} \frac{\mu(S(h,\theta))}{h^p} < \infty, \tag{2.5}$$

and

$$\lim_{h\to 0} \frac{\mu(S(h,\theta))}{h^p} = 0, \tag{2.6}$$

respectively. Observe that $p = 1$ gives the classical Carleson measure and vanishing Carleson measure (see, for example, [Ga] for more information). As Carleson measure (vanishing Carleson measure) can be used to characterize functions in *BMOA* (*VMOA*) (refer to the work of Fefferman, Garcia and Pommerenke [Ba]) bounded p-Carleson measure (compact p-Carleson measure) can be used to characterize functions in Q^p (Q_0^p) for $0 < p < \infty$.

For f analytic in D and $0 < p < \infty$, let μ_f be defined by

$$d\mu_f(z) = |f'(z)|^2\left(1 - |z|^2\right)^p dm(z).$$

For a function $f \in Q^p$, we set

$$\|f\|_{3(Q^p)} = |f(0)| + \left(\sup_I \frac{\mu_f(S(I))}{|I|^p}\right)^{1/2} \tag{2.7}$$

from the discussion in [AuStXi], $\|f\|_{3(Q^p)}$ is a norm of Q^p which is equivalent to the norms $\|f\|_{1(Q^p)}$ and $\|f\|_{2(Q^p)}$ defined by (2.1) and (2.2). For convenience we use $\|\cdot\|_{Q^p}$ to denote all these Q^p norms, even though $\|\cdot\|_{Q^p}$ may have a different meaning at different occurrences.

THEOREM 1 ([Ba, AuStXi]). *Let $p \in (0,\infty)$ and f analytic in D. Then*

(1) *$f \in Q^p$ if and only if $d\mu_f$ is a bounded p-Carleson measure;*
(2) *$f \in Q_0^p$ if and only if $d\mu_f$ is a compact p-Carleson measure.*

From Lemma 1.1 of [AuStXi] and its proof it is easy to show that

THEOREM 2. *Let $\{\mu_b : b \in D\}$ be a collection of positive measures on D. Then, for $0 < p < \infty$,*

(1) $\displaystyle\sup_{\substack{h\in(0,1)\\ \theta\in[0,2\pi),\ b\in D}} \frac{\mu_b(S(h,\theta))}{h^p} < \infty$ *is equivalent to* $\displaystyle\sup_{a,b\in D}\int_D |\sigma_a'(z)|^p\, d\mu_b(z) < \infty$;

(2) $\displaystyle\lim_{h\to 0}\sup_{\substack{\theta\in[0,2\pi)\\ b\in D}} \frac{\mu_b(S(h,\theta))}{h^p} = 0$ *is equivalent to* $\displaystyle\lim_{|a|\to 1}\sup_{b\in D}\int_D |\sigma_a'(z)|^p\, d\mu_b(z) = 0.$

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2.3. Counting function Let φ be analytic in D and denote by $n(\varphi, w)$ the number of roots in D of equation $\varphi(z) = w$, where $w \in \mathbb{C}$. The classical Nevanlinna counting function N_φ for φ was first used to study composition operators on H^2 by Shapiro in [Sh]. In this paper Shapiro also introduced the generalized counting functions for $0 \le p < \infty$ by

$$N_{\varphi,p}(w) = \begin{cases} \sum_{z\in\varphi^{-1}\{w\}} \left[\log(1/|z|)\right]^p, & w \in \varphi(D), \\ 0, & w \in D \setminus \varphi(D) \end{cases}$$

(observe that $N_{\varphi,0}(w) = n(\varphi, w)$), and proved for any positive measurable function on D

$$\int_D (h \circ \varphi)(z)|\varphi'(z)|^2[\log(1/|z|)]^p \, dm(z) = \frac{2^p}{\Gamma(p+1)} \int_D h(w) N_{\varphi,p}(w) \, dm(w).$$

With $\varphi \circ \sigma_a$ replacing φ, we have

$$\int_D (h \circ \varphi)(z)|\varphi'(z)|^2 g(z, a)^p \, dm(z) = \frac{2^p}{\Gamma(p+1)} \int_D h(w) N_{\varphi\circ\sigma_a,p}(w) dm(w). \tag{2.8}$$

Define measure $\mu_{a,p}$ on D by

$$d\mu_{a,p}(w) = N_{\varphi\circ\sigma_a,p}(w) dm(w).$$

From (2.1) and (2.8) we have, for $0 \le p < \infty$

$$\|f \circ \varphi\|_{Q^p} = |f(0)| + \left(\sup_{a\in D} \int_D |f'(w)|^2 \, d\mu_{a,p}(w)\right)^{1/2}. \tag{2.9}$$

3. Composition operators on Q^p

In this section we characterize the compact composition operators on Q^p spaces. Let $D_\delta = \{z \in D : |\varphi(z)| > \delta\}$, $\delta \in (0, 1)$. The characteristic function of D_δ will be denoted by $1_{D_\delta}(z)$. Now we establish the main result of this section.

THEOREM 3.1. *Suppose that* $0 < p < \infty$ *and* φ *is an analytic self-map of* D. *Then the composition operator* C_φ *is compact on* Q^p *if and only if* $\varphi \in Q^p$ *and for every* $\varepsilon > 0$ *there is* δ, $0 < \delta < 1$, *such that*

$$\int_{S(I)} 1_{D_\delta}(z)(1 - |z|^2)^p |f'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) \le \varepsilon |I|^p \tag{3.1}$$

for every arc I *and every* $f \in Q^p$ *with* $\|f\|_{Q^p} \le 1$.

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For the proof of Theorem 3.1 we need the following lemmas.

LEMMA 1. *Let* $X = \mathcal{D}$, *BMOA*, B *or* Q^p. *Then* $C_\varphi : X \to Q^p$ *is a compact operator if and only if for any bounded sequence* (f_n) *in* X *with* $f_n \to 0$ *uniformly on compact subsets of* D *as* $n \to \infty$, $\|C_\varphi f_n\|_{Q^p} \to 0$ *as* $n \to \infty$.

PROOF. From [Zh, page 82] we know that a Bloch function can grow at most as fast as $\log(1/(1-|z|))$:

$$(3.2)\qquad |f_n(z) - f_n(0)| \le C\|f_n\|_B \log\frac{1}{1-|z|} \le C\|f_n\|_{Q^p}\log\frac{1}{1-|z|}.$$

Using [Tj, Lemma 1.10] and (3.2) we only need to prove that the closed unit ball of Q^p is a compact subset of Q^p in the topology of uniform convergence on compact subsets of D.

Let (f_n) be a sequence in the closed unit ball of Q^p, then from (3.2) (f_n) is uniformly bounded on compact subsets of D. By Montel's theorem ([Co, page 137]) there is a subsequence (f_{n_k}) and an analytic function g such that $f_{n_k} \to g$ uniformly on compact subsets of D. We show that $g \in Q^p$:

$$\begin{aligned}\int_D |g'(z)|^2(1-|\sigma_a(z)|^2)^p\,dm(z) &= \int_D \lim_{k\to\infty} |f'_{n_k}(z)|^2(1-|\sigma_a(z)|^2)^p\,dm(z)\\ &\le \liminf_{k\to\infty}\int_D |f'_{n_k}(z)|^2(1-|\sigma_a(z)|^2)^p\,dm(z)\\ &\le \liminf_{k\to\infty}\|f_{n_k}\|^2_{Q^p},\end{aligned}$$

by Fatou's Theorem. This gives $g \in Q^p$. □

LEMMA 2. *Suppose that* $0 < p < \infty$, φ *is an analytic self-map of* D *and* C_φ *is compact on* Q^p. *Then for every* $\varepsilon > 0$ *there is* δ, $0 < \delta < 1$, *such that*

$$(3.3)\qquad \int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p|\varphi'(z)|^2\,dm(z) < \varepsilon|I|^p$$

for all arcs I *on* ∂D.

PROOF. Since C_φ is compact, then for any bounded sequence (f_n) in Q^p, $\|f_n\|_{Q^p} \le C$, converges uniformly to 0 on compact subsets of D, $\|f_n \circ \varphi\|_{Q^p} \to 0$ as $n \to \infty$ by Lemma 1. Set $f_n(z) = z^n$, since z^n is norm bounded in Q^p and converges uniformly to 0 on compact subsets of D, we have $\|\varphi^n\|_{Q^p} \to 0$, as $n \to \infty$. So, given $\varepsilon > 0$, there is an integer $N > 0$ such that if $n \ge N$,

$$n^2\int_{S(I)} |\varphi(z)|^{2n-2}|\varphi'(z)|^2(1-|z|^2)^p\,dm(z) < \varepsilon|I|^p$$

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for all I on ∂D. Given δ, $0 < \delta < 1$, we have

$$N^2\delta^{2N-2}\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p|\varphi'(z)|^2\,dm(z)$$
$$\leq N^2\int_{S(I)} |\varphi(z)|^{2N-2}|\varphi'(z)|^2(1-|z|^2)^p\,dm(z) < \varepsilon|I|^p$$

for all I on ∂D, since $|\varphi(z)| > \delta$ on D_δ. Choosing δ so that $N^2\delta^{2N-2} = 1$, we have (3.3) and the lemma is proved. □

REMARK 1. Sufficiency of Lemma 2 does not hold for $1 \leq p < \infty$. For $p = 1$, we consider univalent function $\varphi_1(z) = 1-(1-z)^{1/2}$ ([BoCiMa]), we know that $\varphi_1 \in BMOA$ and C_{φ_1} is not compact on $BMOA$, however we will show that this φ_1 satisfies (3.3).

Since $\varphi_1'(z) = (1/2)(1-z)^{-1/2}$, then $|\varphi_1'(z)|^2 \leq 1/4(1-|z|)$. Thus

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)|\varphi_1'(z)|^2\,dm(z) \leq \frac{1}{2}\int_{S(I)} 1_{D_\delta}(z)\,dm(z)$$
$$\leq \frac{1}{2}\int_{\theta}^{\theta+|I|}\int_{1-|I|/2\pi}^{1} 1_{D_\delta}(z)\,dm(z).$$

Since $|D_\delta| \to 0$ as $\delta \to 1$, then for every $\varepsilon > 0$ there exists $0 < \delta < 1$ such that $|D_\delta| < \varepsilon$, So

$$\int_{1-|I|/2\pi}^{1} 1_{D_\delta}(z)(re^{i\theta})\,dr \leq |D_\delta| < \varepsilon.$$

Thus

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)|\varphi_1'(z)|^2\,dm(z) \leq \varepsilon\frac{|I|}{2}$$

for all arcs I on ∂D.

For $1 < p < \infty$, $Q^p = B$, we consider function $\varphi_2 = 1-(1/2)(1-z)^{1/2}$, $\varphi_2 \in B_0 \subset B$ and C_{φ_2} is not compact on B (since C_{φ_2} is not compact on B_0 [MaMa]), but with a similar proof as above we can show that φ_2 also satisfies (3.3).

From Remark 1, (3.3) is not sufficient for the compactness of C_φ on Q^p ($1 \leq p < \infty$). In Section 5 we show that (3.3) is not only necessary but also sufficient for C_φ to be compact from a subspace of Q^p to Q^p for $0 < p < \infty$.

PROOF OF THEOREM 3.1. Sufficiency. Suppose that (3.1) holds, we prove that C_φ is compact. Let $\{f_n\} \subset Q^p$ such that $\|f_n\|_{Q^p} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of D as $n \to \infty$. By Lemma 1 we need show that

$$\lim_{n\to\infty} \|C_\varphi(f_n)\|_{Q^p} = 0.$$

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Fix $\varepsilon > 0$ and let δ $(0 < \delta < 1)$ be such that (3.1) holds. Since $\varphi(D \setminus D_\delta)$ is a relatively compact subset of D, $f_n' \circ \varphi$ converges uniformly to 0 on $D \setminus D_\delta$, then there is an integer $N > 0$ such that $|f_n' \circ \varphi|^2 < \varepsilon$ if $n \geq N$ and $z \in D \setminus D_\delta$. So for all $n \geq N$ and I on ∂D

$$\int_{S(I)} 1_{D\setminus D_\delta}(z)(1-|z|^2)^p |f_n'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) \leq \varepsilon \|\varphi\|_{Q^p}^2 |I|^p.$$

From (3.1), for all n and I on ∂D

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |f_n'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) \leq \varepsilon |I|^p.$$

Hence, for $n \geq N$ we obtain

$$\int_{S(I)} (1-|z|^2)^p |f_n'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) \leq (\|\varphi\|_{Q^p}^2 + 1)\varepsilon |I|^p.$$

Since $f_n \circ \varphi(0) \to 0$ as $n \to \infty$, then by (2.7)

$$\|f_n \circ \varphi\|_{Q^p} \to 0, \quad n \to \infty.$$

Necessity. Suppose that C_φ is compact on Q^p, then $C_\varphi(f) \in Q^p$ for all $f \in Q^p$. Set $f = z$, we get $\varphi \in Q^p$. Let $f_s(z) = f(sz)$ for $s \in (0, 1)$, then $f_s \to f$ uniformly on compact subsets of D as $s \to 1$ and the family $\{f_s : 0 < s < 1\}$ is bounded in Q^p, So

$$\|f_s \circ \varphi - f \circ \varphi\|_{Q^p} \to 0, \quad s \to 1.$$

Thus for each $\varepsilon > 0$ there is s, $0 < s < 1$, such that

$$\int_{S(I)} (1-|z|^2)^p |f_s'(\varphi(z)) - f'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) < \frac{\varepsilon}{4} |I|^p. \tag{3.4}$$

Since f_s is analytic on the closed unit disk, then $\sup_s \|f_s'\|_\infty < \infty$, where $\|\cdot\|_\infty = \sup_D |f(\cdot)|$. From Lemma 2, for $\varepsilon/(4\sup_s \|f_s'\|_\infty^2) > 0$, there exists $\delta = \delta(\varepsilon, f) > 0$ such that

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |\varphi'(z)|^2 \, dm(z) \leq \frac{\varepsilon}{4\sup_s \|f_s'\|_\infty^2} |I|^p.$$

So

$$\begin{aligned} &\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |f_s'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) \\ &\leq \|f_s'\|_\infty^2 \int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |\varphi'(z)|^2 \, dm(z) < \frac{\varepsilon}{4} |I|^p. \end{aligned} \tag{3.5}$$

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From (3.4), (3.5) and applying the triangle inequality, for $\varepsilon > 0$ and $f \in Q^p$, there exists $\delta = \delta(\varepsilon, f)$ such that

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |f'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) < \varepsilon |I|^p \tag{3.6}$$

for every arc I on ∂D. Since C_φ is compact on Q^p, then $C_\varphi(\mathbf{B})$ is relatively compact in Q^p, where $\mathbf{B}$ is the unit ball of Q^p. Thus for each $\varepsilon > 0$, there is a $\varepsilon/2$-net: $f_1, f_2, \dots, f_n \in \mathbf{B}$ such that for each $f \in \mathbf{B}$ there exists f_i $(1 \le i \le n)$

$$\int_{S(I)} (1-|z|^2)^p |f'(\varphi(z)) - f_i'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) < \frac{\varepsilon}{2} |I|^p \tag{3.7}$$

for each arc I on ∂D. Using (3.6) for $f_1, \dots, f_n$ and setting $\delta = \max_{1 \le i \le n} \delta(\varepsilon, f_i)$ we get

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |f_i'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) < \frac{\varepsilon}{2} |I|^p \tag{3.8}$$

for each arc I. Applying the triangle inequality again on (3.7) and (3.8) we obtain (3.1). The proof of the theorem is complete. □

Since $\varphi \in H^\infty \subset BMOA \subset B$, we have the following corollaries.

COROLLARY 3.2 ([BoCiMa]). *Suppose that φ is an analytic self-map of D. Then the composition operator C_φ is compact on BMOA if and only if for every $\varepsilon > 0$ there is δ, $0 < \delta < 1$, such that*

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2) |f'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) \le \varepsilon |I|$$

for every arc I and every $f \in BMOA$ with $\|f\|_{BMOA} \le 1$.

COROLLARY 3.3. *Suppose that φ is an analytic self-map of D. Then the composition operator C_φ is compact on B if and only if for every $\varepsilon > 0$ there is δ, $0 < \delta < 1$, such that*

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |f'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) \le \varepsilon |I|^p$$

for every arc I, $f \in B$ with $\|f\|_B \le 1$ and $p \in [1, \infty)$.

The compactness of C_φ on Bloch space B was obtained by Madigan and Matheson [MaMa] (also see [Lo]), Corollary 3.3 gives a different compactness characterization of C_φ on Bloch space B. With a similar proof to Theorem3.1, we can prove the following compactness characterization of C_φ on Q_0^p.

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PROPOSITION 3.4. *Suppose that* $0 < p < \infty$ *and* φ *is an analytic self-map of* D. *Then the composition operator* C_φ *is compact on* Q_0^p *if and only if* $\varphi \in Q_0^p$ *and for every* $\varepsilon > 0$ *there is* δ, $0 < \delta < 1$, *such that*

$$\int_{S(I)} |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^p \, dm(z) \le \varepsilon |I|^p$$

for every arc $I : |I| < \delta$ *and every* $f \in Q_0^p$ *with* $\|f\|_{Q^p} \le 1$.

PROOF. Sufficiency is similar to that of Theorem3.1, we leave the details to readers. Necessity. Since C_φ is compact on Q_0^p, then $C_\varphi(\mathbf{B})$ is relatively compact in Q^p, where $\mathbf{B}$ is a unit ball of Q_0^p. Thus for each $\varepsilon > 0$, there is $f_1, f_2, \dots, f_n \in \mathbf{B}$ such that for each $f \in \mathbf{B}$ (3.7) holds for some f_i and each arc I on ∂D. For $f_i \in \mathbf{B}$, there is $\delta_i = \delta(\varepsilon, f_i)$, such that if $|I| < \delta_i$

$$\int_{S(I)} (1-|z|^2)^p |f_i'(\varphi(z))|^2 |\varphi'(z)|^2 \, dm(z) < \frac{\varepsilon}{2} |I|^p. \tag{3.9}$$

Set $\delta = \min_{1 \le i \le n} \delta_i$, then if $|I| < \delta$ (3.9) holds for any f_i, $i = 1, 2, \dots, n$. Combining (3.9) with (3.7) we get the result. □

COROLLARY 3.5 ([BoCiMa]). *Suppose that* φ *is an analytic self-map of* D. *Then the composition operator* C_φ *is compact on VMOA if and only if* $\varphi \in$ *VMOA and for every* $\varepsilon > 0$ *there is* δ, $0 < \delta < 1$, *such that*

$$\int_{S(I)} |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^p \, dm(z) \le \varepsilon |I|^p$$

for every arc $I : |I| < \delta$ *and every* $f \in$ *VMOA with* $\|f\|_{BMOA} \le 1$.

COROLLARY 3.6. *Suppose that* φ *is an analytic self-map of* D. *Then the composition operator* C_φ *is compact on* B_0 *if and only if* $\varphi \in B_0$ *and for every* $\varepsilon > 0$ *there is* δ, $0 < \delta < 1$, *such that*

$$\int_{S(I)} |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^p \, dm(z) \le \varepsilon |I|^p$$

for every arc $I : |I| < \delta$, $f \in B_0$ *with* $\|f\|_B \le 1$ *and* $p \in [1, \infty)$.

Compare to results in [MaMa] and [Lo], Corollary 3.6 gives a different compactnness characterization of C_φ on little Bloch space B_0. About the compactness of composition operators, we know one way to approach this problem is to relate it to properties of φ. That is to see how fast or how often $\varphi(D)$ touches ∂D. The following result is a natural consequence

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PROPOSITION 3.7. *Suppose that $0 < p < \infty$, and φ is an analytic self-map of D and $\varphi \in Q^p$ with $\|\varphi\|_\infty < 1$. Then the composition operator C_φ is compact on Q^p.*

PROOF. Let $\{f_n\}$ be a bounded sequence in Q^p, and converges to 0 uniformly on compact subsets of D. Given $\varepsilon > 0$, since $\|\varphi\|_\infty < 1$, $\overline{\varphi(D)}$ is a compact subset of D. So there exists integer $N > 0$ such that for all $n \geq N$, $|f_n'(\varphi(z))|^2 < \varepsilon$, if $z \in D$. Thus from (2.2) for all $n \geq N$

$$\text{(3.10)} \quad \int_D |f_n'(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |\sigma_a'(z)|^2)^p \, dm(z) \\ \leq \varepsilon \int_D |\varphi'(z)|^2 (1 - |\sigma_a'(z)|^2)^p \, dm(z) \leq \varepsilon \|\varphi\|_{Q^p}^2.$$

Combining (3.10) with $f_n(\varphi(0)) \to 0$ $(n \to \infty)$, we have $\|C_\varphi(f_n)\|_{Q^p} \to 0$ as $n \to \infty$, hence C_φ is compact on Q^p. □

REMARK 2. The following example shows that C_φ is compact on Q^p but not on Q_0^p for $1 \leq p < \infty$. Consider function $\varphi(z) = (1/2)e^{(z+1)/(z-1)}$, it is obvious that φ is an analytic self-map of D and $\|\varphi\|_\infty < 1$. So, from Proposition 3.7, C_φ is compact on Q^p for $1 \leq p < \infty$. By the definition of B_0, $\varphi \notin B_0$, since $Q_0^p \subset B_0$ for $1 \leq p < \infty$, so $\varphi \notin Q_0^p$ $(1 \leq p < \infty)$. We claim that C_φ is not compact on Q_0^p. In fact, if C_φ is compact on Q_0^p, then C_φ is bounded on Q_0^p. That is $f \circ \varphi \in Q_0^p$ for all $f \in Q_0^p$. Taking $f(z) = z$ we have $z \circ \varphi = \varphi \in Q_0^p$, this is a contradiction.

In [BoCiMa], mean order of contact was introduced to study the compactness of the corresponding composition operator on *BMOA*. For $\alpha > 0$, and G an open subset of D, we say that G contacts ∂D with mean order (at most) $\alpha > 0$ provide that

$$\int_0^{2\pi} 1_G(re^{i\theta}) d\theta = O\left((1-r)^{1/\alpha}\right)$$

as $r \to 1^-$.

The function $\varphi = 1 - (1-z)^{1/2}$ (see [BoCiMa, page 11]) shows that contact of $\varphi(D)$ of mean order 1 is not sufficient to guarantee that C_φ is compact on *BMOA*. However, the following result shows that mean order contact less than 1 does guarantee the compactness not only on *BMOA* but also on Q^p for all $0 < p < \infty$.

PROPOSITION 3.8. *Suppose that $0 < p < \infty$, $\varphi(D)$ is contained in a simply connected region which contacts the unit circle with $\alpha < 1$. Then C_φ is compact on Q^p.*

PROOF. Similar to the proof of Corollary 5.7 of [BoCiMa]. □

Proposition 3.7 shows that if $\varphi \in Q^p$ and $\|\varphi\|_\infty < 1$, then C_φ is compact on Q^p $(0 < p < \infty)$. This is only a sufficient condition. In fact, the example of φ given in

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[BoCiMa, page 14] shows that C_φ is compact on Q^p (Proposition 3.8), $1 \le p < \infty$, but $\overline{\varphi(D)} = \bar{D}$.

In [Sh], Shapiro solves the compactness problem for composition operators on H^2 using the Navanlinna counting function $N_\varphi(w)$. The following theorem gives a sufficient condition for a composition operator to be compact on Q^p spaces.

PROPOSITION 3.9. *Suppose that* $0 < p < \infty$, φ *is an analytic self-map of* D, $\varphi \in Q^p$, *and*

$$\lim_{|w|\to 1} \frac{N_{\varphi\circ\sigma_a,p}(w)}{(1-|\sigma_a(w)|^2)^p} = 0. \tag{3.11}$$

Then the composition operator C_φ *is compact on* Q^p.

PROOF. Let $\{f_n\}$ be a bounded sequence of Q^p, $\|f_n\|_{Q^p} \le C$, such that $f_n \to 0$ uniformly on compact subsets of D. Given $\varepsilon > 0$, (3.11) implies that there is $\delta > 0$ such that if $\delta < |w| < 1$,

$$N_{\varphi\circ\sigma_a,p}(w) < \varepsilon(1-|\sigma_a(w)|^2)^p. \tag{3.12}$$

Set

$$\begin{aligned} &\sup_{a\in D}\int_D |f_n'(w)|^2 N_{\varphi\circ\sigma_a,p}(w)\,dm(w) \\ &\qquad = \sup_a\left(\int_{\delta<|w|<1} + \int_{|w|\le\delta}\right)|f_n'(w)|^2 N_{\varphi\circ\sigma_a,p}(w)\,dm(w) = \mathrm{I}+\mathrm{II}. \end{aligned} \tag{3.13}$$

By (3.12) and the fact that f_n is bounded in Q^p,

$$\mathrm{I} \le \varepsilon \sup_a \int_{\delta<|w|<1} |f_n'(w)|^2(1-|\sigma_a(w)|^2)^p\,dm(w) \le \varepsilon\|f_n\|_{Q^p}^2 \le \varepsilon C. \tag{3.14}$$

Since f_n' converges to 0 uniformly on $|w| \le \delta$, there is $N > 0$ such that for all $n \ge N$ $|f_n'(w)|^2 < \varepsilon$ if $|w| \le \delta$. So using (2.9), we have

$$\mathrm{II} \le \varepsilon \sup_a \int_{|w|\le\delta} N_{\varphi\circ\sigma_a,p}(w)\,dm(w) \le \varepsilon\|C_\varphi(z)\|_{Q^p}^2 = \varepsilon\|\varphi\|_{Q^p}^2. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15) with $f_n(\varphi(0)) \to 0$ ($n \to \infty$), we have

$$\|C_\varphi(f_n)\|_{Q^p} \doteq |f_n(\varphi(0))| + \left(\sup_{a\in D}\int_D |f_n'(z)|^2 N_{\varphi\circ\sigma_a,p}(w)\,dm(w)\right)^{1/2} \to 0$$

as $n \to \infty$. Hence C_φ is compact on Q^p by Lemma 1. □

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4. Composition operators from $\mathscr{D}$ to Q^p and Q_0^p

In this section, motivated by [Tj], we study the boundedness and compactness of composition operators from Dirichlet space $\mathscr{D}$ to Q^p and Q_0^p spaces, which were characterized by the basic conformal automorphism σ_a defined by

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in D.$$

It is easy to check that $\sigma_a \circ \sigma_a(z) = z$ and

$$1 - |\sigma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2} = (1 - |z|^2)|\sigma_a'(z)|. \tag{4.1}$$

THEOREM 4.1. *Suppose that* $0 \le p < \infty$ *and* φ *is an analytic self-map of* D. *Then the composition operator* $C_\varphi : \mathscr{D} \to Q^p$ *is bounded if and only if*

$$\sup_{a \in D} \|C_\varphi(\sigma_a)\|_{Q^p} < \infty. \tag{4.2}$$

PROOF. Necessity. Suppose that $C_\varphi : \mathscr{D} \to Q^p$ is bounded. It is easy to check that $\{\sigma_a, a \in D\}$ is bounded in $\mathscr{D}$, since $\|\sigma_a\|_{\mathscr{D}} = \|z \circ \sigma_a\|_{\mathscr{D}} = \|z\|_{\mathscr{D}}$. So we have $\sup_a \|C_\varphi(\sigma_a)\|_{Q^p} \le C\|\sigma_a\|_{\mathscr{D}} \le C$.

Sufficiency. Suppose that $\sup_a \|C_\varphi(\sigma_a)\|_{Q^p} < \infty$ and $f \in \mathscr{D}$, then from (2.9) we have

$$\int_D |\sigma_a'(w)|^2 d\mu_{b,p}(w) \le C \tag{4.3}$$

for all $a, b \in D$. By Theorem 2, (4.3) is equivalent to

$$\int_{S(h,\theta)} d\mu_{b,p}(w) \le Ch^2 \tag{4.4}$$

for all $h \in (0, 1)$, $\theta \in [0, 2\pi)$ and $b \in D$. Using the Mean Value Theorem and Jensen's inequality we have

$$|f(w)|^2 \le \frac{4}{\pi(1 - |w|)^2} \int_{|w-z|<(1-|w|)/2} |f(z)|^2 \, dm(z), \quad w \in D. \tag{4.5}$$

From the discussion in [Tj, page 36], the inequality $|w - z| < (1 - |w|)/2$ implies $w \in S(2(1 - |z|), \arg z)$ and $1/(1 - |w|)^2 \le C/(1 - |z|)^2$. Combining this with

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Fubini's Theorem, (4.4) and (4.5), we obtain

$$(4.6)\qquad \begin{aligned}\int_D |f'(w)|^2 d\mu_{b,p}(w) &\le \int_D \frac{4}{\pi(1-|w|)^2}\int_{|w-z|<(1-|w|)/2} |f'(z)|^2\, dm(z)\, d\mu_{b,p}(w)\\ &\le C\int_D |f'(z)|^2 \int_D \frac{1}{(1-|w|)^2} 1_{\{z:|w-z|<(1-|w|)/2\}}(z)\, d\mu_{b,p}(w)\, dm(z)\\ &\le C\int_D \frac{|f'(z)|^2}{(1-|z|)^2}\int_{S(2(1-|z|),\arg z)} d\mu_{b,p}(w)\, dm(z)\\ &\le C\left(\int_{|z|>1/2}+\int_{|z|\le 1/2}\right)\frac{|f'(z)|^2}{(1-|z|)^2}\mu_{b,p}(S(2(1-|z|),\arg z))\, dm(z)\\ &= \mathrm{I}+\mathrm{II},\end{aligned}$$

where $1_S(z)$ denotes the characteristic function on S. For $|z| > \frac{1}{2}$, since $2(1-|z|) < 1$, then (4.4) gives

$$(4.7)\qquad \begin{aligned}\mathrm{I} &\le C\int_{|z|>1/2} \frac{|f'(z)|^2}{(1-|z|)^2}(2(1-|z|))^2\, dm(z)\\ &\le C\int_D |f'(z)|^2\, dm(z) \le C\|f\|_{\mathscr{D}}^2.\end{aligned}$$

For $|z| \le 1/2$, since $f \in \mathscr{D} \subset B$, we have $\|f\|_B \le C\|f\|_{\mathscr{D}}$. From (4.2) for any $a \in D$, $\|\sigma_a \circ \varphi\|_{Q^p} \le C$, taking $a = 0$, we have $\varphi \in Q^p$ and $\|\varphi\|_{Q^p} = |\varphi(0)| + (\sup_{b\in D} \mu_{b,p}(D))^{1/2}$ by (2.9), thus

$$(4.8)\qquad \begin{aligned}\mathrm{II} &\le C\int_{|z|\le 1/2} \frac{|f'(z)|^2}{(1-|z|)^2}\mu_{b,p}(D)\, dm(z)\\ &\le C\int_{|z|\le 1/2} \frac{\|f\|_B^2}{(1-|z|)^4}\mu_{b,p}(D)\, dm(z) \le C\|f\|_{\mathscr{D}}^2\|\varphi\|_{Q^p}^2.\end{aligned}$$

Since

$$|f(z)| \le \|f\|_B \log\frac{1}{1-|z|},\qquad \text{for all } z \in D,\ f \in B,$$

then

$$(4.9)\qquad |f(\varphi(0))| \le C\|f\|_{\mathscr{D}} \log\frac{1}{1-|\varphi(0)|},\qquad \varphi(0) \ne 1.$$

From (4.6), (4.7), (4.8) and (4.9) we get

$$\begin{aligned}\|C_\varphi(f)\|_{Q^p} &= |f(\varphi(0))| + \left(\sup_{b\in D}\int_D |f'(w)|^2\, d\mu_{b,p}(w)\right)^{1/2}\\ &\le C\|f\|_{\mathscr{D}}\log\frac{1}{1-|\varphi(0)|} + \left(C\|f\|_{\mathscr{D}}^2 + C\|\varphi\|_{Q^p}^2\|f\|_{\mathscr{D}}^2\right)^{1/2} \le C(\varphi)\|f\|_{\mathscr{D}},\end{aligned}$$

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where $C(\varphi)$ is a constant depending only on φ. Thus C_φ is bounded and the proof is finished. □

The following corollary characterizes boundedness of C_φ from Dirichlet space $\mathscr{D}$ to the well-known spaces $\mathscr{D}$, $BMOA$ and Bloch space B.

COROLLARY 4.2. *Suppose that φ is an analytic self-map of D. Then the composition operator*

(1) $C_\varphi : \mathscr{D} \to \mathscr{D}$ *is bounded if and only if* $\sup_{a\in D} \|C_\varphi(\sigma_a)\|_{\mathscr{D}} < \infty$.
(2) $C_\varphi : \mathscr{D} \to BMOA$ *is bounded if and only if* $\sup_{a\in D} \|C_\varphi(\sigma_a)\|_{BMOA} < \infty$.
(3) $C_\varphi : \mathscr{D} \to B$ *is bounded if and only if* $\sup_{a\in D} \|C_\varphi(\sigma_a)\|_{B} < \infty$.

THEOREM 4.3. *Suppose that $0 \le p < \infty$ and φ is an analytic self-map of D. Then the composition operator $C_\varphi : \mathscr{D} \to Q^p$ is compact if and only if*

$$\|C_\varphi(\sigma_a)\|_{Q^p} \to 0, \quad |a| \to 1$$

PROOF. Necessity. Suppose $C_\varphi : \mathscr{D} \to Q^p$ is compact. Since $\{\sigma_a, a \in D\}$ is bounded in $\mathscr{D}$, and $|\sigma_a - a| = |z|(1-|a|^2)/|1-\bar{a}z|$ converges to 0 uniformly on compact subsets of D as $|a| \to 1$, we have $\|C_\varphi(\sigma_a)\|_{Q^p} \to 0$ as $|a| \to 1$ by Lemma 1.

Sufficiency. Suppose that $\{f_n\}$ is a bounded sequence of $\mathscr{D}$, $\|f_n\|_{\mathscr{D}} \le C$, such that $f_n \to 0$ uniformly on compact subsets of D. Since $\|C_\varphi(\sigma_a)\|_{Q^p} \to 0$ as $|a| \to 1$, then from (2.9) we have

$$\sup_{b\in D} \int_D |\sigma_a'(w)|^2 d\mu_{b,p}(w) \to 0, \quad |a| \to 1. \tag{4.10}$$

By Theorem 2, (4.10) is equivalent to

$$\lim_{h\to 0} \sup_{\substack{b\in D\\ \theta\in[0,2\pi)}} \frac{1}{h^2} \int_{S(h,\theta)} d\mu_{b,p}(w) = 0.$$

Given $\varepsilon > 0$, there is $0 < \delta < 1$ such that for all h, $h < \delta$, $\theta \in [0, 2\pi)$ and $b \in D$

$$\int_{S(h,\theta)} d\mu_{b,p}(w) < \varepsilon h^2. \tag{4.11}$$

As in the proof of Theorem 4.1, we obtain

$$\begin{aligned} \int_D |f_n'(w)|^2\, d\mu_{b,p}(w) &\le \int_D \frac{4}{\pi(1-|w|)^2} \int_{|w-z|<(1-|w|)/2} |f_n'(z)|^2\, dm(z)\, d\mu_{b,p}(w) \\ &\le C \int_D \frac{|f_n'(z)|^2}{(1-|z|)^2} \int_{S(2(1-|z|),\arg z)} d\mu_{b,p}(w)\, dm(z) \end{aligned} \tag{4.12}$$

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$$\leq C\left(\int_{|z|>1-\delta/2}+\int_{|z|\leq 1-\delta/2}\right)\frac{|f_n'(z)|^2}{(1-|z|)^2}\int_{S(2(1-|z|),\arg z)}d\mu_{b,p}(w)\,dm(z)$$
$$=\mathrm{I}+\mathrm{II}.$$

If $|z|>1-\delta/2$, then $2(1-|z|)<\delta$ and (4.11) gives

$$\mathrm{I}\leq \varepsilon C\int_{|z|>1-\delta/2}\frac{|f_n'(z)|^2}{(1-|z|)^2}(2(1-|z|))^2\,dm(z)\leq \varepsilon C\|f_n\|^2_{\mathscr{D}}\leq \varepsilon C. \tag{4.13}$$

If $|z|\leq 1-\delta/2$, since $f_n'\to 0$ uniformly on compact subsets of D, then there is $N>0$ such that if $n\geq N$,

$$|f_n'(z)|^2<\varepsilon,\quad |z|\leq 1-\delta/2. \tag{4.14}$$

By (4.11) $d\mu_{b,p}(w)$ is a compact 2-Careleson measure, it is also a bounded 2-Carleson measure, then for all $h\in(0,1)$, $b\in D$ and $\theta\in[0,2\pi)$, (4.4) holds. From Theorem 4.1, $C_\varphi:\mathscr{D}\to Q^p$ is bounded, so $\varphi\in Q^p$ and (2.9) give $\sup_{b\in D}\int_D d\mu_{b,p}(w)\leq\|\varphi\|^2_{Q^p}<\infty$. Combining this with (4.14) we get, for $n\geq N$,

$$\mathrm{II}\leq C\int_{|z|\leq 1-\delta/2}\frac{|f_n'(z)|^2}{(1-|z|)^2}\,dm(z)\left(\sup_b\int_D d\mu_{b,p}(w)\right)\leq \varepsilon C\|\varphi\|^2_{Q^p}. \tag{4.15}$$

Hence (4.12), (4.13), (4.15) and $f_n(\varphi(0))\to 0$ $(n\to\infty)$ yield that

$$\|C_\varphi(f_n)\|_{Q^p}=|f_n(\varphi(0))|+\left(\sup_{b\in D}\int_D|f_n'(w)|^2\,d\mu_{b,p}(w)\right)^{1/2}<\varepsilon\left(C+C\|\varphi\|^2_{Q^p}\right)^{1/2}$$

for n large enough. So $\|C_\varphi(f_n)\|_{Q^p}\to 0$ as $n\to\infty$. From Lemma 1, $C_\varphi:\mathscr{D}\to Q^p$ is compact. □

COROLLARY 4.4. *Suppose that φ is an analytic self-map of D. Then the composition operator*

(1) *$C_\varphi:\mathscr{D}\to\mathscr{D}$ is compact if and only if $\|C_\varphi(\sigma_a)\|_{\mathscr{D}}\to 0$ as $|a|\to 1$.*
(2) *$C_\varphi:\mathscr{D}\to BMOA$ is compact if and only if $\|C_\varphi(\sigma_a)\|_{BMOA}\to 0$ as $|a|\to 1$.*
(3) *$C_\varphi:\mathscr{D}\to B$ is compact if and only if $\|C_\varphi(\sigma_a)\|_{B}\to 0$ as $|a|\to 1$.*

THEOREM 4.5. *Suppose that $0<p<\infty$ and φ is an analytic self-map of D. Then $C_\varphi(\mathscr{D})\subset Q_0^p$ if and only if $\varphi\in Q_0^p$ and for every $\varepsilon>0$, there exists $\delta>0$ such that for all b ($|b|>\delta$), $\theta\in[0,2\pi)$ and all $h\in(0,1)$*

$$\mu_{b,p}(S(h,\theta))\leq\varepsilon Ch^2. \tag{4.16}$$

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PROOF. Necessity. Suppose that $C_\varphi(\mathscr{D}) \subset Q_0^p$. It is obvious that $C_\varphi(z) = z \circ \varphi = \varphi \in Q_0^p$ as $z \in \mathscr{D}$. Since $\{\sigma_a : a \in D\}$ is bounded in $\mathscr{D}$. Then $C_\varphi(\sigma_a) \in Q_0^p$, that is,

$$\lim_{|b|\to 1} \int_D |\sigma_a'(w)|^2 \, d\mu_{b,p}(w) = 0.$$

Given $\varepsilon > 0$, there is $\delta > 0$, such that for all b, $|b| > 1 - \delta$,

$$\int_D |\sigma_a'(w)|^2 d\mu_{b,p}(w) < \varepsilon. \tag{4.17}$$

For $w \in S(h, \theta)$ and $a = (1-h)e^{i\theta}$, from the discussion in [Tj, page 26], for all $h \in (0, 1)$

$$\frac{1-|a|^2}{|1-\bar{a}w|^2} \geq \frac{1}{4h}.$$

So

$$\begin{aligned}\int_D |\sigma_a'(w)|^2 d\mu_{b,p}(w) &= \int_D \left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^2 d\mu_{b,p}(w) \\ &\geq \inf_{w\in S(h,\theta)} \left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^2 \mu_{b,p}(S(h,\theta)) \geq \frac{\mu_{b,p}(S(h,\theta))}{4^2h^2}.\end{aligned}$$

Thus for all b, $|b| > 1 - \delta$, $h \in (0, 1)$ and $\theta \in [0, 2\pi)$

$$\mu_{b,p}(S(h,\theta)) \leq 4^2h^2 \int_D |\sigma_a'(w)|^2 \, dm_{b,p}(w) < \varepsilon 4h^2.$$

Sufficiency. If $f \in \mathscr{D}$, we show that $C_\varphi(f) \in Q_0^p$. As in the proof of Theorem 4.1.

$$\begin{aligned}&\int_D |f'(w)|^2 d\mu_{b,p}(w) \\ &\leq C\left(\int_{|z|>1/2} + \int_{|z|\leq 1/2}\right) \frac{|f'(z)|^2}{(1-|z|)^2} \mu_{b,p}(S(2(1-|z|), \arg z) \, dm(z) \\ &\leq \mathrm{I} + \mathrm{II}.\end{aligned} \tag{4.18}$$

If $|z| > 1/2$, from (4.16), for every $\varepsilon > 0$ there is $\delta_1 > 0$ such that for $|b| > \delta_1$,

$$\begin{aligned}\mathrm{I} &\leq \varepsilon C \int_{|z|>1/2} \frac{|f'(z)|^2}{(1-|z|)^2} (2(1-|z|))^2 \, dm(z) \\ &\leq \varepsilon C \int_D |f'(z)|^2 \, dm(z) \leq \varepsilon C \|f\|_{\mathscr{D}}^2.\end{aligned} \tag{4.19}$$

If $|z| \leq 1/2$, since $f \in \mathscr{D} \subset B$, we have $\|f\|_B \leq C\|f\|_{\mathscr{D}}$. By $\varphi \in Q_0^p$, (2.9) and the definition of Q_0^p we have $\lim_{|b|\to 1} \int_D d\mu_{b,p}(w) = 0$. So there exists $\delta_2 > 0$ such that

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if $|b| > \delta_2$, $\mu_{b,p}(D) < \varepsilon$, thus

$$\text{(4.20)} \qquad \begin{aligned} \text{II} &\le C \int_{|z|\le 1/2} \frac{|f'(z)|^2}{(1-|z|)^2} \mu_{b,p}(D)\, dm(z) \\ &\le C \int_{|z|\le 1/2} \frac{\|f\|_B^2}{(1-|z|)^4} \mu_{b,p}(D)\, dm(z) \le \varepsilon C \|f\|_{\mathscr{D}}^2. \end{aligned}$$

Taking $\delta = \max\{\delta_1, \delta_2\}$, if $|b| > \delta$, from (4.18), (4.19) and (4.20), we get

$$\int_D |f'(w)|^2 d\mu_{b,p}(w) < \varepsilon C \|f\|_{\mathscr{D}}^2,$$

that is, $C_\varphi(f) \in Q_0^p$. □

Combining Theorem 4.1, Theorem 4.3 and Theorem 4.5 we obtain the boundedness and compactness of $C_\varphi : \mathscr{D} \to Q_0^p$, where the boundedness means $C_\varphi(\mathscr{D}) \subset Q_0^p$ and $C_\varphi : \mathscr{D} \to Q^p$ is bounded, and compactness means that $C_\varphi(\mathscr{D}) \subset Q_0^p$ and $C_\varphi : \mathscr{D} \to Q^p$ is compact.

COROLLARY 4.6. *Suppose that* $0 < p < \infty$ *and* φ *is an analytic self-map of* D. *Then the composition operator*

(1) $C_\varphi : \mathscr{D} \to Q_0^p$ *is bounded if and only if* $\sup_{a\in D} \|C_\varphi(\sigma_a)\|_{Q^p} < \infty$ *and the sufficient condition* (4.16) *of Theorem* 4.5 *holds.*
(2) $C_\varphi : \mathscr{D} \to Q_0^p$ *is compact if and only if* $\lim_{|a|\to 1} \|C_\varphi(\sigma_a)\|_{Q^p} = 0$ *and the sufficient condition* (4.16) *of Theorem* 4.5 *holds.*

5. Composition operators from B^0 to Q^p and Q_0^p

In Remark 1 of Section 3, we point out that (3.3) is not sufficient for the compactness of C_φ on Q^p spaces. In this section we show that (3.3) is necessary and sufficient for the compactness of C_φ from a subspace B^0 of Q^p to Q^p, where B^0 is a space of analytic functions f with $f' \in H^\infty$, and $\|f\|_{B^0} = |f(0)| + \|f'\|_\infty$.

THEOREM 5.1. *Suppose that* $0 < p < \infty$ *and* φ *is an analytic self-map of* D. *Then the composition operator* $C_\varphi : B^0 \to Q^p$ *is bounded if and only if* $\varphi \in Q^p$.

PROOF. Suppose that $\varphi \in Q^p$, and $f \in B^0$, we show that $f \circ \varphi \in Q^p$. From $\varphi \in Q^p$ and $f \in B^0$ we have

$$\begin{aligned} &\int_{S(I)} |(f\circ\varphi)'(z)|^2 (1-|z|^2)^p\, dm(z) \\ &\qquad = \int_{S(I)} |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^p\, dm(z) \end{aligned}$$

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$$\leq \|f\|_{B^0}^2 \int_{S(I)} |\varphi'(z)|^2(1-|z|^2)^p \, dm(z) \leq \|f\|_{B^0}^2 \|\varphi\|_{Q^p}^2 |I|^p$$

for all I on ∂D. Since $f \in B^0 \subset \mathscr{D}$, then (4.9) gives

$$|f(\varphi(0))| \leq C\|f\|_{B^0} \log \frac{1}{1-|\varphi(0)|}.$$

So, from (2.7),

$$\|C_\varphi f\|_{Q^p} \leq C(\varphi)\|f\|_{B^0}, \quad f \in B^0,$$

where $C(\varphi)$ is a constant depending only on φ. If $C_\varphi : B^0 \to Q^p$ is bounded, then $C_\varphi(f) \in Q^p$ for all $f \in B^0$. Taking $f = z$, we have $\varphi \in Q^p$. □

COROLLARY 5.2. *If φ is an analytic self-map of D, then the composition operator $C_\varphi : B^0 \to BMOA(B)$ is bounded.*

THEOREM 5.3. *Suppose that $0 < p < \infty$ and φ is an analytic self-map of D. Then the composition operator $C_\varphi : B^0 \to Q^p$ is compact if and only if $\varphi \in Q^p$ and for every $\varepsilon > 0$ there is δ, $0 < \delta < 1$, such that*

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |\varphi'(z)|^2 \, dm(z) < \varepsilon |I|^p \tag{5.1}$$

for all arcs I on ∂D.

PROOF. If $C_\varphi : B^0 \to Q^p$ is compact, then $\varphi \in Q^p$ by Theorem 5.1. From Lemma 1 and for any $f_n \in B^0$, $\|f_n\|_{B^0} \leq C$ and converges uniformly to 0 on compact subsets of D, we have $\|f_n \circ \varphi\|_{Q^p} \to 0$ as $n \to \infty$. Set $f_n(z) = z^n/n$, since z^n/n is norm bounded in B^0 and converges uniformly to 0 on compact subsets of D, we have

$$\left\| \frac{\varphi^n}{n} \right\|_{Q^p} \to 0, \quad n \to \infty.$$

Hence, given $\varepsilon > 0$, there is $N > 0$ such that if $n \geq N$, then

$$\frac{1}{n} \int_{S(I)} n^2 |\varphi(z)|^{2n-2} |\varphi'(z)|^2 (1-|z|^2)^p \, dm(z) < \varepsilon |I|^p$$

for all I. Given δ, $0 < \delta < 1$,

$$N\delta^{2N-2} \int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |\varphi'(z)|^2 \, dm(z)$$
$$\leq N \int_{S(I)} |\varphi(z)|^{2N-2} |\varphi'(z)|^2 (1-|z|^2)^p \, dm(z) < \varepsilon |I|^p$$

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for all I, since $|\varphi(z)| > \delta$ on D_δ. Choosing δ so that $N\delta^{2N-2} = 1$, we obtain (5.1).

To prove that C_φ is compact, let $\{f_n\} \subset B^0$ be such that $\|f_n\|_{B^0} \le C$ and converges to 0 uniformly on compact subsets of D. We show that

$$\|C_\varphi(f_n)\|_{Q^p} \to 0, \quad n \to \infty.$$

Fix $\varepsilon > 0$ and let $\delta, 0 < \delta < 1$, such that (5.1). Since $\varphi(D \setminus D_\delta)$ is a relatively compact subset of D, $f_n' \circ \varphi$ converges uniformly to 0 on $D \setminus D_\delta$, then there is $N \ge 0$ such that $|f_n' \circ \varphi|^2 < \varepsilon$ if $n \ge N$ and $z \in D \setminus D_\delta$. So for all $n \ge N$ and I on ∂D

$$\text{(5.2)} \qquad \int_{S(I)} 1_{D\setminus D_\delta}(z)(1-|z|^2)^p |f_n'(\varphi(z))|\varphi'(z)|^2\, dm(z) \le \varepsilon \|\varphi\|^2_{Q^p} |I|^p$$

and

$$\text{(5.3)} \qquad \int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |f_n'(\varphi(z))|^2|\varphi'(z)|^2\, dm(z)$$
$$\le \|f_n\|^2_{B^0} \int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |\varphi'(z)|^2\, dm(z) \le \varepsilon C |I|^p.$$

Hence, combining (5.2) with (5.3), we obtain,

$$\int_{S(I)} (1-|z|^2)^p |f_n'(\varphi(z))|\varphi'(z)|^2\, dm(z) \le \varepsilon(C + \|\varphi\|^2_{Q^p})|I|^p$$

for all I on ∂D and $n \ge N$. Since $f_n \circ \varphi(0) \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|C_\varphi(f_n)\|_{Q^p} = 0$$

The proof of Theorem 5.3 is complete. □

When $p = 1$, we get the compactness of composition operator C_φ from B^0 to *BMOA* and Bloch space B.

COROLLARY 5.4. *Suppose that φ is an analytic self-map of D. Then $C_\varphi : B^0 \to$ BMOA is compact if and only if for every $\varepsilon > 0$ there is δ, $0 < \delta < 1$, such that*

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)|\varphi'(z)|^2\, dm(z) < \varepsilon |I|$$

for all arcs I on ∂D.

COROLLARY 5.5. *Suppose that φ is an analytic self-map of D. Then $C_\varphi : B^0 \to B$ is compact if and only if for every $\varepsilon > 0$ there is δ, $0 < \delta < 1$, such that*

$$\int_{S(I)} 1_{D_\delta}(z)(1-|z|^2)^p |\varphi'(z)|^2\, dm(z) < \varepsilon |I|^p$$

for all arcs I on ∂D and $p \in [1, \infty)$.

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THEOREM 5.6. *Suppose that* $0 < p < \infty$ *and* φ *is an analytic self-map of* D. *Then the following statements are equivalent*:

(1) $\varphi \in Q_0^p$.
(2) $C_\varphi : B^0 \to Q_0^p$ *is bounded.*
(3) $C_\varphi : B^0 \to Q_0^p$ *is compact.*

PROOF. (3) implies (2) is obvious.

(2) implies (1). If C_φ is bounded, then $f \circ \varphi \in Q_0^p$ for all $f \in B^0$. Set $f(z) = z$, we obtain $\varphi \in Q_0^p$.

(1) implies (2). If $\varphi \in Q_0^p$, then $C_\varphi : B^0 \to Q^p$ is bounded by Theorem 5.1. So it is enough to show that $C_\varphi(B^0) \subset Q_0^p$. Since $\varphi \in Q_0^p$, then for every $\varepsilon > 0$, there exists $\delta > 0$

$$\begin{aligned}\int_{S(I)} &|(f \circ \varphi)'(z)|^2(1-|z|^2)^p \, dm(z) \\ &= \int_{S(I)} |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^p \, dm(z) \\ &\le \|f\|_{B^0}^2 \int_{S(I)} |\varphi'(z)|^2(1-|z|^2)^p \, dm(z) \le \varepsilon \|f\|_{B^0}^2 \|\varphi\|_{Q^p}^2 |I|^p\end{aligned}$$

for all I, $|I| \le \delta$, and $f \in B^0$, that is, $C_\varphi(f) = f \circ \varphi \in Q_0^p$.

(1) implies (3). Suppose that $\varphi \in Q_0^p$. To prove that C_φ is compact, we need to show that $C_\varphi(B^0) \subset Q_0^p$ and $C_\varphi : B^0 \to Q^p$ is compact. The first inclusion is obvious from (1) implies (2). Now we prove compactness of C_φ. Let $\{f_n\} \subset B^0$ such that $\|f_n\|_{B^0} \le C$, and converges to 0 uniformly on compact subsets of D. It is enough to show that

$$\|C_\varphi(f_n)\|_{Q^p} \to 0, \quad n \to \infty.$$

Since $\varphi \in Q_0^p$, from Theorem 1 and (2.6), for $\varepsilon > 0$, there is $0 < \delta < 1$ such that for $h < \delta$ and $\theta \in [0, 2\pi)$,

$$\int_{S(h,\theta)} |\varphi'(z)|^2(1-|z|^2)^p \, dm(z) \le \varepsilon h^p. \tag{5.4}$$

For h, $h < \delta$, $\theta \in [0, 2\pi)$, from (5.4) and $\|f_n\|_{B^0} \le C$, we have

$$\begin{aligned}\int_{S(h,\theta)} &|(f_n \circ \varphi)'(z)|^2(1-|z|^2)^p \, dm(z) \\ &= \int_{S(h,\theta)} |f_n'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^p \, dm(z) \\ &\le \|f_n\|_{B^0}^2 \int_{S(h,\theta)} |\varphi'(z)|^2(1-|z|^2)^p \, dm(z) \le \varepsilon C h^p.\end{aligned} \tag{5.5}$$

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For $h > \delta$, choose $h_0 < \delta$, $\theta \in [0, 2\pi)$. From the definition of $S(h, \theta)$, it is obvious that there exist $\theta_1, \dots, \theta_m \in [0, 2\pi)$ and a compact subset K of D such that

$$S(h,\theta) = K \cup \left(\bigcup_{i=1}^{m} S(h_0, \theta_i) \right). \tag{5.6}$$

Since f_n' converges to 0 uniformly on a compact subset K, then there exists $N > 0$ such that for all $n \geq N$ and $h \in (0, 1)$

$$\int_K |(f_n \circ \varphi)'(z)|^2 (1 - |z|^2)^p \, dm(z) \leq \varepsilon \int_K (1 - |z|^2)^p \leq \varepsilon C h^p. \tag{5.7}$$

For $S(h_0, \theta_i)$, $i = 1, \dots, m$, using (5.5), we have

$$\int_{S(h_0,\theta_i)} |(f_n \circ \varphi)'(z)|^2 (1 - |z|^2)^p \, dm(z) \leq \varepsilon h_0^p. \tag{5.8}$$

From (5.6), (5.7) and (5.8), we have for $h > \delta$ and $n \geq N$

$$\begin{aligned} &\int_{S(h,\theta)} |(f_n \circ \varphi)'(z)|^2 (1 - |z|^2)^p \, dm(z) \\ &\quad \leq \left(\int_K + \int_{\sum_{i=1}^m S(h_0,\theta_i)} \right) |(f_n \circ \varphi)'(z)|^2 |(1 - |z|^2)^p \, dm(z) \\ &\quad \leq C\varepsilon h^p + \sum_{i=1}^{m} \int_{S(h_0,\theta_i)} |(f_n \circ \varphi)'(z)|^2 (1 - |z|^2)^p \, dm(z) \\ &\quad \leq C\varepsilon h^p + C \sum_{i=1}^{m} \varepsilon h_0^p \leq C\varepsilon h^p. \end{aligned} \tag{5.9}$$

Combining (5.5) with (5.9) we get for all $n \geq N$, $h \in (0, 1)$ and all $\theta \in [0, 2\pi)$

$$\int_{S(h,\theta)} |(f_n \circ \varphi)'(z)|^2 (1 - |z|^2)^p \, dm(z) \leq C\varepsilon h^p.$$

Hence

$$\|C_\varphi(f_n)\|_{Q^p} \to 0 \quad n \to \infty.$$

by Theorem 1 and (2.6). The proof is finished. □

COROLLARY 5.7. *If φ is an analytic self-map of D, then the following statements are equivalent*:

(1) $\varphi \in VMOA$.

(2) $C_\varphi : B^0 \to VMOA$ *is bounded.*

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(3) $C_\varphi : B^0 \to VMOA$ *is compact.*

COROLLARY 5.8. *If φ is an analytic self-map of D, then the following statements are equivalent*:

(1) $\varphi \in B_0$.
(2) $C_\varphi : B^0 \to B_0$ *is bounded.*
(3) $C_\varphi : B^0 \to B_0$ *is compact.*

Corollary 5.8 shows that Theorem 4.1 in [Lo] holds for $\alpha = 0$ and $\beta = 1$.

6. Composition operators from B to Q^p and Q_0^p

In [SmZh], Smith and Zhao have studied the compactness of composition operators C_φ from Bloch space B to Q^p and Q_0^p spaces, see [SmZh, Theorem 1.6 and Proposition 6.5]. In this section, we give different compact characterizations of $C_\varphi : B \to Q^p(Q_0^p)$. In [ArFiPe], the following result was proved

THEOREM 6.1. *Let μ be a positive measure on D and $0 \le p < \infty$. Then*

$$\int_d |f'(w)|^p d\mu(w) \le C\|f\|_B$$

for all $f \in B$, if and only if

$$\int_D \frac{d\mu(w)}{(1-|w|^2)^p} < \infty.$$

Combining Theorem 6.1 with (2.9), yields the following characterization of bounded composition operator from Bloch space to Q^p spaces for $0 \le p < \infty$.

THEOREM 6.2. *Suppose that $0 \le p < \infty$ and φ is an analytic self-map of D. Then the composition operator $C_\varphi : B \to Q^p$ is bounded if and only if*

$$\sup_{a\in D} \int_D \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}(1-|\sigma_a(z)|^2)^p \, dm(z) < \infty.$$

Smith and Zhao [SmZh] proved Theorem 6.2 for $0 < p < \infty$ using a different idea. For $1 < p < \infty$, we know by Schwarz-Pick lemma that C_φ is bounded on B for any analytic self-map φ . For $p = 0, 1$, we get the following corollary (Note that when $p = 0$, $N_{\varphi,0}(w) = n(\varphi, w)$.

COROLLARY 6.3. *Suppose that φ be is analytic self-map of D. Then the composition operator*

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(1) $C_\varphi : B \to \mathcal{D}$ *is bounded if and only if*

$$\int_D \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) < \infty. \tag{6.1}$$

(2) $C_\varphi : B \to BMOA$ *is bounded if and only if*

$$\sup_{a\in D} \int_D \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}(1-|\sigma_a(z)|^2)\, dm(z) < \infty.$$

THEOREM 6.4. *Suppose that* $0 < p < \infty$ *and* φ *is an analytic self-map of* D. *Then* $C_\varphi : B \to Q^p$ *is compact if and only if* $\varphi \in Q^p$ *and for every* $\varepsilon > 0$ *there is* $\delta : 0 < \delta < 1$ *such that*

$$\int_{S(I)} 1_{D_\delta}(z)|f'(\varphi(z))|^2|\varphi(z)|^2(1-|z|^2)^p\, dm(z) \le \varepsilon |I|^p \tag{6.2}$$

for every arc I *and every* $f \in B$ *with* $\|f\|_B \le 1$.

PROOF. Similar to the proof of Theorem 3.1, we omit the details. □

PROPOSITION 6.5. *Suppose that* $0 < p \le \infty$, φ *is an analytic self-map of* D *and satisfies*

$$\int_D \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) < \infty. \tag{6.3}$$

Then $C_\varphi : B \to Q^p$ *is compact.*

PROOF. For $0 < p < \infty$, since

$$\int_D 1_{D_\delta}(z)\frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) \to 0, \quad \delta \to 1.$$

We have

$$\begin{aligned}
&\frac{1}{|I|^p}\int_{S(I)} 1_{D_\delta}(z)|f'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^p\, dm(z) \\
&\quad \le \int_{S(I)} 1_{D_\delta}(z)\frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) \\
&\quad \le \int_D 1_{D_\delta}(z)\frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) \to 0 \quad (\delta \to 1)
\end{aligned}$$

for all arc I on ∂D. By Theorem 6.4 C_φ is compact.

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For $p = 0$, the proof is standard. Let (f_n) be a bounded sequence in B, $\|f_n\|_B^2 \le C$, and converges to 0 uniformly on compact subsets. From hypothesis (6.3)

$$\int_{\{\delta<|\varphi|<1\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) \to 0, \quad \delta \to 1.$$

Set

$$(6.4) \qquad \|C_\varphi(f_n)\|_{\mathscr{D}}^2 = \int_D |f_n'(\varphi(z))|^2 |\varphi'(z)|^2\, dm(z)$$
$$= \left(\int_{\{\delta<|\varphi|<1\}} + \int_{\{|\varphi|\le\delta\}}\right) |f_n'(\varphi(z))|^2 |\varphi'(z)|^2\, dm(z) = \mathrm{I} + \mathrm{II}.$$

So for any $\varepsilon > 0$, there exists $\delta < 1$, such that

$$(6.5) \qquad \mathrm{I} \le \|f_n\|_B^2 \int_{\{\delta<|\varphi|<1\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) \le C\varepsilon.$$

Since f_n' converges to 0 uniformly on $\{z \in D : |\varphi| \le \delta\}$, there is $N > 0$ such that if $n \ge N$,

$$(6.6) \qquad \mathrm{II} \le \varepsilon \int_{\{|\varphi|\le\delta\}} |\varphi'(z)|^2\, dm(z) \le \varepsilon \|\varphi\|_{\mathscr{D}}^2,$$

here $\varphi \in \mathscr{D}$, because under the condition of Proposition 6.5, $C_\varphi : B \to \mathscr{D}$ is bounded by Corollary 6.3 (1), so $\varphi \in \mathscr{D}$. Combining (6.4), (6.5) with (6.6) we have $\|C_\varphi(f_n)\|_{\mathscr{D}} \to 0$ $(n \to \infty)$ and $C_\varphi : B \to \mathscr{D}$ is compact by Lemma 1. □

Combining Corollary 6.3 (1) with Proposition 6.5 we get the following result,

COROLLARY 6.6. *Suppose that φ is an analytic self-map of D. Then the following statements are equivalent*:

(1) *$C_\varphi : B \to \mathscr{D}$ is compact.*
(2) *$C_\varphi : B \to \mathscr{D}$ is bounded.*
(3) $\displaystyle\int_D \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dm(z) < \infty.$

THEOREM 6.7. *Suppose that $0 < p < \infty$ and φ is an analytic self-map of D. Then $C_\varphi : B \to Q_0^p$ is compact if and only if*

$$(6.7) \qquad \lim_{|a|\to 1} \sup_{\{f\in B:\|f\|_B<1\}} \int_D |f'(\varphi(z))|^2 |\varphi(z)|^2 (1-|\sigma_a(z)|^2)^p\, dm(z) = 0.$$

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PROOF. Necessity. Suppose that $C_\varphi : B \to Q_0^p$ is compact, then $C_\varphi(\mathbf{B})$ is relatively compact in Q_0^p, $\mathbf{B}$ is the unit ball of B. Let $\varepsilon > 0$, then there is $(\varepsilon/4)$-net $f_1, \dots, f_m$ in $\mathbf{B}$. For f_i, $i = 1, 2, \dots, m$, there is $\delta > 0$, if $|z| > \delta$,

$$\int_D |(f_i \circ \varphi)'(z)|^2 |(1 - |\sigma_a(z)|^2)^p \, dm(z) < \frac{\varepsilon}{4}.$$

For any $f \in \mathbf{B}$, there existes $f_i \in \mathbf{B}$, $i \in \{1, \dots, m\}$ such that

$$\|(f - f_i) \circ \varphi\|_{Q^p} < \varepsilon/4.$$

So we get

$$\begin{aligned}
&\int_D |(f \circ \varphi)'(z)|^2 |(1 - |\sigma_a(z)|^2)^p \, dm(z) \\
&\quad \le 2 \int_D |(f \circ \varphi - f_i \circ \varphi)'(z)|^2 |(1 - |\sigma_a(z)|^2)^p \, dm(z) \\
&\quad\quad + 2 \int_D |(f_i \circ \varphi)'(z)|^2 |(1 - |\sigma_a(z)|^2)^p \, dm(z) < 2\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon,
\end{aligned}$$

if $|a| > \delta$ and for all $f \in B$ with $\|f\|_B < 1$. So (6.7) is proved.

Sufficiency. Suppose that $(f_n) \subset B$ with $\|f_n\|_B < 1$ and converges to 0 uniformly on compact subsets of D, we prove

$$\lim_{k \to \infty} \|C_\varphi(f_n)\|_{Q^p} = 0. \tag{6.8}$$

Let $\varepsilon > 0$, from (6.1), there is $\delta > 0$, such that for all f_n, $\|f_n\|_B < 1$,

$$\sup_{\delta < |a| < 1} \int_D |(f_n \circ \varphi)'(z)|^2 |(1 - |\sigma_a(z)|^2)^p \, dm(z) < \varepsilon. \tag{6.9}$$

For $a \in D$, $t \in (0, 1)$ and $D_t = \{z \in D : |\varphi(z)| > t\}$, set

$$T_t(a) = \int_{D_t} |(f_n \circ \varphi)'(z)|^2 (1 - |\sigma_a(z)|^2)^p \, dm(z).$$

Since $f_n \circ \varphi \in Q^p$, then $\lim_{t \to 1} T_t(a) = 0$. For each $a \in D$, there exists t_a such that $T_{t_a}(a) < \varepsilon$. The same as in the proof of Lemma 1.3 of [SmZh], $T_t(a)$ is a continuous function of a, so there is a neighbourhood $N(a) \subset D$ of a such that $T_{t_a}(z) < \varepsilon$, for all $z \in N(a)$. Since $\{a : |a| \le \delta\} \subset \bigcup_{a \in \{a : |a| \le \delta\}} N(a)$ and $\{a : |a| \le \delta\}$ is closed, there exist $N(a_1), \dots, N(a_m)$ such that $\{a : |a| \le \delta\} \subset \bigcup_{i=1}^m N(a_i)$. For a_i, $i = 1, \dots, m$, there exists t_{a_i} such that $T_{t_{a_i}}(z) < \varepsilon$, $z \in N(a_i)$, $i = 1, \dots, m$. Setting $t_0 = \max\{t_{a_1}, \dots, t_{a_m}\}$, $T_{t_0}(a) < \varepsilon$ for all $|a| \le \delta$. That is

$$\sup_{|a| \le \delta} \int_{D_{t_0}} |(f_n \circ \varphi)'(z)|^2 |(1 - |\sigma_a(z)|^2)^p \, dm(z) < \varepsilon. \tag{6.10}$$

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On the other hand, since f_n converges to 0 uniformly on compact subsets of D, there exists N, such that for all $n \geq N$, if $|w| \leq t_0$, $|f_n'(w)|^2 < \varepsilon$. Set $f = z$ in (6.7) we have $\varphi \in Q^p$, so

$$\sup_{|a|\leq\delta} \int_{D\setminus D_{t_0}} |(f_n \circ \varphi)'(z)|^2 (1 - |\sigma_a(z)|^2)^p \, dm(z) \leq \sup_{|a|\leq\delta} \varepsilon \int_D |\varphi'(z)|^2 (1 - |\sigma_a(z)|^2)^p \, dm(z) \leq \varepsilon \|\varphi\|_{Q^p}^2. \tag{6.11}$$

From (6.10) and (6.11), we get, for $n \geq N$,

$$\sup_{|a|\leq\delta} \int_D |(f_n \circ \varphi)'(z)|^2 (1 - |\sigma_a(z)|^2)^p \, dm(z) \leq (1 + \|\varphi\|_{Q^p}^2)\varepsilon. \tag{6.12}$$

Combining (6.9) with (6.12) we have $\|C_\varphi(f_n)\|_{Q^p} \to 0$ as $n \to \infty$. This completes the proof. □

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