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Equivalent Definitions of Infinite Positive Elements in Simple C^*-algebras

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Abstract. We prove the equivalence of three definitions given by different comparison relations for infiniteness of positive elements in simple C^*-algebras.

1 Introduction

J. Cuntz [4, 5] considered two comparison relations of arbitrary elements in a simple C^*-algebra A, which we denote by $a \lessapprox b$ and $a \lesssim b$, where a and b are in A. H. Lin and S. Zhang [15] introduced a comparison relation of positive elements a and b in a simple C^*-algebra, which we denote by $a \overset{\approx}{<} b$. In [15], Lin–Zhang gave the definition of an infinite positive element in a simple C^*-algebra by the comparison relation "$\lesssim$". Following the lines of Cuntz , Lin [13] defined another comparison relation of positive elements a and b in C^*-algebras, which we denote by $[a] \leq [b]$. The relation $[a] \leq [b]$ is a very useful tool for the classification of C^*-algebras, especially for the C^*-algebras with tracial topological rank zero. The comparison relations of elements in C^*-algebras have been studied and applied by many mathematicians (see [1–3, 6–12, 14, 16–20]). For the positive elements in a C^*-algebra, all the comparison relations are not equivalent to each other. In Section 2 we establish the relationship of the four comparison relations.

Inspired by Lin–Zhang [15], we can think of several definitions for the infiniteness of positive elements in simple C^*-algebras. In Section 3, we show that the definitions of infinite positive elements in a simple C^*-algebra defined by different comparison relations are equivalent.

Throughout this paper, we denote by A_+ the positive cone of a C^*-algebra A, by $\mathrm{Her}(a)$ the hereditary C^*-subalgebra of A generated by a, and by A^{**} the enveloping von Neumann algebra of A.

2 Comparisons of Positive Elements in C^*-algebras

First, we give various comparison relations of positive elements in C^*-algebras as follows.

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Definition 2.1 (i) ([4]) For any two elements a and b in a C^*-algebra A, we write $a \lessapprox b$ if there exist x and y in A such that $a = xby$. Write $a \approx b$ if $a \lessapprox b$ and $b \lessapprox a$.

(ii) ([5]) For any two elements a and b in A, we write $a \lesssim b$ if there are sequences $\{x_n\}$ and $\{y_n\}$ in A such that $x_n b y_n \to a$. Write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. In particular, if $a, b \in A_+$, then we can choose $y_n = x_n^*$.

(iii) ([15]) For any two positive elements a and b in A, we write $a \overset{\approx}{<} b$ if there is $r \in A$ such that $a \leq rbr^*$.

(iv) ([13, Definition 3.5.2]) Let a and b be two positive elements in A. We write $[a] \leq [b]$ if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in \mathrm{Her}(a)$, $v^*c,\ cv \in A$, $vv^* = p_a$, where p_a is the range projection of a in A^{**}, and $v^*cv \in \mathrm{Her}(b)$. We write $[a] = [b]$ if $v^*\,\mathrm{Her}(a)v = \mathrm{Her}(b)$.

Remark For the relation $[a] \leq [b]$, it would be convenient to use its equivalent definition, *i.e.*, there is $x \in A$ such that $x^*x = a$ and $xx^* \in \mathrm{Her}(b)$.

Lemma 2.2 *Let A be a C^*-algebra, $a, b \in A_+$, then the following statements hold:*

$$a \lessapprox b \Rightarrow a \overset{\approx}{<} b; \quad a \overset{\approx}{<} b \Rightarrow [a] \leq [b]; \quad [a] \leq [b] \Rightarrow a \lesssim b.$$

Proof (i) Since $a \lessapprox b$, there are $x, y \in A$ such that $a = xby$. Put $r = \frac{1}{\sqrt{2}}(x + y^*) \in A$; then

$$rbr^* = \frac{1}{2}(x + y^*)b(x + y^*)^* = \frac{1}{2}(xbx^* + y^*bx^* + xby + y^*by)$$
$$\geq \frac{1}{2}(xby + y^*bx^*) = \frac{1}{2}(a + a^*) = a,$$

and so $a \overset{\approx}{<} b$.

(ii) Since $a \overset{\approx}{<} b$, there is an $r \in A$ such that $a \leq rbr^*$, therefore $[a] \leq [rbr^*] \leq [b]$, that is, $[a] \leq [b]$.

(iii) Since $[a] \leq [b]$, there is $x \in A$ such that $x^*x = a$ and $xx^* \in \mathrm{Her}(b)$. Since $\{x^*[b(b + \frac{1}{n})^{-1}]x\}_{n \geq 1}$ converges to x^*x in the norm topology, we have

$$\lim_{n \to \infty} x^*[b(b + \frac{1}{n})^{-1}]x = x^*x = a,$$

that is, $a \lesssim b$. ■

Given any positive number ε, a continuous function f_ε is defined on the real line $\mathbb{R}$ by

$$f_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 2^{-1}\varepsilon \\ 2\varepsilon^{-1}(t - 2^{-1}\varepsilon) & \text{if } 2^{-1}\varepsilon \leq t \leq \varepsilon \\ 1 & \text{if } \varepsilon \leq t \end{cases}$$

Lemma 2.3 ([15, Lemma 1.3]) *Let A be a C^*-algebra, $a, b \in A_+$, if $a \lesssim b$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f_\varepsilon(a) \overset{\approx}{<} f_\delta(b)$.*

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Lemma 2.4 *Let A be a C^*-algebra, $a, b \in A_+$, if $a \lesssim b$, then for any $\varepsilon > 0$, $f_\varepsilon(a) \overset{\approx}{<} b$.*

Proof By Lemma 2.3, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f_\varepsilon(a) \overset{\approx}{<} f_\delta(b)$. So we need only to show that for the above δ, $f_\delta(b) \overset{\approx}{<} b$.

In fact, for $\delta > 0$, there is a non-negative continuous function g on $\mathbb{R}$ such that $f_\delta(t) = (g(t))^{\frac{1}{2}} t (g(t))^{\frac{1}{2}}$. Then $f_\delta(b) = (g(b))^{\frac{1}{2}} b (g(b))^{\frac{1}{2}}$ by continuous functional calculus. Therefore $f_\delta(b) \lessapprox b$, and hence $f_\delta(b) \overset{\approx}{<} b$. ■

Lemma 2.5 ([4, Proposition 1.3]) *Let $x \in A$ with polar decomposition $x = u|x|$, where $u \in A^{**}$. Then $uf(|x|)$ is in A for any continuous function f on $\mathbb{R}$ that vanishes in 0.*

3 Equivalent Definitions of Infinite Positive Elements

In this section we give the main result of this paper.

For an arbitrary positive element in the Pedersen ideal $P(A)$ (minimal dense two-sided ideal) of a simple C^*-algebra A we give the following definition of infinite positive elements.

Definition 3.1 ([15, Definition 1.1]) A positive element a in $P(A)$ is called *infinite*, if there are nonzero positive elements $b, c \in P(A)$ such that $bc = cb = 0$, (*i.e.*, $b \perp c$), $b + c \lesssim c$ and $b + c \lesssim a$. A non-positive element a in $P(A)$ is called infinite if a^*a is infinite.

Proposition 3.2 *Let A be a simple C^*-algebra and $P(A)$ be the Pedersen ideal of A. If $a \in P(A)$ is a positive infinite element, then $d \overset{\approx}{<} a$ for any positive element d in $P(A)$.*

Proof Since $a \in P(A)$ is an infinite positive element, there are two nonzero positive elements $b, c \in P(A)$ such that $b \perp c$, $b + c \lesssim c$ and $b + c \lesssim a$. Take $0 < \delta_0 < 1$ such that $f_{\delta_0}(b) \neq 0$. We can also take $0 < \delta < 1$ such that $f_\delta(f_{\delta_0}(b)) \neq 0$. Since $d \in P(A)$, there are $x_i, y_i \in A (i = 1, 2, \dots, n)$ such that

$$d = \sum_{i=1}^{n} x_i f_\delta(f_{\delta_0}(b)) y_i.$$

We prove this proposition in four steps:

Step 1. Construct positive elements $b_1, b_2, \dots, b_n$ in $\mathrm{Her}(c)$ such that $b_i \perp b_j$ if $i \neq j$.

Since $b + c \lesssim c$, there exists δ_1 with $0 < \delta_1 < \delta_0 < 1$ such that $f_{\delta_0}(b+c) \overset{\approx}{<} f_{\delta_1}(c)$ by Lemma 2.3. Then by Lemma 2.2 $[f_{\delta_0}(b + c)] \leq [f_{\delta_1}(c)] \leq [c]$. Similarly for δ_1, there exists δ_2 with $0 < \delta_2 < \delta_1 < \delta_0 < 1$ such that $[f_{\delta_1}(b + c)] \leq [f_{\delta_2}(c)] \leq [c]$.

Repeating this argument, there are

$$0 < \delta_n < \delta_{n-1} < \delta_{n-2} < \cdots < \delta_2 < \delta_1 < \delta_0 < 1$$

such that $[f_{\delta_i}(b + c)] \leq [f_{\delta_{i+1}}(c)] \leq [c] (i = 0, 1, 2, \dots, n - 1)$.

Since $[f_{\delta_{n-1}}(b + c)] \leq [f_{\delta_n}(c)]$, there is $x_1 \in A$ such that $x_1^* x_1 = f_{\delta_{n-1}}(b + c)$, $x_1 x_1^* \in \mathrm{Her}(f_{\delta_n}(c))$. Suppose that $x_1 = v_1|x_1|$ is the polar decomposition of x_1, where

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$v_1 \in A^{**}$. Then there is a $*$-isomorphism ϕ_1 from $\mathrm{Her}(f_{\delta_{n-1}}(b+c))$ into $\mathrm{Her}(f_{\delta_n}(c))$ defined by $\phi_1(x) = v_1 x v_1^*$ for any $x \in \mathrm{Her}(f_{\delta_{n-1}}(b+c))$.

Similarly there are $*$-isomorphisms ϕ_i from $\mathrm{Her}(f_{\delta_{n-i}}(b+c))$ into $\mathrm{Her}(f_{\delta_{n-i+1}}(c))$, $x_i \in A$ and $v_i \in A^{**}$ $(i = 2, 3, \dots, n)$ such that $\phi_i(x) = v_i x v_i^*$.

Since $b \perp c$, $f_{\delta_0}(b) + f_{\delta_0}(c) = f_{\delta_0}(b+c)$. Since $f_{\delta_0}(b) \leq f_{\delta_0}(b+c) \leq f_{\delta_{n-i}}(b+c)$ for each $i (1 \leq i \leq n)$, $\phi_i(f_{\delta_0}(b))$ is a well defined element in $\mathrm{Her}(f_{\delta_{n-i+1}}(c))$. Since $\phi_i(f_{\delta_0}(b)) \in \mathrm{Her}(f_{\delta_{n-i+1}}(c)) \subseteq \mathrm{Her}(f_{\delta_{n-i+1}}(b+c))$, $\phi_{i-1}\phi_i(f_{\delta_0}(b))$ is well defined.

Set $b_1 = \phi_1(f_{\delta_0}(b)) \in \mathrm{Her}(f_{\delta_n}(c))$. Then $b \perp b_1$. Set $b_2 = \phi_1\phi_2(f_{\delta_0}(b)) \in \mathrm{Her}(f_{\delta_n}(c)) \subseteq \mathrm{Her}(c)$. Since $\phi_2(f_{\delta_0}(b)) \in \mathrm{Her}(f_{\delta_{n-1}}(c))$ and $f_{\delta_0}(b) \perp \mathrm{Her}(f_{\delta_{n-1}}(c))$, $\phi_2(f_{\delta_0}(b)) \perp f_{\delta_0}(b)$. Hence $\phi_1\phi_2(f_{\delta_0}(b)) \perp \phi_1(f_{\delta_0}(b))$, that is $b_2 \perp b_1$. Since

$$b_2 = \phi_1\phi_2(f_{\delta_0}(b)) \in \mathrm{Her}(c), \quad b_2 \perp b.$$

Proceeding recursively, we obtain positive elements

$$b_i = \phi_1\phi_2\phi_3\cdots\phi_i(f_{\delta_0}(b)) \in \mathrm{Her}(f_{\delta_n}(c)) \subseteq \mathrm{Her}(c), i = 1, 2, 3, \dots, n,$$

then $b \perp b_i, b_i \perp b_j (i \neq j)$.

Step 2. For all the $b_i \in \mathrm{Her}(c)$ defined as above and $\delta > 0$, we have $f_\delta(f_{\delta_0}(b)) = V_i^* f_\delta(b_i) V_i$, where $V_i = v_1 v_2 \cdots v_{i-1} v_i$, $i = 1, 2, \dots, n$.

In fact, for any $i (1 \leq i \leq n)$, $b_i = \phi_1\phi_2\phi_3\cdots\phi_i(f_{\delta_0}(b)) = V_i f_{\delta_0}(b) V_i^*$. Since

$$v_i^* v_i = P_{|x_i|} = P_{(x_i^* x_i)^{\frac{1}{2}}} = P_{(f_{\delta_{n-i}}(b+c))^{\frac{1}{2}}} = P_{f_{\delta_{n-i}}(b+c)} \geq P_{f_{\delta_0}(b)},$$

where $P_{|x|}$ denote the range projection of $|x|$ in A^{**}, $v_i^* v_i f_{\delta_0}(b) = f_{\delta_0}(b)$. Since

$$v_i v_i^* = P_{|x_i^*|} = P_{(x_i x_i^*)^{\frac{1}{2}}} \leq P_{(f_{\delta_{n-i+1}}(c))^{\frac{1}{2}}} \leq P_{f_{\delta_{n-i+1}}(b+c)} = v_{i-1}^* v_{i-1},$$

$v_i v_i^* v_{i-1}^* v_{i-1} = v_i v_i^*$.

Since v_i is a partial isometry,

$$\begin{aligned} V_i^* V_i &= v_i^* v_{i-1}^* \cdots v_2^* v_1^* v_1 v_2 \cdots v_{i-1} v_i \\ &= v_i^* v_{i-1}^* \cdots v_3^* v_2^* v_2 v_2^* v_1^* v_1 v_2 v_3 \cdots v_{i-1} v_i \\ &= v_i^* v_{i-1}^* \cdots v_3^* v_2^* v_2 v_3 \cdots v_{i-1} v_i \\ &\ \ \vdots \\ &= v_i^* v_i. \end{aligned}$$

Then $b_i^2 = V_i f_{\delta_0}(b) V_i^* V_i f_{\delta_0}(b) V_i^* = V_i f_{\delta_0}(b) v_i^* v_i f_{\delta_0}(b) V_i^* = V_i f_{\delta_0}^2(b) V_i^*$.

In this way we have $b_i^m = V_i f_{\delta_0}^m(b) V_i^*$ for any positive integer $m \geq 2$. Since $f_\delta(t) = \lim_{M\to\infty} \sum_{m=1}^M c_m t^m$, where $c_m \in \mathbb{R}$,

$$f_\delta(b_i) = \lim_{M\to\infty} \sum_{m=1}^M c_m (b_i)^m = V_i \Big(\lim_{M\to\infty} \sum_{m=1}^M c_m (f_{\delta_0}(b))^m \Big) V_i^* = V_i f_\delta(f_{\delta_0}(b)) V_i^*$$

by functional calculus. And so $f_\delta(f_{\delta_0}(b)) = V_i^* V_i f_\delta(f_{\delta_0}(b)) V_i^* V_i = V_i^* f_\delta(b_i) V_i$.

Step 3. For $b_i \in \mathrm{Her}(c)$ and $\delta > 0$ defined as above, $d \stackrel{\approx}{<} (\sum_{i=1}^n f_\delta(b_i))^{\frac{1}{2}}$, where $d \in P(A)_+$, $i = 1, 2, \ldots, n$.

By *Step 2*, we have already proved

$$
\begin{aligned}
d &= \sum_{i=1}^n x_i f_\delta(f_{\delta_0}(b)) y_i = \sum_{i=1}^n x_i V_i^* f_\delta(b_i) V_i\, y_i \\
&= (\sum_{i=1}^n x_i V_i^* f_\delta^{\frac{1}{4}}(b_i))(\sum_{i=1}^n f_\delta(b_i))^{\frac{1}{2}}(\sum_{i=1}^n f_\delta^{\frac{1}{4}}(b_i) V_i\, y_i).
\end{aligned}
$$

Set $x_0 = \sum_{i=1}^n x_i V_i^* f_\delta^{1/4}(b_i)$ and $y_0 = \sum_{i=1}^n f_\delta^{1/4}(b_i) V_i\, y_i$. So it suffices to prove $x_0, y_0 \in A$. Therefore we need only to show $f_\delta^{1/4}(b_i) V_i \in A$ for each $i = 1, 2, \ldots, n$, and hence to show $b_i V_i \in A$ since $f_\delta^{1/4}(0) = 0$.

Since $b_i = V_i f_{\delta_0}(b) V_i^*$, $b_i V_i = V_i f_{\delta_0}(b) V_i^* V_i = V_i f_{\delta_0}(b) v_i^* v_i = V_i f_{\delta_0}(b)$. So it suffices to prove $V_i f_{\delta_0}(b) \in A (i = 1, 2, \ldots, n)$.

For this purpose, we first prove $v_i \,\mathrm{Her}(f_{\delta_{n-i}}(b+c)) \subseteq \overline{\mathrm{Her}(f_{\delta_{n-i+1}}(b+c))A}$ for each $i = 1, 2, \ldots, n$.

Since $v_i f_{\delta_{n-i}}^{\frac{1}{2}}(b+c) \in A$ and

$$
v_i f_{\delta_{n-i}}^{\frac{1}{2}}(b+c) v_i^* = H_i(f_{\delta_{n-i}}^{\frac{1}{2}}(b+c)) \in \mathrm{Her}(f_{\delta_{n-i+1}}(c)) \subseteq \mathrm{Her}(f_{\delta_{n-i+1}}(b+c)),
$$

we have

$$
\begin{aligned}
&v_i f_{\delta_{n-i}}(b+c) = \\
&\qquad v_i f_{\delta_{n-i}}^{\frac{1}{2}}(b+c) v_i^* v_i f_{\delta_{n-i}}^{\frac{1}{2}}(b+c) \in \mathrm{Her}(f_{\delta_{n-i+1}}(b+c))A \subseteq \overline{\mathrm{Her}(f_{\delta_{n-i+1}}(b+c))A}.
\end{aligned}
$$

Then there are $\{x_k\}_{k\geq 1} \subseteq A$ and $\{a_k\}_{k\geq 1} \subseteq A$ such that

$$
v_i f_{\delta_{n-i}}(b+c) = \lim_{k\to\infty} f_{\delta_{n-i+1}}(b+c) x_k f_{\delta_{n-i+1}}(b+c) a_k.
$$

Since $v_{i-1} f_{\delta_{n-i+1}}(b+c) \in \mathrm{Her}(f_{\delta_{n-i+2}}(b+c))A$ and $x_k f_{\delta_{n-i+1}}(b+c) a_k \in A$,

$$
v_{i-1} v_i f_{\delta_{n-i}}(b+c) = \lim_{k\to\infty} v_{i-1} f_{\delta_{n-i+1}}(b+c) x_k f_{\delta_{n-i+1}}(b+c) a_k \in \overline{\mathrm{Her}(f_{\delta_{n-i+2}}(b+c))A}.
$$

Proceeding recursively, we obtain

$$
V_i f_{\delta_{n-i}}(b+c) = v_1 v_2 \cdots v_{i-1} v_i f_{\delta_{n-i}}(b+c) \in \overline{\mathrm{Her}(f_{\delta_n}(b+c))A}.
$$

Then for any $y \in V_i \,\mathrm{Her}(f_{\delta_{n-i}}(b+c))$, there are $\{y_k\}_{k\geq 1} \subseteq A$ such that

$$
y = \lim_{k\to\infty} V_i f_{\delta_{n-i}}(b+c) y_k f_{\delta_{n-i}}(b+c) \in \overline{\mathrm{Her}(f_{\delta_n}(b+c))A} \subseteq A.
$$

Thus $V_i \,\mathrm{Her}(f_{\delta_{n-i}}(b+c)) \subseteq \overline{\mathrm{Her}(f_{\delta_n}(b+c))A} \subseteq A$. Since $f_{\delta_0}(b) \in \mathrm{Her}(f_{\delta_{n-i}}(b+c))$, $V_i f_{\delta_0}(b) \in A$.

Step 4. At last, we prove that for $b_i \in \mathrm{Her}(c)$ defined as above and $\delta > 0$,

$$\sum_{i=1}^{n} f_{\frac{\delta}{2}}(b_i) \overset{\approx}{<} a, \; i = 1, 2, \ldots, n.$$

Since $\sum_{i=1}^{n} b_i \in \mathrm{Her}(c)$, there is a sequence $\{z_m\}_{m\geq 1} \subseteq A$ such that

$$\Big(\sum_{i=1}^{n} b_i\Big)^2 = \lim_{m\to\infty} cz_m c^2 z_m c.$$

Then $\sum_{i=1}^{n} b_i \sim (\sum_{i=1}^{n} b_i)^2 \precsim c^2 \sim c$. Since $c \precsim b + c \precsim a$, $\sum_{i=1}^{n} b_i \precsim a$. By Lemma 2.4, for the above $\delta > 0$, $\sum_{i=1}^{n} f_{\frac{\delta}{2}}(b_i) = f_{\frac{\delta}{2}}(\sum_{i=1}^{n} b_i) \overset{\approx}{<} a$.

Since $f_{\delta}^{\frac{1}{2}}(b_i) = f_{\delta}^{\frac{1}{4}}(b_i) f_{\frac{\delta}{2}}(b_i) f_{\delta}^{\frac{1}{4}}(b_i)$ for each $i = 1, 2, \ldots, n$ and $b_i \perp b_j$ $(i \neq j)$,

$$\begin{aligned}\Big(\sum_{i=1}^{n} f_{\delta}(b_i)\Big)^{\frac{1}{2}} &= \sum_{i=1}^{n} f_{\delta}^{\frac{1}{2}}(b_i) = \sum_{i=1}^{n} f_{\delta}^{\frac{1}{4}}(b_i) f_{\frac{\delta}{2}}(b_i) f_{\delta}^{\frac{1}{4}}(b_i) \\ &= \Big(\sum_{i=1}^{n} f_{\delta}^{\frac{1}{4}}(b_i)\Big)\Big(\sum_{i=1}^{n} f_{\frac{\delta}{2}}(b_i)\Big)\Big(\sum_{i=1}^{n} f_{\delta}^{\frac{1}{4}}(b_i)\Big) \\ &= f_{\delta}^{\frac{1}{4}}\Big(\sum_{i=1}^{n} b_i\Big)\Big(\sum_{i=1}^{n} f_{\frac{\delta}{2}}(b_i)\Big) f_{\delta}^{\frac{1}{4}}\Big(\sum_{i=1}^{n} b_i\Big).\end{aligned}$$

Then $(\sum_{i=1}^{n} f_{\frac{\delta}{2}}(b_i))^{\frac{1}{2}} \overset{\approx}{<} \sum_{i=1}^{n} f_{\frac{\delta}{2}}(b_i)$. Therefore

$$d \overset{\approx}{<} \Big(\sum_{i=1}^{n} f_{\delta}(b_i)\Big)^{\frac{1}{2}} \overset{\approx}{<} \sum_{i=1}^{n} f_{\frac{\delta}{2}}(b_i) = f_{\frac{\delta}{2}}\Big(\sum_{i=1}^{n} b_i\Big) \overset{\approx}{<} a,$$

and so $d \overset{\approx}{<} a$. ■

Theorem 3.3 *Let A be a simple C^*-algebra and $P(A)$ be the Pedersen ideal of A, then the following definitions of the infinite positive element $a \in P(A)$ are equivalent:*

(i) *There are nonzero positive elements b and c in $P(A)$ such that $bc = cb = 0$ (i.e., $b \perp c$), $b + c \precsim c$ and $b + c \precsim a$.*

(ii) *There are nonzero positive elements b and c in $P(A)$ such that $bc = cb = 0$ (i.e., $b \perp c$), $b + c \overset{\approx}{<} c$ and $b + c \overset{\approx}{<} a$.*

(iii) *There are nonzero positive elements b and c in $P(A)$ such that $bc = cb = 0$ (i.e., $b \perp c$), $[b + c] \leq [c]$ and $[b + c] \leq [a]$.*

Proof (i) $\Rightarrow$ (ii) If a is an infinite positive element, then there are nonzero positive elements b and c in $P(A)$ such that $bc = cb = 0$, $b + c \precsim c$ and $b + c \precsim a$. Clearly, c is also an infinite element. Then it follows from Proposition 3.2, $b + c \overset{\approx}{<} c$ and $b + c \overset{\approx}{<} a$ since $b + c \in P(A)_+$.

(ii) $\Rightarrow$ (iii) This is obvious by Lemma 2.2.

(iii) $\Rightarrow$ (i) This is also obvious by Lemma 2.2. ■

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Remark The authors were told by the referee that by using [18, Proposition 2.4(iv)] one can similarly provide a slightly shorter proof of the Proposition 3.2 for the comparison "$\lessapprox$", and so the equivalence of the infiniteness with the comparison "$\lessapprox$" to the other comparisons in Theorem 3.3.

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