RELATIONSHIP BETWEEN THE SYSTEM OF ASTRONOMICAL CONSTANTS AND THE RADAR DETERMINATIONS OF THE ASTRONOMICAL UNIT

By D. O. MUHLEMAN,

Jet Propulsion Laboratory,
California Institute of Technology (1)
and The Harvard College Observatory,
Harvard University.

RÉSUMÉ. — Cette Communication est consacrée à la discussion de la détermination de l'unité astronomique à l'aide de radar et à une analyse approfondie des erreurs provenant des techniques employées. On utilise les relations théoriques entre les constantes pour construire un ensemble cohérent de valeurs de constantes à partir de ces résultats.

ABSTRACT. — This paper is devoted to an exact discussion of the determination of the astronomical unit with radar and to an extensive error analysis of the technique. Theoretical relationships between the constants are used to construct a consistent set of numerical values of constants based on these results.

ZUSAMMENFASSUNG. — Diese Arbeit enthält eine genaue Untersuchung der Bestimmung der Astronomischen Einheit mit Radar und eine eingehende Fehleranalyse ihrer technischen Durchführung. Unter Benutzung der theoretischen Beziehungen zwischen den Konstanten wird aus diesen Ergebnissen ein widerspruchsfreies System numerischer Werte der Konstanten aufgestellt.

Резюме. — Это сообщение посвящено обсуждению определения астрономической единицы при помощи радара и полному анализу погрешностей этой техники. Исходя из теоретических соотношений между постоянными, автор дает связную систему значений астрономических постоянных.

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(1) This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.

1. **Introduction.** — A powerful new technique for the determination of some of the astronomical constants is offered by the development of Radar Astronomy methods. Important radar experiments have been successfully completed on the Moon, Venus, Mercury, and Mars by investigators at the Jet Propulsion Laboratory in particular. This paper deals with the analysis of these observations with the special attention to the determination of the AU. An attempt has been made to accomplish a detailed error analysis of the methods employed. The major new contribution of this paper is a verification of the results of Muhleman, *et al.* [1] with further radar observations of Venus and Mercury.

2. **Velocity of light.** — A precise value of the velocity of light has not been a particular concern to astronomical questions until the present time. The adopted value of c as given in the *Nautical Ephemeris* is a very old determination by Newcomb and is well known to be grossly in error. The radar determinations of the astronomical unit and the determination of associated constants by radar and radio-tracking of artificial space vehicles are intimately concerned with a precise measurement of the velocity of light, however. It will be shown that, even though the modern value of c is known reliably to six figures, the uncertainty in the light-velocity determinations is the major single source of error in the radar measurements when used in terms of kilometers.

An excellent survey of the classical determinations has been given by Bergstrand [2]. A recent survey of the important light-velocity determinations since 1946 has been given by DuMond [3]. His results are shown in table I.

TABLE I.

Modern velocity of light determinations.

Author.	Date	Method.	c. (km/s.)
Aslakson [4]	1949	Shoran	299 792 ± 3.5
Hansen and Bol [5]	1950	Cavity resonance	299 789.3 ± 1.3
Essen [6]	1950	» »	299 792.5 ± 1.0
Bergstrand [7]	1951	Geodimeter	299 793.1 ± 0.32
Froome [8]	1952	Microwave interferometer	299 792.6 ± 0.7
Mackenzie [9]	1953	Geodimeter	299 792.4 ± 0.4
Froome [10]	1954	Microwave interferometer	299 792.7 ± 0.3
Plyler *et al.* [11]	1955	Infrared spectrometer	299 792 ± 6
Florman [12]	1955	Microwave interferometer	299 795.1 ± 1.9
Bergstrand [13]	1957	Geodimeter	299 792.8 ± 0.34
		Geodimeter survey	299 792.85 ± 0.16
Froome [14]	1958	Microwave	299 792.50 ± 0.15

https://doi.org/10.1017/S0074180900104930 Published online by Cambridge University Press

The best single determination is apparently the value found by Froome of

$$299\,792.50 \pm 0.10 \text{ km/s}.$$

which he obtained by a microwave interferometer technique at 74 500 Mc. I have computed the mean value from table I, weighting the values with the reciprocal-squares of the quoted uncertainties, and found

$$299\,792.63 \pm 0.08 \text{ km/s}.$$

This result is in excellent accord with Froome's individual measurement which is partially due to the large weight assigned to Froome's 1958 determination. The general agreement to a few parts in 10^6 of all of the modern values shown in table I is reassuring, and it appears highly unlikely that a systematic error larger than 0.3 km/s could exist.

The International Union of Geodesy and Geophysics, on the recommendation of the XII General Assembly of the International Scientific Radio Union, has adopted the value of

$$299\,792.5 \pm 0.4 \text{ km/s}.$$

This value has been used in the radar determinations of the AU.

3. **Determination of the AU by radar at the 1961 inferior conjunction of Venus.** — Radar observations have been obtained for Venus around the 1961 inferior conjunction by several groups. The resulting values for the astronomical unit are shown in table II. All the determinations are in agreement. However, Newcomb's tables of the Sun and Venus were employed in all cases, which, if they cause an important error at all, would affect each determination in essentially the same way. A detailed discussion of these effects is presented below.

TABLE II.

Astronomical unit determinations from radar observations of Venus (*).

Good radar methods (**).	AU (km).	$\pi_{\odot}$(s).
D. Muhleman *et al.*	149 598 640 ± 250	8.7941379 ± 0.000015
G. Pettinghill *et al.*	149 597 850 ± 400	8.7941849 ∓ 0.000026
D. Muhleman (revision of Pettinghill's value)	149 598 100 ± 400	8.7941705 ± 0.000026
Marginal radar methods.		
Thompson *et al.*	149 601.000 ± 5000	8.7940 ± 0.0003
Maron *et al.*	149 596 000	8.7943
Kotellnikov	149 599 500 ± 800	8.7941 ± 0.00005

(*) Muhleman [15].
(**) Those that observed Venus over a sufficiently long arc to remove the major part of the errors from the ephemerides.

A. *Instrumentation.* — Details of the computations of Muhleman *et al.* [1]are described in this section. A complete discussion of Pettingill's result can be found in Pettingill. *et al.* [17]. The observations reported in the latter paper have been used to compute a slightly revised value of the AU. The observations of Muhleman *et al.*, were made at the Goldstone station of the Jet Propulsion Laboratory, California Institute of Technology, with three fundamentally different radar receiving systems. The observations consisted of the Doppler frequency shift on the 2 388 Mc carrier and measurements of the propagation time to Venus and back to Earth by modulating the carrier with either a regular square wave or a pseudo-random code.

The frequency reference for the Doppler velocity measurements was an Atomichron cæsium-resonance line which had a measured stability of 1 or 2 parts in 10^{10} over a period of about 5 mn. All other reference frequencies in the receiver were coherently derived from the standard in such a manner that frequency errors introduced into the system were subsequently subtracted out at some other point in the closed-loop system. Consequently, the measurements of the Doppler frequency shift are probably accurate to better than 1 part in 10^7. This uncertainty is far smaller than that due to the velocity of light.

The systems of modulation employed by the two methods of measuring the propagation time were designed to have a range resolution of about 100 km. The overall accuracies of this system are on the order of 100 km except for the uncertainty of *c*, i. e., about 0.0003 s for the Earth-Venus distance.

B. *Preparation of the ephemeris.* — The Doppler frequency shift and the propagation time must be computed from the ephemerides with precision for the comparison with observations. The total propagation time is given by :

1. the time for the signal to travel from the position of the transmitting antenna at time 1 to the *surface* of Venus at time 2;

2. plus the time for the signal to travel from the surface of Venus at time 2 to the position of the receiving antenna at time 3.

The actual epoch for each observation was taken to be time 3 and the arguments for entries into the tables of the Sun and Venus were computed with a simple iteration scheme. The Doppler frequency shift is a function of

1. the velocity of the *center of mass* of Venus at the instant the wave front strikes the surface of the planet with respect to the position and velocity of the transmitting station at time 1, $\dot{R}_{12}$;

2. the velocity and position of the receiving station at the instant the reflected wave front reaches the receiving station, with respect to

the velocity of the center of mass of Venus at the instant of reflection, time 2, $\dot{R}_{23}$.

The equation for the conversion of the ephemeris velocities, $\dot{R}_{12}$ and $\dot{R}_{23}$, to Doppler frequency shift has been derived by Muhleman [1] to the second ordre in $\frac{v}{c}$ and is

$$(1) \qquad (\tilde{\nu} - \nu) = -\nu\left(\frac{\dot{R}_{12}}{C} + \frac{\dot{R}_{23}}{C} - \frac{\dot{R}_{12}\dot{R}_{23}}{C^2} - \frac{\dot{R}_{23}^2}{C^2}\right),$$

where ν is transmitter frequency and $\tilde{\nu}$ is the received frequency at time 3.

The actual values used in the analysis of the radar observations were computed with a tracking program written for the I. B. M. 7090 computer. The co-ordinates to be smoothed were obtained directly from Newcomb's tables of the Sun and Venus with corrections for known errors. In particular, a correction of $-4''.78$ T was applied to the mean anomaly of the Sun after Clemence [18]. An n-body numerical integration, starting with injection position and velocity, was compared with the co-ordinates written on a magnetic tape from the Newcomb tables, and corrections to the injection conditions were derived using a least-squares iterative procedure. Several iterations yielded the best injection values over a 120-day arc for Venus and a 70-day arc for the Earth. These residuals were reduced to a few parts in 10^7 which is consistent with the roundoff in the tabulated data. Velocity data were obtained at each epoch of interest as a consequence of the Runge-Kutta numerical integration procedure. The velocities obtained in this manner are smooth to seven figures and probably accurate to a few parts in 10^6. The ephemerides obtained with the above technique are considered a smooth equivalent to the numerical tables of Newcomb, including only the change in the argument M referred to above. Subsequently in this paper, the ephemerides will be referred to as the Newcomb ephemerides.

Duncombe [19] has obtained a set of corrections to Newcomb's elements from the Venus observations over a period from 1795 to 1949. The published corrections are :

for Earth :

$$\begin{aligned}
\Delta e_{\oplus} &= -0''.10 \pm 0''.01 + 0''.00\,\mathrm{T},\\
\Delta \varepsilon &= +0''.04 \pm 0''.01 - (0''.29 \pm 0''.03)\,\mathrm{T},\\
\Delta \mathrm{L}_{\oplus} &= -0''.39 \pm 0''.05 + (0''.45 \pm 0''.15)\,\mathrm{T},\\
e\Delta\varpi &= -0''.07 \pm 0''.03 - 0''.09\,\mathrm{T};
\end{aligned}$$

for Venus :

$$
\begin{aligned}
\Delta l_{♀} &= +0''.10 \pm 0''.06 + (0''.53 \pm 0''.18)\,\mathrm{T},\\
\Delta e_{♀} &= -0''.12 \pm 0''.03 + 0''.01\,\mathrm{T},\\
e_{♀}\,\Delta\varpi_{♀} &= +0''.01 \pm 0''.04 + 0''.04\,\mathrm{T},\\
\Delta i_{♀} &= +0''.08 \pm 0''.03 - 0''.02\,\mathrm{T},\\
\sin_{♀}\,\Delta\Omega_{♀} &= +0''.21 \pm 0''.03 + 0''.02\,\mathrm{T}.
\end{aligned}
$$

The corrections actually used were supplied by Duncombe [20] and are only slightly different :

for the Earth :

$$\Delta e_{\oplus} = -0''.113\,\mathrm{T},$$
$$\Delta\varepsilon = +0''.045 - 0''.29\,\mathrm{T},$$
$$\Delta M_{\oplus} = +4''.78\,\mathrm{T} \text{ (already applied in the Newcomb ephemerides);}$$

for Venus :

same as above.

The Duncombe corrections were incorporated into the program which evaluated the Newcomb theory, and a new ephemeris was generated utilizing the same technique as before. This ephemeris has been called the Duncombe ephemeris.

C. *Results.* — Observations of Venus were made at 10-s intervals over continuous periods of from 5 mn to 1 h. This was normally done daily for the Doppler measurements and the two ranging-systems measurements. Each set of observations was used to compute a separate estimate of the AU, which was computed with an iterative least-squares procedure which minimized the observations minus the calculated value by computing a correction to the AU value used in the previous iteration. The calculations were performed for both the Newcomb ephemeris and the Duncombe ephemeris. The r. m. s. residuals for the velocity observations were about $\pm$ 0.1 m/s, and about $\pm$ 200 km was obtained for the range residuals. Actually the residuals varied somewhat with the distance to Venus because of the decrease in the radar-echo power with distance.

The computed AU estimates from the velocity observations are shown in figure 1. This figure shows that the estimates of the AU rapidly diverge downward as conjunction (April 11) is approached from the east and return from above immediately after conjunction has passed. The effect of the Duncombe corrections was to raise the estimates on March 23 by 1200 km and on April 7 by about 7 000 km. Similarly, on April 13 the estimate was lower by 8 900 km and on May 3, by 400 km.

https://doi.org/10.1017/S0074180900104930 Published online by Cambridge University Press

Clearly the effect is due to the sensitivity of the Doppler velocity (range rate) to errors in the ephemerides as the velocity gets small. The primary correction of Duncombe is to advance the longitude of Venus by about $0''.55$ relative to that of the Earth. This was apparently not enough to completely straighten the curve. Muhleman *et al.* [1], have shown that the effect of an error in the longitudes of Venus and the Earth in the determination of the AU is approximately (near conjunction)

$$\delta(\mathrm{AU}) \simeq A_{\oplus} \cot an(l_{\female} - l_{\oplus})\, \delta(l_{\female} - l_{\oplus}) \tag{2}$$

which is very similar to the behavior shown in figure 1. A more exact analysis of this problem will be given below.

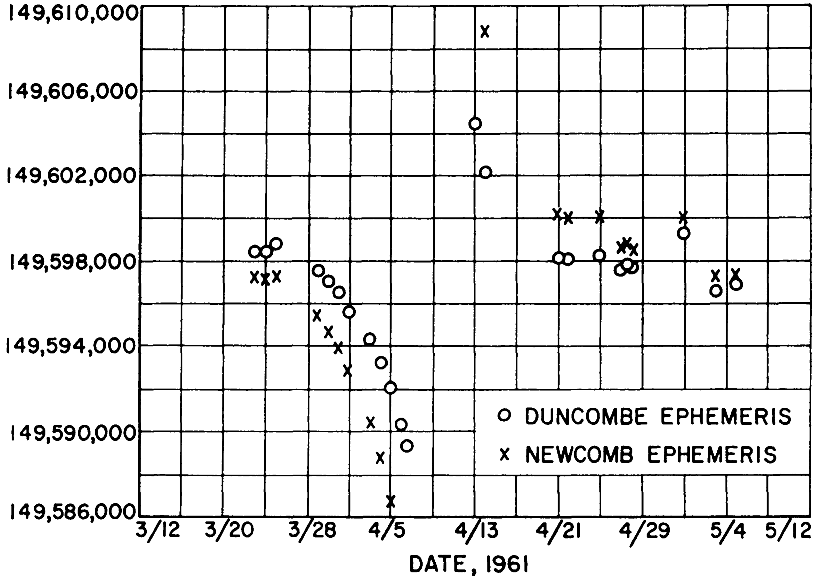

Fig. 1. — The astronomical unit computed from the Goldstone velocity observations.

The estimates of the AU computed from the range measurements from the system employing the pseudo-random code modulation are shown in figure 2. These observations are all post-conjunction. A linear trend with date is evident from the figure, the slope of which was decreased by applying the Duncombe corrections. Muhleman *et al.* [1], have shown that the effect on the AU determinations from range data due to only an error in the relative planetary longitudes is approximately

$$\delta(\mathrm{AU}) \simeq A_{\oplus}\left(\frac{r_{\female} r_{\oplus}}{r^2}\right) \sin(l_{\female} - l_{\oplus})\, \delta(l_{\female} - l_{\oplus}), \tag{3}$$

where $r_♀$ and $r_⊕$ are the heliocentric distances to the planets and r is the distance between them. The equation agrees well with the effect observed in figure 2.

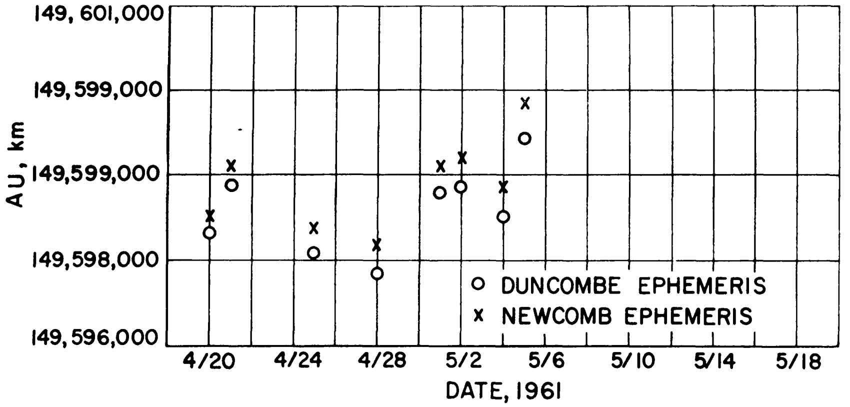

Fig. 2. — The astronomical unit derived from the Goldstone range observations.

The measured radar propagation times to Venus published by Pettingill *et al.* [17] were used to compute the estimates of the AU shown in figure 3. The agreement between these estimates and those computed by Pettingill is excellent. A trend similar to that predicted by equation (3) is again evident in the estimates.

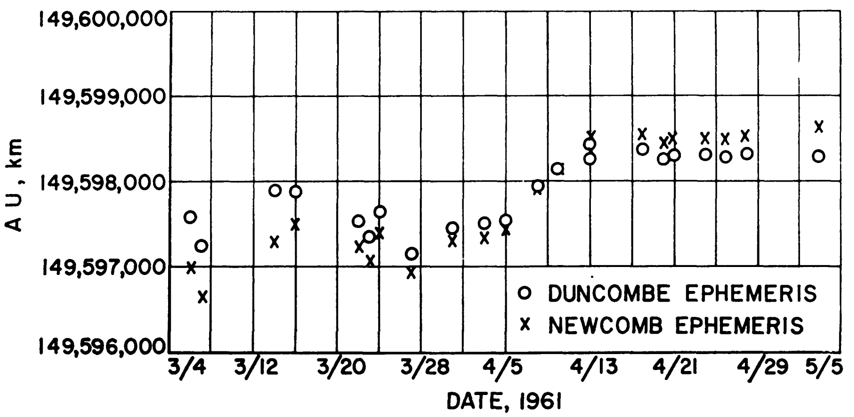

Fig. 3. — The astronomical unit computed from the Millstone observations.

The reduction of all of the AU estimates to a single result is a considerable task. Because of the apparent errors in the ephemerides (after Duncombe's corrections) it is necessary to proceed somewhat arbitrarily.

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Equation (2) was used to extrapolate the Doppler-AU estimates toward the east and west elongations where errors in longitude would have a minimal effect. However, an error in $e''\Delta\varpi''$ may be significant at these points. Equation (3) was employed to interpolate the range-AU estimates at conjunction (clearly, the total effect of the Duncombe corrections is nearly zero at conjunction). The results of this procedure are :

1.	Doppler near eastern conjunction..........	149 598 750 ± 200 km
2.	Doppler near western elongation...........	149 598 000 ± 1000 »
3.	Range at conjunction......................	149 598 500 ± 150 »
4.	Range at conjunction......................	149 598 800 ± 150 »

where the value 4 was computed from range observations from the second ranging system which was independent of the first system to a large degree. The uncertainties attached to the above values are estimates based primarily on the scattering in the estimates. The systematic errors will be considered below.

The final value of the AU is the mean of the four figures above with weights equal to the reciprocal variances :

$$149\,598\,640 \pm 200 \text{ km}.$$

The value computed from Pettingill's observations utilizing equation (3) for interpolation to conjunction is

$$149\,598\,100 \pm 400 \text{ km},$$

where the uncertainty was taken from Pettingill *et al.* [17].

4. **Determination of the AU by radar at the 1962 inferior conjunction of Venus.** — The observational program on Venus for 1961 was repeated around the 1962 inferior conjunction. The techniques that were employed in the latter observations were somewhat different. In 1961 two antennas separated by 10 km were operated as a transmitter and receiver pair and consequently yielded continuous runs of data. However, it was necessary to use a single antenna in 1962 as both the transmitter and the receiver. This was done by transmitting for the propagation time from the Earth to Venus and switching to the receiver mode for a similar length of time. This reduced the observation time by one-half. Furthermore, it was decided that a comparison ephemeris should be constructed over an arc much longer than the 100-day arcs utilized in the previous analysis in order to cover both observational periods with one fit. The ephemeris was prepared in essentially the manner described above, but 10-year arcs were employed as reported by Peabody and Block [21]. The residuals in positions relative to the Newcomb tables (after a correction of $M'' = +4''.78\,T$) exhibited oscillations as large as 5×10^{-7} AU in the radius vectors and $0''.1$ in the

longitudes and latitudes with the sidereal periods. These residuals have had serious effects on the AU results. Primarily for this reason the 1962 results reported here as to be considered as preliminary. However, in all cases the values of the AU deduced agree to within the accuracy of the analysis to those found in 1961.

A. *Calculation of the astronomical unit.* — The AU has been obtained by comparing the observations to the values computed from the astronomical tables using a first guess of the AU for entry into the tables and then computing a second estimate of the AU from the differences by the classical least-squares technique. The process is repeated until the r. m. s. differences (residuals) obtained in the n-th iteration are not significantly smaller than those obtained in the $(n-1)$th iteration. Thus the AU is found by assuming that the astronomical tables are correct except for one parameter, the AU. In general, a given residual is given by (after a Taylor's expansion to first order)

$$(4) \qquad (R_0 - R_c)_i = \left(\frac{\partial R_c}{\partial \alpha_1}\right)_i \delta\alpha_1 + \left(\frac{\partial R_c}{\partial \alpha_2}\right)_i \delta\alpha_2 + \ldots + \left(\frac{\partial R_c}{\partial \alpha_n}\right)_i \delta\alpha_n,$$

where R_0 is the observed range (for example) and R_c is the range computed from the tables with an assumed value of the AU. The $\delta\alpha$'s are the (unknown) errors in the significant parameters of the planetary theory *including* the AU. Thus, the method employed here assumes that all of the $\delta\alpha$'s are zero except δ (AU). When the set of equations (4) (the normal equations) are solved in a least-squares sense the resulting correction for the AU in the case where all of the other $\delta\alpha$'s are zero is

$$(5) \qquad \delta(\mathrm{AU}) = \frac{-\sum_i \left(\frac{\partial R_c}{\partial(\mathrm{AU})}\right)_i (R_M - R_c)_i}{\sum_i \left(\frac{\partial R_c}{\partial(\mathrm{AU})}\right)_i^2}.$$

A similar expression can be written for δ(AU) for the Doppler observations. The solution for a general set of $\delta\alpha$'s merely involves an inversion of the matrix of coefficient from equation (4).

A total of 52 Doppler runs was made over the period from October 11 to December 17, 1962. The average number of samples per run was 141 and the *average* standard deviation of the final residuals for each run was 2.54 c/s. The actual standard deviations are a function of signal-to-noise ratio and they vary from about 3.5 c/s at the beginning and end of the observational period to about 1.2 c/s at the time of conjunction. Clearly, the uncertainty in a given estimate of the AU from any single run depends further on the total Doppler shift at that time and is widely variable. At the points of greatest interest in the case of the Doppler, i.e., the farthest way from conjunction where the Doppler shift is the greatest, the following uncertainties in the AU have been computed *based entirely on the above internal statistics* assuming no correlation between samples :

October 21 : $\sigma_{AU} = 195$ km ;
December 12 : $\sigma_{AU} = 209$ km.

The resulting estimates of the AU using the Newcomb ephemerides are shown in figures 4 and 5.

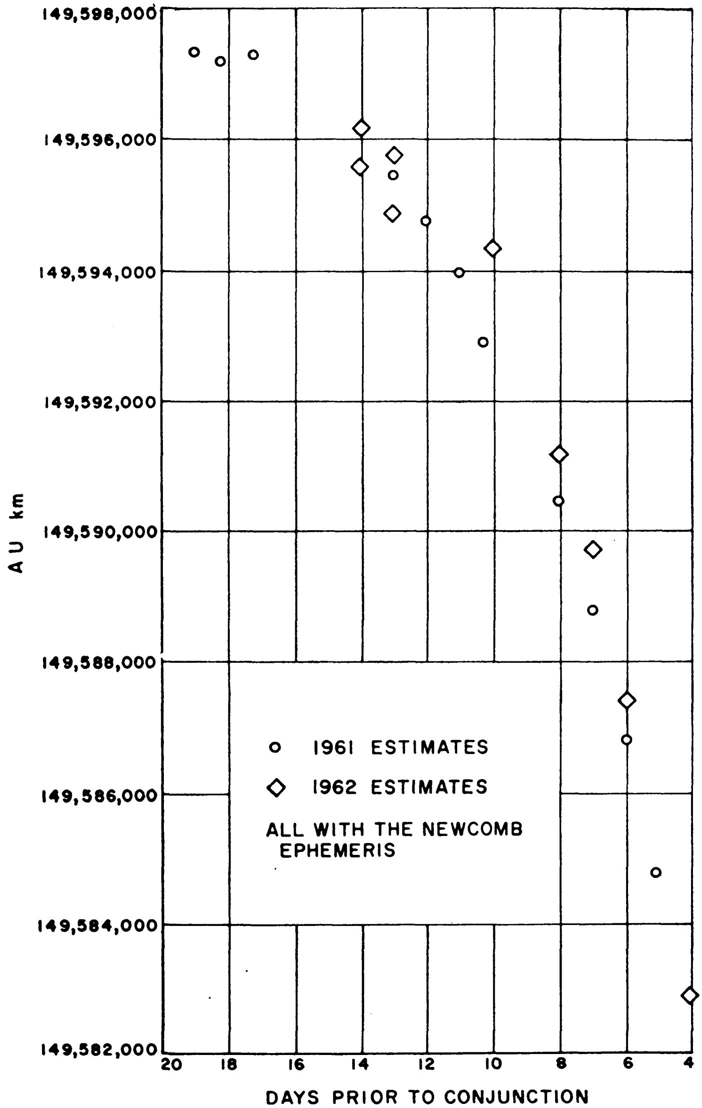

Fig. 4. — Comparison between the 1961 and 1962 determinations of the astronomical unit by Doppler velocity.

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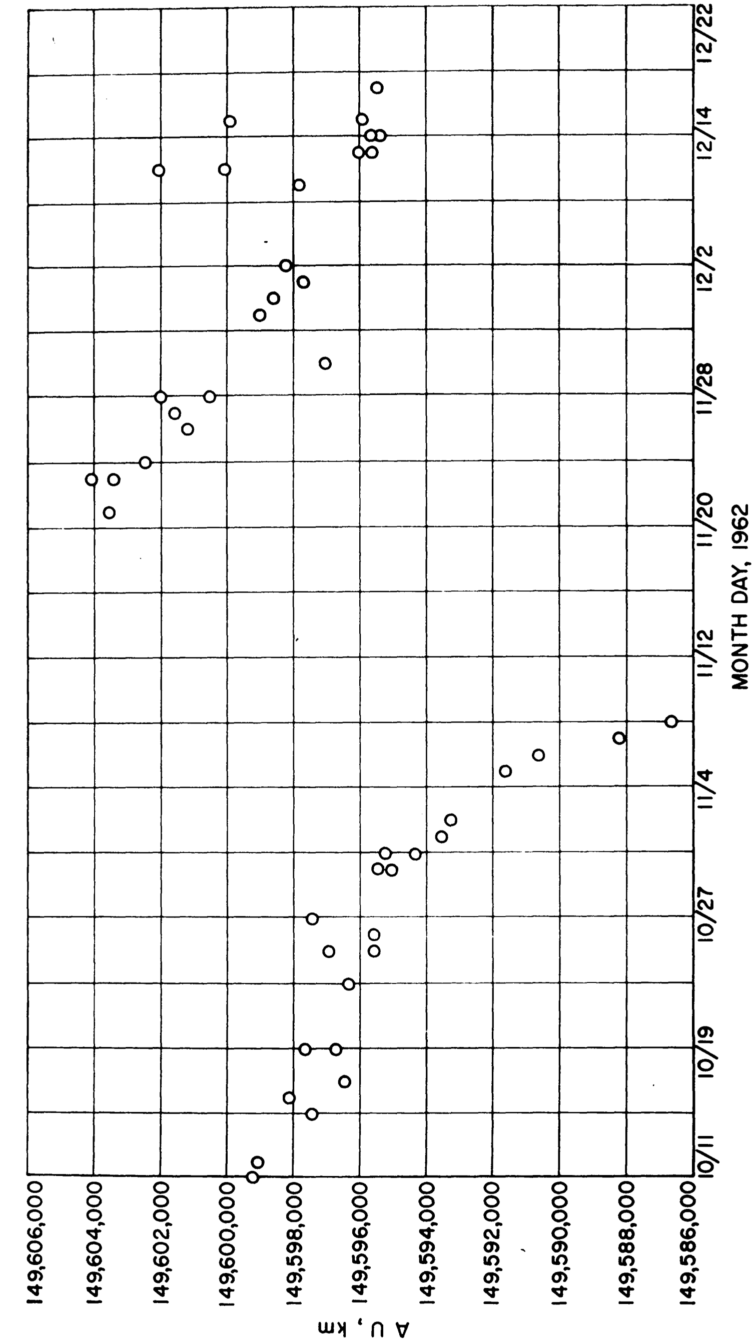

Fig. 5. — The astronomical unit estimates from the 1962 Doppler observations using the Newcomb ephemerides.

A total of ten estimates of the AU have been made from the range data over a period from November 8 to December 15, 1962. The average number of samples per run was 472, and the average standard deviation was 614 μs round-trip propagation time. However, the range residuals are highly correlated. If it is assumed that the residuals are correlated over, say, 25 points the average run has an uncertainty of 614 times the square-root of 472/25 or 141 μs, which corresponds to 42.3 km in round-trip range. Adopting this value for the range uncertainty for a measurement at conjunction gives 79 km in the AU *based on these statistics alone.* The resulting estimates of the AU are shown in figure 6.

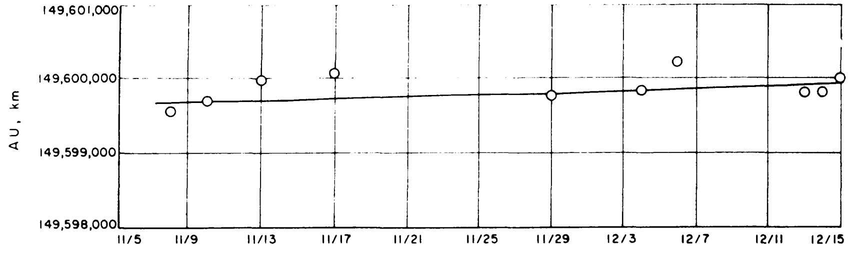

Fig. 6. — The astronomical unit estimates from range observations. The solid line is a weighted linear fit.

B. *Range and Doppler* AU *results.* — The Doppler AU results shown in figures 4 and 5 exhibit exactly the same variation with date as those reported by Muhleman *et al.* [1], for 1961. It is certain that this variation is due to errors in the orbital elements of the Earth and Venus employed in Newcomb's tables. In particular, small changes in the mean longitudes and/or the perihelia of the Earth and Venus would essentially remove this variation.

A Duncombe ephemeris for the 1962 observations has not been computed as yet. Consequently, it was necessary to analytically compute the change in the AU estimate resulting from the Duncombe corrections at each point of interest. It turns out that the effect of the corrections is smallest at specific times in the observational period, i. e., at the points furthest from conjunction for the Doppler data and the point at conjunction for the range data. Since these points are the least sensitive to the corrections, they are probably the most accurate estimates of the AU, at least for the types of errors considered. The correction procedure follows from equation (4). If δc_1 is identified with the correction to the AU, the result, upon solving equation (4) for δc_1, is

$$\delta c_1 = \delta(\mathrm{AU}) = \frac{\left\{ (R_M - R_0) - \frac{\partial R_0}{\partial c_2}\delta c_2 - \dots - \frac{\partial R_0}{\partial c_M}\delta c_M \right\}}{\left(\frac{\partial R_0}{\partial c_1}\right)_1}. \tag{6}$$

But the term $(R_M - R_0)$ has been iterated to zero. Therefore

$$\delta(\mathrm{AU}) = \frac{\left\{ -\frac{\partial R_0}{\partial c_2}\,\partial c_2 - \ldots - \frac{\partial R_0}{\partial c_M}\,\delta c_M \right\}}{\frac{\partial R_0}{\partial(\mathrm{AU})}}, \tag{7}$$

where $\delta c_2 = \Delta L''$, $\delta c_3 = \Delta e''$, etc. The partial derivatives in (7) have been computed from analytical expressions utilizing a digital computer program. An expression similar to (7) can be written for the Doppler data. The individual terms in δ(AU) are shown in table III for the range observation of November 12, 1962. The actual AU estimates listed in table IV were obtained by computing the weighted mean of the estimates near the date of interest. It is clear from table IV, as well as figure 5, that the value for December 12 is anomalously low. A similar effect but of much smaller magnitude was observed one month after conjunction in 1961. Figure 5 suggests that the observations in this region may have been faulty, but no explanation can be offered to support this conjecture. Some insight can be gained by the following analysis, however.

TABLE III.

The effect of the Duncombe corrections on the AU.

	Range November 12.
$\Delta L''$	+ 4
$\Delta e''$	191
$e''\Delta\varpi''$	−319
Δl	+ 1
Δe	+ 40
$e\Delta\varpi$	+ 45
Δp	+ 67
Δq	+ 8
TOTAL	37

TABLE IV.

AU 1962 *results.*

	Newcomb ephemerides (*).	Duncombe ephemerides (**).
Doppler, October 12	149 599 060 km	—
Range, November 12	149 599 730 »	149 599 374 km
Doppler, December 12	149 596 452 »	—

(*) Newcomb ephemerides means Newcomb's tables with a mean anomaly correction of $\Delta M_{\oplus} = +4''.78\ T$.

(**) Duncombe ephemerides means here that only $\Delta e''$ has been applied for the Earth plus all of the Venus corrections.

The true longitude of the Sun, λ, is computed from Newcomb's tables using the equation

$$\lambda = L'' - (f'' - M'') + \text{perturbation terms}, \tag{8}$$

where f'' and M'' are the true anomaly and mean anomaly of the Sun, respectively. To first order in e'',

$$f'' - M'' = 2e'' \sin M''. \tag{9}$$

Then from (8)

$$\lambda = L'' - 2e'' \sin M'' + \text{perturbation terms}. \tag{10}$$

Now the only change that was made to Newcomb's tables was $\Delta M'' = -4''.78\ T$. From (10) for a change of M'' only, the result is

$$\Delta\lambda = 2e'' \Delta M'' \cos M''.$$

Actually there is a slight change in the perturbation terms due to a change in ΔM, but it is negligible. It turns out that $\cos M''$ for October 12 is 0.135 whereas for December 12, $\cos M'' = 0.922$. Thus any change in M'' has about 7 times the effect on the latter date than on the former date. Actually the inclusion of $-4''.78\ T$ had an effect on the AU estimate for October 12 of + 13 km and for December 12 of + 111 km. Clearly, it is possible to raise the AU estimate of December 12 by a very large amount without lowering the estimate of October 12 significantly with a correction to M'' (or $e''\Delta\varpi''$). However, an impossibly large $\Delta M''$ is required to bring the two estimates into complete agreement. It may be concluded from this that the ephemeris errors introduced into the AU computations are probably large compared to the uncertainties of the fundamental radar observations. These errors include those in the Newcomb tables, Duncombe corrections to this table, and probably the most significant, errors in our numerical representation of the ephemerides.

C. *Weighted mean results and comparison with previous radar results.* — We shall adopt the mean of AU estimates reported in table IV weighted by estimated variances based on the noise in figures 5 and 6 and estimated ephemeris uncertainties. Adopting

October 12, 1962	149 599 060 ± 1000 km
November 12, 1962	149 599 374 ± 1000 »
December 12, 1962	149 596 452 ± 2000 »

the preliminary 1962 result is

$$149\,598\,900 \pm 670 \text{ km}$$

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The final AU results from the 1961 observations reported by Muhleman [16] are shown in table V.

TABLE V.

1961 *Radar results.*

1. Doppler near eastern elongation...........	149 598 750 ± 200 km
2. Doppler near western elongation...........	149 598 000 ± 1000 »
3. Range at conjunction (closed loop).........	149 598 500 ± 150 »
4. Range at conjunction (radiometer).........	149 598 800 ± 150 »
5. Millstone result..........................	149 597 850 ± 400 »
6. Muhleman's rework of Millstone data.......	149 598 100 ± 400 »
7. Weighted mean of 1, 2, 3, and 4...........	149 598 640 ± 200 »

D. *Conclusions concerning the* AU. — The preliminary best value of the astronomical unit from the observations of Venus around the 1962 inferior conjunction is

$$149\,598\,900 \pm 670 \text{ km}$$

where most of the uncertainties are due to ephemeris errors. This result is in complete agreement with the 1961 Goldstone radar result of

$$149\,598\,640 \pm 200 \text{ km}$$

as well as with the results from the 1961 Millstone radar observations.

The remaining uncertainties are linked primarily to the uncertainties in the ephemerides of the Earth and Venus and are of such a nature that the radar observations will ultimately yield definitive corrections to the fundamental ephemerides. This ultimate result is difficult to obtain from an analytical standpoint and will evolve slowly. While it is clear that the observations available at this time are of sufficient quality and quantity to accomplish a good measure of this goal, it should be realized that observations distant from conjunction are required to solve for certain of the corrections that are strongly correlated. In particular, radar observations from the Earth on other planets (or from asteroids) are highly desirable for the separation of the effects of the Earth's orbit from those of the orbit of Venus.

5. **Error analysis.** — A. *Velocity of light.* — The uncertainty in the vacuum velocity of light was shown to be $\pm$ 0.3 km/s and this appears pessimistic. The effect on the radar values of the AU is then approximately $\pm 0.3 \times 500$ s or 150 km.

B. *Dispersion and refraction.* — The effects of signal delays and refraction in the Earth's atmosphere are completely negligible at the frequencies of operation utilized by the Goldstone group (2 300 Mc) and Pettingill (440 Mc). The effect of refraction in the atmosphere of

Venus is probably negligible because the echo power primarily passes through the Venusian atmosphere at normal incidence.

The question of possible delays in the Venusian atmosphere is much more complex, however. An exhaustive discussion of the point has been given by Muhleman [16]. Briefly, the effecto f any delay in the atmosphere is to make the propagation time longer than that for the vacuum case and hence, cause the determined value of the AU to be larger. Furthermore, according to the modern theories of propagation, any delaying medium would have an effect increasing with decreasing frequency; thus the value of the AU determined from a radar at 440 Mc should be larger than that computed from observations at 2 300 Mc. In fact, it has been shown that if the value of the AU from the 2 300 Mc observations is in error by 100 km, then the value measured at 440 Mc should be larger by about 7 000 km, whereas the value determined above is actually smaller at 440 Mc by 540 km than the value at 2 300 Mc. Thus, it is unlikely that there is any delay effect at all.

C. *The radius of Venus.* — The uncertainty in the radius of Venus does not effect the value of the AU determined from the Doppler frequency. The effect on the range measurements is equal to the radius uncertainty. If the uncertainty of the Venusian radius is taken to be 25 km, the effect on the AU is about 89 km.

D. *The ephemerides.* — The only reasonable estimate of the ephemeris errors are the Duncombe corrections themselves. It is difficult to see how the errors in the ephemerides *after* corrections could be as large as the corrections themselves. Consequently, Duncombe's values can logically be taken as upper bounds on the errors, but this appears too pessimistic. Therefore, it is desirable first to analyze the range case, and second, the Doppler observations.

The range between Venus and the Earth, r, given by

$$r^2 = r_{\varphi}^2 + r_{\oplus}^2 - 2 r_{\varphi} r_{\oplus} \cos\theta, \tag{11}$$

where r_{φ} and $r_{\oplus}$ are the solar distances to the planets, and θ is the heliocentric angle between the Earth and Venus given by

$$\cos\theta = \cos(l_{\varphi} - \Omega_{\varphi}) \cos(l_{\oplus} - \Omega_{\varphi}) + \sin(l_{\varphi} - \Omega_{\varphi}) \sin(l_{\oplus} - \Omega_{\varphi}) \cos i_{\varphi}. \tag{12}$$

Thus, r is a function of the eccentricities and the arguments of the perihelia through equation (11) and the equations of elliptical motion, and r is a function of l_{φ}, $l_{\oplus}$, Ω_{φ} and i_{φ} through equation (12). The uncertainty in the obliquity is neglected because its effect on r is very small. Therefore,

$$r = r(l_{\oplus}, l_{\varphi}, e_{\oplus}, e_{\varphi}, \varpi_{\oplus}, \varpi_{\varphi}, \Omega_{\varphi}, i_{\varphi}) \tag{13}$$

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where it is assumed that

$$r_\oplus = \frac{a_\oplus(1 - e_\oplus^2)}{1 + e_\oplus \cos(l_\oplus - \varpi_\oplus)}. \tag{14}$$

The quantities $a_\oplus$ and $a_♀$ will be assumed precisely known in astronomical units. Then from equation (11):

$$r dr = r_♀ \left(\frac{\partial r_♀}{\partial e_♀} de_♀ + \ldots \right) + r_\oplus \left(\frac{\partial r_\oplus}{\partial e_\oplus} de_\oplus + \ldots \right) + \ldots \tag{15}$$

All of the partial derivatives are then computed from equations (12) and (14). Now, the error in the AU due to an error *dr* is

$$\delta(\text{AU}) = A_\oplus \frac{dr}{r}, \tag{16}$$

where $A_\oplus$ is the value of the AU in kilometers. The expression for δ(AU) may then be written for small errors in the elements utilizing the partials. Since the primary interest is in the value of δ(AU) at the 1961 inferior conjunction of Venus, the general expression will be given with all of the expressions evaluated at that epoch. The result is

$$\begin{aligned} \delta(\text{AU}) = 9\,680\ \text{km} \{ & 0.031\, de_\oplus + 0.0047\, dl_\oplus \\ & - 0.276\, e_\oplus\, d\varpi_\oplus - 0.028\, de_♀ + 0.0014\, dl_♀ \\ & - 0.198\, e_♀\, d\varpi_♀ - 0.029\, di_♀ - 0.13\, d\Omega_♀ \}. \end{aligned} \tag{17}$$

With the Duncombe corrections inserted for the differentials,

$$\delta(\text{AU}) = \{ -30 - 5 + 322 + 33 + 6 + 19 + 19 - 47.4 \}\ \text{km},$$
$$\delta(\text{AU}) = +317\ \text{km}.$$

Thus, if the ephemerides are in error after correction by as much as the corrections themselves, the error in the AU from the range observations is about 317 km.

The case for the Doppler observations is far more complicated. Since the points of interest in this case are toward the east and west elongations, it can be shown that the terms involving $\sin i_♀$ are negligible to first order, and a first order analysis can be carried out in two dimensions. Since the analysis has been carried out in the plane of the ecliptic, the effects of the obliquity are also ignored. Then the range rate (or Doppler velocity) is approximately

$$\dot{r} \simeq V_♀(\sin\alpha_♀ - \gamma_♀ \cos\alpha_♀) - V_\oplus(\sin\alpha_\oplus + \gamma_\oplus \cos\alpha_\oplus), \tag{18}$$

where

$V_\oplus$, $V_♀$, orbital speeds of the Earth and Venus;

$\alpha_♀$, the angle between the Sun and the Earth at Venus, similarly for $\alpha_\oplus$;

$\gamma_\oplus$, $\gamma_♀$, the angles of the Earth and Venus velocity vectors from the perpendicular to the radius vectors in the orbital planes.

From well known equations of celestial mechanics, to first order in the eccentricities

(19) $$V_\oplus \simeq n_\oplus a_\oplus [1 + e_\oplus \cos(l_\oplus - \varpi_\oplus)]$$

and

(20) $$\gamma_\oplus \simeq e_\oplus \sin(l_\oplus - \varpi_\oplus).$$

Thus, from equations (14), (18), (19), and (20), $\dot{r}$ can be expressed in terms of the elements and the partial derivatives taken. The results are too complex to profitably write down, and only the resulting expression for the δ (AU) will be presented with all of the expressions evaluated at the epoch March 23, 1961, the date of observation nearest the eastern elongation and consequently, the point of greatest interest :

$$d\dot{r} = 35.05 \text{ km/s} \{ 0.13\, de_♀ - 1.96\, dl_♀ - 1.18\, e_♀ d\varpi_♀ \} - 29.8 \text{ km/s} \{ -1.0\, de_\oplus - 1.34\, dl_\oplus - 1.74\, e_\oplus d\varpi_\oplus \}.$$

Since

(21) $$\delta(AU) = \frac{A_\oplus}{\dot{r}} d\dot{r}$$

inserting the Duncombe corrections,

$$\delta(AU) = -1350 \text{ km}.$$

This value is, of course, very large and probably equally pessimistic. If the uncertainties of the corrections are used, the largest term is due to the uncertainty in the longitude of Venus and is 620 km. It is not possible to combine the individual terms in a meaningful statistical manner because the correlation coefficient between the terms may even approach unity. However, it appears safe to say that the error in the AU from the Doppler observations is less than 620 km. If this circumstance is correct, the Doppler value of the AU has been weighted twice as heavily as it should have been in the final reduction to a single result.

6. **Radar measurements of Mercury.** — Unequivocal radar contact of Mercury has been accomplished by the Jet Propulsion Laboratory. The observations have been made by transmitting a pure continuous wave with the Venus radar equipment. The echo signal has been

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detected by computing the power spectral density of the received signal in a digital computer. The signal spectrum was shifted down near *dc* by continuously adjusting the receiver local oscillator to the ephemeris Doppler frequency plus an offset of about 100 c/s. An example of such a spectrum taken by R. Carpenter of JPL is shown in figure 7. The ephemeris was prepared in the same way as the Venus ephemeris. The vertical center line in figure 7 indicates the frequency about which the

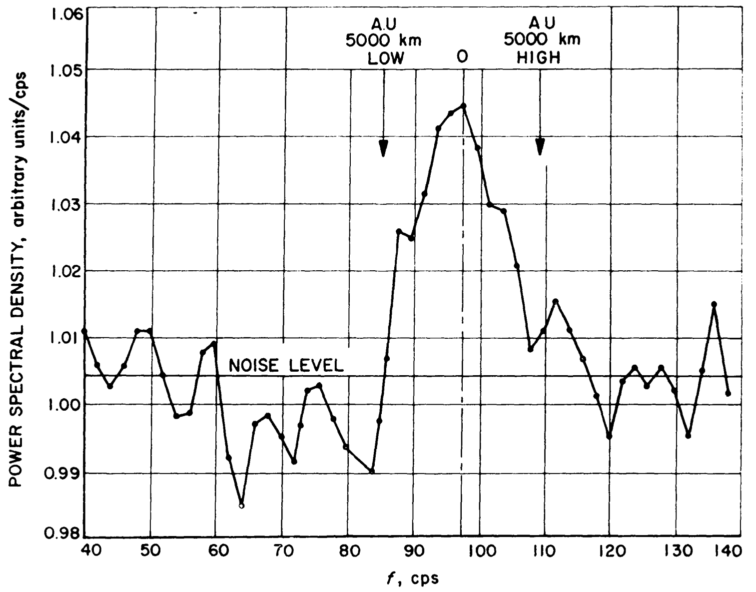

Fig. 7. — The spectrum of a Mercury radar echo. The center line is at the frequency where the spectrum would fall if AU = 149 598 640 km.

observed spectrum would be centered if the ephemeris were perfect and the value used for the AU = 149 598 640 km were correct. The arrows indicate the amount that the spectrum would be shifted for an error in the AU of ± 5 000 km for the observation date of May 8, 1963.

Some error in the measurement of the center frequency is to be expected due to errors in positioning the local oscillator on the order of 1 or 2 c/s. Known errors of the ephemerides would have a similar effect. Thus unless the spectrum in figure 7 was positioned fortuitously the observations yield an excellent verification of the radar value of the AU.

Range measurements to Mercury have been accomplished by R. Goldstein of the Jet Propulsion Laboratory concurrently with the Doppler measurements. He has made two measurements both of which are within about 100 km of the ephemeris values. The ephemeris was computed, of course, using 149 598 640 km for the AU. Doppler measurements of the kind shown in figure 7 were made on 10 different days. The differences between the spectral center frequencies and the ephemeris Doppler shifts are shown in figure 8, where the circles are measurements of R. Carpenter and the squares are those of R. Goldstein. The solid lines in figure 8 represent the error in Doppler frequency for an error in the mean anomaly of Mercury of $\Delta M = -2''.8$ and an error in the

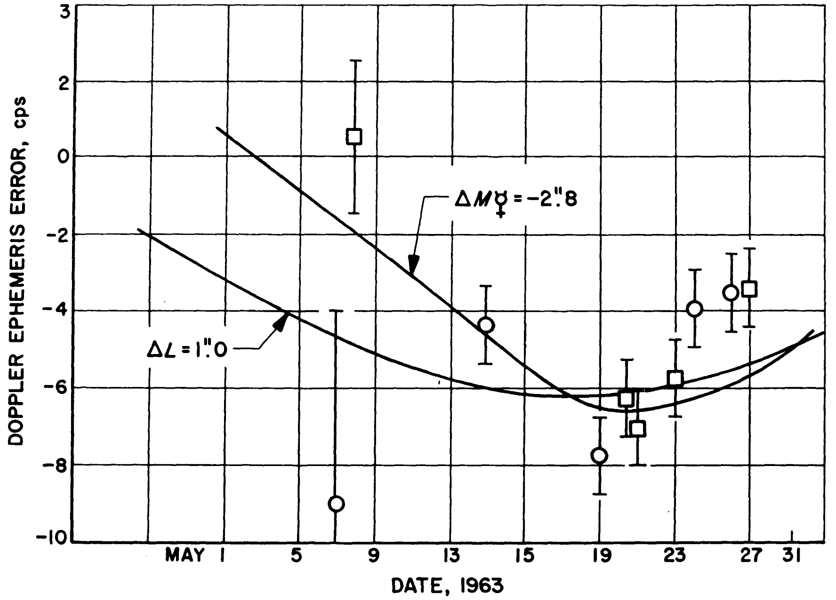

Fig. 8. — Observed frequency shift from ephemeris.

relative mean longitudes of Mercury and the Earth of $1''.0$. Therefore, the residuals can easily be explained by the hypothesis of reasonable errors in the Mercury and Earth ephemerides.

7. **The related astronomical constants.** — The relationships existing between the astronomical unit and related astronomical constants may now be utilized to construct a consistent set of some of the constants based on the AU result of 149 598 640 ± 250. Using R = 6 347 166 km, the result for the solar parallax is

$$\pi_{\odot} = 8''.794139 \pm 0''000015.$$

The light-time for unit distance is

$$\tau = 499.0073 \pm 0.0007 \text{ s.}$$

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It should be realized that τ is the most fundamental result from the radar work because it is independent of the speed of light. The aberration constant is also independent of c when the radar value of the AU is used,

$$K = 20''.49562 \pm 0.00003.$$

The Earth-Moon mass ratio can be obtained from the lunar inequality equation

$$L = \left(\frac{\mu}{1+\mu}\right)\frac{a_{☾}}{AU}$$

and the dependence on c is again removed from the radar results if Yaplee's radar value of $a_{☾}$ is corrected to the same value of c. Using $L = 6.4378 \pm 0.0002$ [22] and $a_{☾} = 388\,400.4$,

$$\mu^{-1} = 81.327 \pm 0.025$$

where the uncertainty is due to that of L.

The coefficient of the parallactic inequality, where again c factors out if radar values of $a_{☾}$ and AU are used is :

$$P = -124''.987 \pm 0.001.$$

Finally, a consistent value of the mass of the Earth plus Moon can be obtained from an expression given by Brouwer [23] :

$$\frac{S}{E+M} = 0.00558001\,40\,\frac{(AU)^3}{(a_{☾})^3},$$

where Brouwer has obtained the constant term from modern measurements of the Earth constants. Note again that for radar values of AU and $a_{☾}$ the errors due to c are removed and the result is

$$(E+M)^{-1} = 328\,903.2.$$

The values above cannot be considered definitive until the ephemeris errors are removed from the radar values, but it is clear that all the above constants except $\pi_{\odot}$ are free from the error in the radar AU introduced by using a specific value of c. Thus, from this standpoint, the major criticism of the radar method, namely the uncertainty of the propagation velocity, is destroyed.

Acknowledgments. — The author wishes to acknowledge the extensive assistance of D. B. Holdridge, Dr. P. Peabody, N. Block, and M. Easterling of JPL; Professor D. Brouwer of Yale; and Dr. G. M. Clemence and Dr. R. L. Duncombe of the U. S. Naval Observatory.

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