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ON A MULTIPLE INTEGRAL USEFUL IN ORDER STATISTICS DISTRIBUTION THEORY

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Summary. A known multiple integral from the difference calculus is used to derive certain distributions of ordered statistics from the truncated exponential, rectangular, and ordered random intervals populations.

1. **Introduction.** Let $x_{(1)}<x_{(2)}<\cdots<x_{(N)}$ be an ordered sample of size N from a population whose density function is $f(x)$. Then the density function of the variate $u=g(x_{(1)}, x_{(2)},\ldots, x_{(N)})$ is given by the integral

$$f(u) = N! \int_R f(x_{(1)})\ldots f(x_{(N)})\, dx_{(1)}\ldots dx_{(N)}, \tag{1}$$

where the range R is determined by the conditions,

$$-\infty < x_{(i)} < \infty, \quad i = 1,\ldots, N; \qquad x_{(1)} <\cdots< x_{(N)}, \quad u = g(x_{(1)},\ldots, x_{(N)}).$$

The evaluation of the multiple integrals of above types which occur in order statistics distribution theory is involved due to the fact that the integration is to be carried over an ordered range of the variables of integration. This difficulty is sometimes completely obviated by transforming the ordered variates to the unordered ones. Such transformations are available in the theory of Multiple Integrals. In previous papers, Kabe [3], [4], the author used one such transformation, namely the Dirichlet's transformation, and gave simpler proofs of several known results in order statistics distribution theory from the exponential and power function distribution theory. This is yet another attempt in the same direction. Here a known multiple integral is used to derive certain ordered statistics distributions.

Some useful results are stated in the next section and their applications are outlined in §3.

2. **Some useful results.** If the $(N-1)$th derivative of $f(x)$ is continuous, then the $(N-1)$th divided difference of $f(x)$ may be expressed as a multiple integral, see, e.g., Milne-Thomson [6, p. 18, Example 6]

$$\int_{y_1+\cdots+y_N=1} f^{(N-1)}(\alpha_1 y_1+\ldots+\alpha_N y_N)\, dy_1\ldots dy_N = \sum_{i=1}^{N} \frac{f(\alpha_i)}{(\alpha_i-\alpha_1)\ldots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\ldots(\alpha_i-\alpha_N)}. \tag{2}$$

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It follows from (2) that

$$\int_{y_1+\cdots+y_N=t} f(\alpha_1 y_1+\cdots+\alpha_N y_N)\,dy_1\ldots dy_N$$

(3)

$$= \sum_{j=1}^{N} \frac{(\alpha_j)^{N-1}\int f(\alpha_j t)(dt)^{N-1}}{(\alpha_j-\alpha_1)\ldots(\alpha_j-\alpha_{j-1})(\alpha_j-\alpha_{j+1})\ldots(\alpha_j-\alpha_N)}$$

where $\int(dt)^{N-1}$ denotes the iterated $(N-1)$-fold indefinite integral operator, i.e., $\int f^{(N-1)}(t)(dt)^{N-1}=f(t)$. If the joint density of y's is $f(\alpha_1 y_1+\cdots+\alpha_N y_N)$, then obviously the right-hand side of (3) represents the density of the variate t.

In particular, if $f(t)=\exp(-t)$, then we have from (3) that

(4)

$$R(N,t) = \int_{y_1+\cdots+y_N=t} \exp\{-(\alpha_1 y_1+\cdots+\alpha_N y_N)\}\,dy_1\ldots dy_N$$

$$= \sum_{j=1}^{N} \frac{(-1)^{N-1}\exp(-\alpha_j t)}{(\alpha_j-\alpha_1)\ldots(\alpha_j-\alpha_{j-1})(\alpha_j-\alpha_{j+1})\ldots(\alpha_j-\alpha_N)}.$$

Incidentally, we note that $\Phi(N,t)=\alpha_1\ldots\alpha_{N-1}R(N,t)$ satisfies the difference-differential equation

(5)
$$\frac{d\Phi(N,t)}{dt} = \alpha_N\Phi(N,t)-\alpha_{N-1}\Phi(N-1,t),$$

which is analogous to the difference-differential equation of the pure birth process, see e.g., Prabhu [7, p. 135, equation (6.5)].

Again if $f(t)=(-1)^{N-1}(\Gamma(N))^{-1}(1-t)^{N-1}$, then we may deduce from (3) that

$$\int_{y_1+\cdots+y_N=t} (1-\alpha_1 y_1-\cdots-\alpha_N y_N)^0\,dy_1\ldots dy_N$$

(6)

$$= \sum_{j=1}^{N} \frac{(-1)^{N-1}(\Gamma(N))^{-1}(1-\alpha_j t)^{N-1}}{(\alpha_j-\alpha_1)\ldots(\alpha_j-\alpha_{j-1})(\alpha_j-\alpha_{j+1})\ldots(\alpha_j+\alpha_N)}.$$

The transformation

(7)
$$x_{(i)} = \sum_{j=1}^{N} y_j,\ i,j = 1,2,\ldots,N,$$

with Jacobian unity transforms the ordered x variates to the unordered y variates.

Now we proceed with the applications of the results of this section.

3. **Some illustrative examples.** *Example* 1. This example illustrates an application of (3) to order statistic distribution theory from a truncated exponential population

(8)
$$f(x) = (1-\exp(-a))^{-1}\exp\{-x\},\quad 0 < x < a.$$

The truncated exponential population is nowadays finding wide applications in Reliability Theory, see e.g., Holla [2]. Let $x_{(1)} < \cdots < x_{(N)} < a$ be an ordered sample from (8), then by using the transformation (7), and writing $C = N!\,(1-\exp(-a))^{-N}$, we find that the joint density of $x_{(i)}$ and $x_{(j)}$ is

$$
\begin{aligned}
f(x_{(i)}, x_{(j)}) = C \int_0^{a-x_{(j)}} dw \int_{R_1} \exp\left\{-\sum_{k=1}^{i} (N-k+1)y_k\right\} dy_1 \ldots dy_i \\
\times \int_{R_2} \exp\left\{-\sum_{s=i+1}^{j} (N-s+1)y_s\right\} dy_{i+1} \ldots dy_j \\
\times \int_{R_3} \exp\left\{-\sum_{r=j+1}^{N} (N-r+1)y_r\right\} dy_{j+1} \ldots dy_N,
\end{aligned}
\tag{9}
$$

where the regions of integrations R_1, R_2, and R_3 are respectively determined by the conditions

$$
\begin{aligned}
y_1 + \cdots + y_i &= x_{(i)}, & 0 < y_k < a, \quad & k = 1, \ldots, i; \\
y_{i+1} + \cdots + y_j &= x_{(j)} - x_{(i)}, & 0 < y_s < a, \quad & s = i+1, \ldots, j; \\
y_{j+1} + \cdots + y_N &= w, & 0 < y_r < a, \quad & r = j+1, \ldots, N.
\end{aligned}
$$

The integrals over R_1, R_2, and R_3 may be evaluated by using the result (4), and we find that

$$
\begin{aligned}
f(x_{(i)}, x_{(j)}) = C \sum_{k=1}^{i} \sum_{s=i+1}^{j} \sum_{r=j+1}^{N} \frac{(-1)^{i+j+N+k+s+r}}{(k-1)!\,(i-k)!} \\
\times \frac{\exp\{-(s-k)x_{(i)} - (N-s+1)x_{(j)}\}\{1-\exp(-(N-r+1)(a-x_{(j)}))\}}{(s-i-1)!(j-s)!(N-j-1)!(N-r+1)!}.
\end{aligned}
\tag{10}
$$

Similarly the distribution function of $x_{(N-k+1)}$ may be shown to be

$$
\begin{aligned}
P\{x_{(N-k+1)} \geq t\} = \binom{N}{k} \sum_{j=k}^{N} (-1)^{j-k} \binom{N-k}{j-k} \binom{k}{j} \\
\times (1-\exp(-a))^{-j} (\exp(-t) - \exp(-a))^j, \quad 0 \leq t \leq a.
\end{aligned}
\tag{11}
$$

In general it is difficult to find the distributions of linear functions of ordered variables from (8), although the characteristic function of any linear function may be found by using (4).

Example 2. Let $0 < x_{(1)} < \cdots < x_{(N)} < 1$ be an ordered sample of N values from $f(x) = 1, 0 < x < 1$. Now we wish to find the distribution of the linear function $u = \sum_{i=1}^{N} d_i x_{(i)}$. We set $\sum_{i=j}^{N} d_i = \alpha_j$ and using (4) and (7) find the Mellin transform or the sth moment μ_s' of u to be

$$
\begin{aligned}
\mu_s' &= N! \int_{y_1 + \cdots + y_N \leq 1} (\alpha_1 y_1 + \cdots + \alpha_N y_N)^s \, dy_1 \ldots dy_N \\
&= N! \sum_{j=1}^{N} \frac{\alpha_j^{s+N-1}\, \Gamma(s+1)/\Gamma(s+N+1)}{(\alpha_j - \alpha_1) \ldots (\alpha_j - \alpha_{j-1})(\alpha_j - \alpha_{j+1}) \ldots (\alpha_j - \alpha_N)}.
\end{aligned}
\tag{12}
$$

On inverting the Melling transform (12) we have that

$$(13)\qquad f(u) = \sum_{j=1}^{N} \frac{N(\alpha_j-u)^{N-1}}{\alpha_j(\alpha_j-\alpha_1)\ldots(\alpha_j-\alpha_{j-1})(\alpha_j-\alpha_{j+1})\ldots(\alpha_j-\alpha_N)},$$

where the summation in (13) stops as soon as (α_j-u) ceases to be positive.

Example 3. Let $0<x_{(1)}<\cdots<x_{(N)}<1$ be as in example 2, and let $y_{(1)}<y_{(2)}<\cdots<y_{(N)}$ be ordered values of the random intervals $y_j=x_{(j)}-x_{(j-1)}, j=1,\ldots,N$, $x_{(0)}=0$. It is required to find the density function of, say, $y_{(N)}$ see Wilks [8, p. 252, Example 8.42]. By using (7) we find the joint density of $y_1,\ldots,y_N$ to be

$$(14)\qquad f(y_1,\ldots,y_N) = N!\,(1-\sum y_j)^0, \quad \sum y_j < 1.$$

The joint density of the ordered y's is

$$(15)\qquad f(y_{(1)},\ldots,y_{(N)}) = (N!)^2(1-\sum y_{(j)})^0, \quad y_{(1)}<\cdots<y_{(N)} < 1.$$

Now by setting $y_{(j)}=\sum_{i=1}^{j} w_i$, we observe that the density of $y_{(N)}=\sum w_j$ is

$$(16)\qquad f(y_{(N)}) = (N!)^2 \int_{w_1+\cdots+w_N=y_{(N)}} \left(1-\sum_{j=1}^{N}(N-j+1)w_j\right)^0 dw_1\ldots dw_N.$$

The integral (16) is evaluated by using (6), and we find that

$$(17)\qquad f(y_{(N)}) = \sum_{i=1}^{N} \frac{(N!)^2\,(-1)^{N-i}(1-(N-i+1)y_{(N)})^{N-1}}{(i-1)!(N-i)!(N-1)!},$$

where the summation in (17) stops as soon as $(1-(N-i+1)y_{(N)})$ ceases to be positive.

The distributions of linear functions of ordered random intervals or ordered rectangular variates may be obtained by noting that the distributions of random intervals are identical with those of reduced exponential statistics, see Karlin [5, p. 263, Example 7].

Example 4. Barlow, Gupta, and Panchapakesan [1] in their studies of ranking and selection procedures require the distribution of $x_{(N)}$, when the joint distribution of the ordered sample $x_{(1)}<\cdots<x_{(n)}<\infty$ is

$$(18)\qquad f(x_{(1)},\ldots,x_{(N)}) = (1+x_{(1)}+\cdots+x_{(N)})^{-(N+1)}(N!)^2.$$

By using (7) and (18) we find that

$$(19)\qquad f(x_{(N)}) = \int_{y_1+\cdots+y_N=x_{(N)}} (N!)^2\left(1+\sum_{j=1}^{N}(N-j+1)y_j\right)^{-(N+1)} \pi\, dy_j$$

$$= \sum_{j=1}^{N} \frac{(-1)^{N+j}(1+(N-j+1)x_{(N)})^{-2}(N!)^2}{N!(j-1)!(N-j)!}, \quad 0 < x_{(N)} < \infty.$$

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The integral (19) is evaluated by using (4). The distribution (19) is obtained by Barlow *et al.* [1, p. 920, equation 2.14] by another method.

References

1. R. E. Barlow, S. S. Gupta and S. Panchapakesan, *On the distribution of the maximum and minimum of ratios of order statistics*, Ann. Math. Statist. **40** (1964), 918–934.

2. M. S. Holla, *Reliability estimation of the truncated exponential model*, Technometrics **9** (1967), 332–336.

3. D. G. Kabe, *Dirichlet's transformation and distributions of linear functions of ordered gamma variates*, Ann. Inst. Statist. Math. **18** (1966), 367–374.

4. ———, *Some distribution problems of order statistics from exponential and power function distributions*, Canad. Math. Bull. **11** (1968), 263–274.

5. Samuel Karlin, *A First Course in stochastic processes*, Academic Press, New York, 1966.

6. L. M. Milne-Thomson, *The calculus of finite differences*, Macmillan, London, England, 1951.

7. N. W. Prabhu, *Stochastic processes*, Macmillan, New York, 1965.

8. S. S. Wilks, *Mathematical statistics*, John Wiley, New York, 1962.

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