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BOUNDS FOR CERTAIN LINEAR COMBINATIONS OF THE FABER COEFFICIENTS OF FUNCTIONS ANALYTIC IN AN ELLIPSE

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Abstract Let Ω be a bounded, simply connected domain in $\mathbb{C}$ with $0 \in \Omega$ and $\partial\Omega$ analytic. Let $S(\Omega)$ denote the class of functions $F(z)$ which are analytic and univalent in Ω with $F(0) = 0$ and $F'(0) = 1$. Let $\{\Phi_n(z)\}_{n=0}^{\infty}$ be the Faber polynomials associated with Ω. If $F(z) \in S(\Omega)$, then $F(z)$ can be expanded in a series of the form

$$F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z), \quad z \in \Omega,$$

in terms of the Faber polynomials. Let

$$E_r = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{(1 + (1/r^2))^2} + \frac{y^2}{(1 - (1/r^2))^2} < 1\right\},$$

where $r > 1$.

In this paper, we obtain sharp bounds for certain linear combinations of the Faber coefficients of functions $F(z)$ in $S(E_r)$ and in certain related classes.

Keywords: Faber polynomials; Faber coefficients; Jacobi elliptic sine function

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1. Introduction

Let S denote the class of functions f analytic and univalent in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The Bieberbach conjecture [**2**] asserts that if $f \in S$, then $|a_n| \leqslant n$ $(n \geqslant 2)$. This famous conjecture was proved by de Branges [**4**] in 1984. It was also shown that equality holds if and only if f is a rotation of the Koebe function

$$k(z) = \frac{z}{(1 - z)^2}. \tag{1.2}$$

In this paper, we investigate bounds for certain linear combinations of the Faber coefficients in domains other than the unit disk $\mathbb{D}$, in particular an elliptical domain.

Let Ω be a bounded, simply connected domain in $\mathbb{C}$ with capacity 1 and $0 \in \Omega$. Let $g(z)$ be the unique, one-to-one, analytic mapping of $\Delta = \{z : |z| > 1\}$ onto $\mathbb{C} \setminus \bar{\Omega}$ with

$$g(z) = z + \sum_{n=0}^{\infty} \frac{c_n}{z^n} \quad (c > 0,\ z \in \Delta). \tag{1.3}$$

The *Faber polynomials*, $\{\Phi_n(z)\}_{n=0}^{\infty}$, associated with Ω (or $g(z)$) are defined by the generating function relation [**5**, p. 218]

$$\frac{\eta g'(\eta)}{g(\eta) - z} = \sum_{n=0}^{\infty} \Phi_n(z)\eta^{-n} \quad (\eta \in \Delta). \tag{1.4}$$

Faber polynomials play an important role in the theory of functions of a complex variable and in approximation theory. On a simply connected domain Ω with at least two boundary points the Faber polynomials, $\{\Phi_n(z)\}_{n=0}^{\infty}$, play a role analogous to that of $\{z^n\}_{n=0}^{\infty}$ in $\mathbb{D}$. If $\partial\Omega$ is analytic and $F(z)$ is analytic in Ω, then $F(z)$ can be expanded into a series of the form

$$F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z), \quad z \in \Omega, \tag{1.5}$$

in terms of the Faber polynomials. This series is called the *Faber series* and it converges uniformly on compact subsets of Ω. The coefficients A_n, which can be computed via the formula

$$A_n = \frac{1}{2\pi \mathrm{i}} \int_{|z|=\rho} F(g(z)) z^{-n-1} \mathrm{d}z$$

with $\rho < 1$ and close to 1, are called the *Faber coefficients* of $F(z)$ [**11**, p. 42].

Let $\phi(z)$ be the unique, one-to-one, analytic mapping of Ω onto $\mathbb{D}$ with $\phi(0) = 0$ and $\phi'(0) > 0$. Thus a function $F(z)$ which is analytic and univalent in Ω and normalized by the conditions $F(0) = 0$ and $F'(0) = 1$ may be written as

$$F(z) = \frac{f(\phi(z))}{\phi'(0)} \tag{1.6}$$

for some $f \in S$. The Faber coefficients $\{A_n\}_{n=0}^{\infty}$ of a function $F(z)$ of the form (1.6) will be denoted by $\{A_n(f)\}_{n=0}^{\infty}$ to indicate the dependence on $f \in S$.

In order to investigate the Faber coefficients $A_n(f)$, it will be convenient to work with a domain Ω for which the Faber polynomials $\Phi_n(z)$ may be computed via the formula (1.4) in terms of the *exterior* mapping $g(z)$ given by (1.3). We then express the *interior* functions $F(z)$ given by (1.6) in terms of the interior mapping $\phi(z)$. However, it is not easy to deal with both exterior and interior mappings at the same time, so we restrict our interest to the elliptical domain

$$E_r = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{(1 + (1/r^2))^2} + \frac{y^2}{(1 - (1/r^2))^2} < 1\right\},$$

where $r > 1$, for which both of these functions are manageable.

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The function $g(z) = z + (1/r^2 z)$, $r > 1$, is analytic and univalent in Δ and maps Δ onto $\mathbb{C} \setminus \bar{E}_r$. After doing necessary calculations, we obtain from (1.4) that the Faber polynomials, $\{\Phi_n(z)\}_{n=0}^{\infty}$, associated with E_r are given by

$$\Phi_n(z) = 2^n r^{-n} P_n(\tfrac{1}{2}rz) \quad (n = 0, 1, 2, \dots).$$

Here $\{P_n(z)\}_{n=0}^{\infty}$ are the monic Chebyshev polynomials of degree n, which are given by

$$P_0(z) = 1$$

and

$$P_n(z) = 2^{-n}\{[z + \sqrt{z^2 - 1}]^n + [z - \sqrt{z^2 - 1}]^n\} \quad (n = 1, 2, 3, \dots).$$

Let $sn(z; q)$ be the Jacobi elliptic sine function with nome q and modulus k_0, and let [**8**, Chapter 2]

$$K = \int_0^1 \frac{1}{\sqrt{1 - t^2}\sqrt{1 - k_0^2 t^2}} \, \mathrm{d}t.$$

Then the function

$$\varphi(z) = \sqrt{k_0}\left(\frac{2K}{\pi} \sin^{-1} \frac{rz}{2}; \frac{1}{r^4}\right)$$

is the one-to-one mapping of E_r onto $\mathbb{D}$ with $\varphi(0) = 0$ and $\varphi'(0) = rK\sqrt{k_0}/\pi > 0$ [**10**, p. 296].

We define $S(E_r)$ to be the class of functions $F(z)$ that are analytic and univalent in E_r and normalized by the conditions $F(0) = 0$ and $F'(0) = 1$. We define two subclasses of $S(E_r)$ as

$$C(E_r) = \{F(z) \in S(E_r) : F(E_r) \text{ is convex}\}$$

and

$$S^{(2)}(E_r) = \{F(z) \in S(E_r) : F(z) \text{ is odd}\}.$$

In addition, we let $P(E_r)$ denote the class of functions analytic in E_r and satisfying the conditions $F(0) = 1/\varphi'(0) = \pi/rK\sqrt{k_0}$ and $\mathrm{Re}\{F(z)\} > 0$. (The condition $F(0) = 1/\varphi'(0)$ is imposed for convenience.)

Note that if $F(z)$ is in one of the classes $S(E_r)$, $C(E_r)$, $S^{(2)}(E_r)$ or $P(E_r)$, then $F(z)$ may be written as in (1.6) for some $f(z)$ in the classes S, convex functions C, odd functions $S^{(2)}$ or functions with positive real part P defined for $\mathbb{D}$.

It has been conjectured in [**7**] that if $F(z) \in S(E_r)$, then

$$|A_0(f)| \leqslant A_0(k) = \frac{\pi^3}{8rK^3\sqrt{k_0}(1 - k_0)^2 \ln r}$$

and

$$|A_n(f)| \leqslant A_n(k) = \frac{\pi^3 n}{4rK^3\sqrt{k_0}(1 - k_0)^2(1 - r^{-2n})} \quad (n \geqslant 1),$$

whose special case for $r \to \infty$ is the famous Bieberbach conjecture.

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In this paper, we obtain sharp upper bounds for certain linear combinations of the Faber coefficients for functions in the classes $S(E_r)$, $C(E_r)$ and $P(E_r)$. Special cases yield bounds for $A_0(f)$, $A_1(f)$ and $A_2(f)$. In each case, there are two extremal functions, which are given by

$$f(z) = k(z)$$

or

$$f(z) = -k(-z),$$

where $k(z)$ is the Koebe function given by (1.2),

$$f(z) = c(z) = \frac{z}{1-z} \tag{1.7}$$

or

$$f(z) = -c(-z) = \frac{z}{1+z},$$

and

$$f(z) = p(z) = \frac{1+z}{1-z} \tag{1.8}$$

or

$$f(z) = p(-z) = \frac{1-z}{1+z}, \tag{1.9}$$

respectively, for the classes $S(E_r)$, $C(E_r)$ and $P(E_r)$. Here it is important that the number of extremal functions is the same as the number of invariant rotations of the elliptical domain E_r. For functions in $S^{(2)}(E_r)$ a sharp bound for a linear combination of $A_1(f)$ and $A_{2n+1}(f)$, whose special case yields a sharp upper bound for $A_1(f)$ is obtained. The corresponding extremal function in $S^{(2)}$ is shown to be

$$o(z) = \frac{z}{1-z^2}. \tag{1.10}$$

2. Main results

We begin with the following lemma.

Lemma 2.1. *Let $F(z)$ be analytic in E_r and have the Faber series given by (1.5). Then the Faber coefficients $\{A_n\}_{n=0}^{\infty}$ of $F(z)$ are given by the formula*

$$A_n = \frac{r^n}{\pi}\int_0^{\pi} F\left(\frac{2\cos\theta}{r}\right)\cos n\theta \,\mathrm{d}\theta \quad (n = 0, 1, 2, \dots).$$

Proof. Letting $z = 2\cos\theta/r$ in (1.5) and using $\Phi_n(2\cos\theta/r) = 2r^{-n}\cos n\theta$ yields

$$F\left(\frac{2\cos\theta}{r}\right) = 2\sum_{n=0}^{\infty} A_n r^{-n}\cos n\theta. \tag{2.1}$$

Multiplying (2.1) by $\cos m\theta$ and then integrating from 0 to π gives the desired result. □

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As a consequence of this representation it can be shown that if $F(z)$ has the representation (1.6) and belongs to one of the classes $S(E_r)$, $C(E_r)$, $S^{(2)}(E_r)$ or $P(E_r)$, then the Faber coefficients are given by

$$A_n(f) = \frac{r^{n-1}}{K\sqrt{k_0}} \int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) \cos n\theta \, d\theta \quad (n = 0, 1, 2, \dots). \tag{2.2}$$

In addition, as was shown in [**6**], if $F(z) \in S^{(2)}(E_r)$, then

$$A_{2n}(f) = 0 \quad (n = 0, 1, 2, \dots).$$

Another representation formula for the Faber coefficients, $\{A_n(f)\}_{n=0}^{\infty}$, is given in the following corollary.

Corollary 2.2. *The Faber coefficients, $\{A_n(f)\}_{n=0}^{\infty}$, of functions in the classes $S(E_r)$, $C(E_r)$, $S^{(2)}(E_r)$ and $P(E_r)$ are given by*

$$A_n(f) = \frac{2^n n! r^{n-1}}{K\sqrt{k_0}(2n)!} \int_0^{\pi} (f(\varphi(x)))^{(n)}|_{x=2\cos\theta/r} \sin^{2n}\theta \, d\theta \quad (n = 0, 1, 2, \dots). \tag{2.3}$$

Proof. Since $P_n(\cos\theta) = 2^{1-n}\cos n\theta$, formula (2.2) becomes

$$A_n(f) = \frac{2^{n-1} r^{n-1}}{K\sqrt{k_0}} \int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) \cos n\theta \, d\theta \quad (n = 0, 1, 2, \dots).$$

Making the change of variable $x = \cos\theta$ yields

$$A_n(f) = \frac{2^{n-1} r^{n-1}}{K\sqrt{k_0}} \int_{-1}^{1} f\left(\varphi\left(\frac{2x}{r}\right)\right) \frac{P_n(x)}{\sqrt{1-x^2}} \, dx.$$

Multiplying the identity [**1**, p. 785]

$$\frac{P_n(x)}{\sqrt{1-x^2}} = \frac{(-1)^n 2^{1-n}}{1 \cdot 3 \cdot \dots \cdot (2n-1)} \frac{d^n}{dx^n}[(1-x^2)^{n-1/2}]$$

by $f(\varphi(2x/r))$ and then integrating from -1 to 1 we obtain

$$\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-x^2}} f\left(\varphi\left(\frac{2x}{r}\right)\right) dx = \frac{(-1)^n 2^{1-n}}{1 \cdot 3 \cdot \dots \cdot (2n-1)} \int_{-1}^{1} \frac{d^n}{dx^n}[(1-x^2)^{n-1/2}]\left(f\left(\varphi\left(\frac{2x}{r}\right)\right)\right) dx. \tag{2.4}$$

Integrating the right-hand side of (2.4) by parts n times results in

$$\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-x^2}} f\left(\varphi\left(\frac{2x}{r}\right)\right) dx = \frac{(-1)^n 2^{1-n}}{1 \cdot 3 \cdot \dots \cdot (2n-1)} \int_{-1}^{1} (1-x^2)^{n-1/2} \left(f\left(\varphi\left(\frac{2x}{r}\right)\right)\right)^{(n)} dx. \tag{2.5}$$

The result follows from (2.2) by letting $x = \cos\theta$ in (2.5), and carrying through the details of this change of variables. □

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Theorem 2.3. *If $k(z)$, $c(z)$ and $p(z)$ are given by (1.2), (1.8) and (1.10), respectively, then we have, for $n = 0, 1, 2, \dots$,*

$$|A_0(f) \pm r^{-2n} A_{2n}(f)| \leqslant A_0(k) \pm r^{-2n} A_{2n}(k), \quad f \in S, \tag{2.6}$$

$$|A_0(f) \pm r^{-2n} A_{2n}(f)| \leqslant A_0(c) \pm r^{-2n} A_{2n}(c), \quad f \in C, \tag{2.7}$$

and

$$|A_0(f) \pm r^{-2n} A_{2n}(f)| \leqslant A_0(p) \pm r^{-2n} A_{2n}(p), \quad f \in P. \tag{2.8}$$

Equalities occur in (2.6)–(2.8) if and only if $f(z) = k(z)$ or $f(z) = -k(-z)$, $f(z) = c(z)$ or $f(z) = -c(-z)$ and $f(z) = p(z)$ or $f(z) = p(-z)$, respectively.

Proof. To prove (2.6), let $f \in S$ be given by (1.1) and consider

$$I_n(f) = \int_0^{\pi} f\biggl(\varphi\biggl(\frac{2\cos\theta}{r}\biggr)\biggr)(1 \pm \cos 2n\theta)\,\mathrm{d}\theta \quad (n = 0, 1, 2, \dots). \tag{2.9}$$

Then

$$\begin{aligned} I_n(f) &= \int_0^{\pi/2} \biggl[f\biggl(\varphi\biggl(\frac{2\cos\theta}{r}\biggr)\biggr) + f\biggl(-\varphi\biggl(\frac{2\cos\theta}{r}\biggr)\biggr)\biggr](1 \pm \cos 2n\theta)\,\mathrm{d}\theta \\ &= 2\int_0^{\pi/2} \biggl[\sum_{m=1}^{\infty} a_{2m}\varphi^{2m}\biggl(\frac{2\cos\theta}{r}\biggr)\biggr](1 \pm \cos 2n\theta)\,\mathrm{d}\theta. \end{aligned}$$

Thus

$$|I_n(f)| \leqslant 2\int_0^{\pi/2} \biggl[\sum_{m=1}^{\infty} |a_{2m}|\varphi^{2m}\biggl(\frac{2\cos\theta}{r}\biggr)\biggr](1 \pm \cos 2n\theta)\,\mathrm{d}\theta, \tag{2.10}$$

since $\varphi(x) \geqslant 0$ for $x \in [0, 2/r]$. Using the de Branges theorem in (2.10) results in

$$|I_n(f)| \leqslant 2\int_0^{\pi/2} \biggl[\sum_{m=1}^{\infty} 2m\varphi^{2m}\biggl(\frac{2\cos\theta}{r}\biggr)\biggr](1 \pm \cos 2n\theta)\,\mathrm{d}\theta = I_n(k) = -I_n(-k(-z)).$$

Hence there are two extremal functions, which are given by $f(z) = k(z)$ and $f(z) = -k(-z)$.

In a similar way, the proof (2.7) follows from the coefficient estimate $|a_n| \leqslant 1$ ($n = 2, 3, \dots$) for the class C [**9**].

Now substituting

$$f(z) = 1 + \sum_{m=1}^{\infty} b_m z^m \in P \tag{2.11}$$

into (2.9) gives

$$I_n(f) = 2\int_0^{\pi/2} \biggl[1 + \sum_{m=1}^{\infty} b_{2m}\varphi^{2m}\biggl(\frac{2\cos\theta}{r}\biggr)\biggr](1 \pm \cos 2n\theta)\,\mathrm{d}\theta.$$

Thus

$$|I_n(f)| \leqslant 2\int_0^{\pi/2}\left[1+\sum_{m=1}^{\infty}|b_{2m}|\varphi^{2m}\left(\frac{2\cos\theta}{r}\right)\right](1\pm\cos 2n\theta)\,\mathrm{d}\theta, \tag{2.12}$$

since $\varphi(x) \geqslant 0$ for $x \in [0, 2/r]$. Using the coefficient estimate $|b_n| \leqslant 2$ $(n = 1, 2, \dots)$ for the class P [**3**] in (2.12) yields (2.8) as

$$|I_n(f)| \leqslant 2\int_0^{\pi/2}\left[1+\sum_{m=1}^{\infty}2\varphi^{2m}\left(\frac{2\cos\theta}{r}\right)\right](1\pm\cos 2n\theta)\,\mathrm{d}\theta = I_n(p) = I_n(p(-z)),$$

where $p(z)$ is given in (1.8) as desired. □

Remark 2.4. The case $n = 0$ in Theorem 2.3 yields sharp bounds for the Faber coefficient $A_0(f)$ in the classes S, C and P, respectively.

Theorem 2.5. *If $k(z)$, $c(z)$ and $p(z)$ are defined as in Theorem 2.3 and $o(z)$ is given by (1.10), then we have, for $n = 0, 1, 2, \dots$,*

$$|A_1(f) \pm \tfrac{1}{2}r^{-2n}A_{2n+1}(f) \pm \tfrac{1}{2}r^{-2n+2}A_{2n-1}(f)| \leqslant A_1(k) \pm \tfrac{1}{2}r^{-2n}A_{2n+1}(k) \pm \tfrac{1}{2}r^{-2n+2}A_{2n-1}(k), \quad f \in S, \tag{2.13}$$

$$|A_1(f) \pm \tfrac{1}{2}r^{-2n}A_{2n+1}(f) \pm \tfrac{1}{2}r^{-2n+2}A_{2n-1}(f)| \leqslant A_1(c) \pm \tfrac{1}{2}r^{-2n}A_{2n+1}(c) \pm \tfrac{1}{2}r^{-2n+2}A_{2n-1}(c), \quad f \in C, \tag{2.14}$$

$$|A_1(f) \pm \tfrac{1}{2}r^{-2n}A_{2n+1}(f) \pm \tfrac{1}{2}r^{-2n+2}A_{2n-1}(f)| \leqslant A_1(p) \pm \tfrac{1}{2}r^{-2n}A_{2n+1}(p) \pm \tfrac{1}{2}r^{-2n+2}A_{2n-1}(p), \quad f \in P, \tag{2.15}$$

and

$$|A_1(f) \pm \tfrac{1}{2}r^{-2n}A_{2n+1}(f) \pm \tfrac{1}{2}r^{-2n+2}A_{2n-1}(f)| \leqslant A_1(o) \pm \tfrac{1}{2}r^{-2n}A_{2n+1}(o) \pm \tfrac{1}{2}r^{-2n+2}A_{2n-1}(o), \quad f \in S^{(2)}, \tag{2.16}$$

where $A_{-1}(f) = 0$. The extremal functions in (2.13)–(2.15) are identical to those given in the statement of Theorem 2.3. In (2.16) equality holds if and only if $f(z) = o(z)$.

Proof. Let $f \in S$ be given by (1.1) and consider

$$L_n(f) = \int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right)\cos\theta(1\pm\cos 2n\theta)\,\mathrm{d}\theta \quad (n = 0, 1, 2, \dots).$$

Then

$$\begin{aligned} L_n(f) &= \int_0^{\pi/2}\left[f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right) - f\left(-\varphi\left(\frac{2\cos\theta}{r}\right)\right)\right]\cos\theta(1\pm\cos 2n\theta)\,\mathrm{d}\theta \\ &= 2\int_0^{\pi/2}\left[\varphi\left(\frac{2\cos\theta}{r}\right) + \sum_{m=1}^{\infty}a_{2m+1}\varphi^{2m+1}\left(\frac{2\cos\theta}{r}\right)\right]\cos\theta(1\pm\cos 2n\theta)\,\mathrm{d}\theta. \end{aligned}$$

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Hence

$$|L_n(f)| \leqslant 2\int_0^{\pi/2}\left[\varphi\left(\frac{2\cos\theta}{r}\right)+\sum_{m=1}^{\infty}|a_{2m+1}|\varphi^{2m+1}\left(\frac{2\cos\theta}{r}\right)\right]\cos\theta(1\pm\cos 2n\theta)\,\mathrm{d}\theta. \tag{2.17}$$

Using the de Branges theorem in (2.17) yields (2.13). As in Theorem 2.3, it is easy to observe that the two extremal functions are $f(z) = k(z)$ and $f(z) = -k(-z)$.

Proofs of (2.14) and (2.15) are given in a similar way.

If $f \in S^{(2)}$, then

$$L_n(f) = 2\int_0^{\pi/2} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right)\cos\theta(1\pm\cos 2n\theta)\,\mathrm{d}\theta.$$

Thus

$$|L_n(f)| \leqslant 2\int_0^{\pi/2} \left|f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right)\right|\cos\theta(1\pm\cos 2n\theta)\,\mathrm{d}\theta. \tag{2.18}$$

Using the distortion theorem

$$|f(z)| \leqslant \frac{|z|}{1-|z|^2}, \quad f \in S^{(2)},$$

[**5**, p. 70] in (2.18), one obtains that

$$|L_n(f)| \leqslant 2\int_0^{\pi/2} \frac{\varphi(2\cos\theta/r)}{1-\varphi^2(2\cos\theta/r)}\cos\theta(1\pm\cos 2n\theta)\,\mathrm{d}\theta = L_n(o),$$

because $0 \leqslant \varphi(x) < 1$ for $x \in [0, 2/r]$. □

Remark 2.6. The case $n = 0$ in Theorem 2.5 yields sharp bounds for the Faber coefficient $A_1(f)$ in the classes S, C, P and $S^{(2)}$, respectively.

Theorem 2.7. *If $k(z)$, $c(z)$, $p(z)$ and $o(z)$ are as in Theorem 2.5, then we have, for $n = 0, 1, 2, \dots,$*

$$\begin{aligned}
|(2n+1)A_1(f) \pm r^{-2n}A_{2n+1}(f)| &\leqslant (2n+1)A_1(k) \pm r^{-2n}A_{2n+1}(k), \quad f \in S,\\
|(2n+1)A_1(f) \pm r^{-2n}A_{2n+1}(f)| &\leqslant (2n+1)A_1(c) \pm r^{-2n}A_{2n+1}(c), \quad f \in C,\\
|(2n+1)A_1(f) \pm r^{-2n}A_{2n+1}(f)| &\leqslant (2n+1)A_1(p) \pm r^{-2n}A_{2n+1}(p), \quad f \in P,\\
|(2n+1)A_1(f) \pm r^{-2n}A_{2n+1}(f)| &\leqslant (2n+1)A_1(o) \pm r^{-2n}A_{2n+1}(o), \quad f \in S^{(2)}.
\end{aligned}$$

Extremal functions are as in Theorem 2.5.

Proof. Let

$$M_n(f) = \int_0^{\pi} f\left(\varphi\left(\frac{2\cos\theta}{r}\right)\right)[(2n+1)\cos\theta \pm \cos(2n+1)\theta]\,\mathrm{d}\theta.$$

Then Theorem 2.7 follows by using the argument of Theorem 2.5 and noting that

$$(2n+1)\cos\theta \pm \cos(2n+1)\theta \geqslant 0, \quad \theta \in [0, \pi/2].$$

$\square$

Remark 2.8. The case $n = 0$ in Theorem 2.7 again yields bounds for the Faber coefficient $A_1(f)$ in the classes S, C, P and $S^{(2)}$, respectively.

Remark 2.9. More coefficient estimates may be obtained this way by considering different trigonometric inequalities. For instance, the trigonometric inequality $\cos^4\theta \geqslant 0$ on $[0, \pi/2]$ gives rise to a sharp estimate of a linear combinations of the Faber coefficients $A_0(f)$, $A_2(f)$ and $A_4(f)$ in the classes $S(E_r)$, $C(E_r)$ and $P(E_r)$ since $\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$.

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