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SMALLEST REGULAR GRAPHS WITHOUT NEAR 1-FACTORS

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Abstract

A near 1-factor of a graph of order $2n \geqslant 4$ is a subgraph isomorphic to $(n-2)K_2 \cup P_3 \cup K_1$. Wallis determined, for each $r \geqslant 3$, the order of a smallest r-regular graph of even order without a 1-factor; while for each $r \geqslant 3$, Chartrand, Goldsmith and Schuster determined the order of a smallest r-regular, $(r-2)$-edge-connected graph of even order without a 1-factor. These results are extended to graphs without near 1-factors. It is known that every connected, cubic graph with less than six bridges has a near 1-factor. The order of a smallest connected, cubic graph with exactly six bridges and no near 1-factor is determined.

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Introduction

A *factor* of a graph G is a spanning subgraph of G; an r-regular factor is called an *r-factor*. A great deal of research has involved the existence or non-existence of 1-factors in graphs. Graphs that contain 1-factors have been characterized by Tutte [5].

THEOREM A (Tutte [5]). *A graph G has a 1-factor if and only if for every proper subset S of $V(G)$, the number of odd components of $G - S$ does not exceed $|S|$.*

Of particular interest has been the determination of those r-regular graphs, $r \geqslant 3$, that contain 1-factors. Of course, any graph having a 1-factor must have even order. An important and famous result along this line for 3-regular graphs is due to Petersen [3]. A *bridge* in a connected graph is an edge whose deletion results in a disconnected graph.

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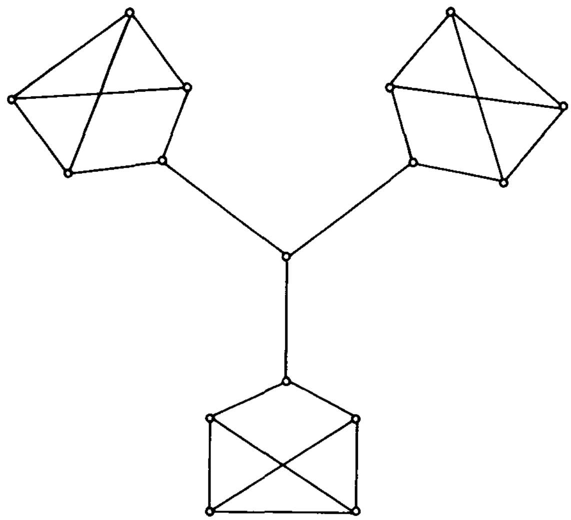

FIGURE 1. The smallest cubic graph without a 1-factor

THEOREM B (Petersen [3]). *Every cubic graph with at most two bridges contains a 1-factor.*

This result is best possible since cubic graphs with three bridges and no 1-factors exist. The graph of Figure 1 is the unique smallest such graph. Theorem B also implies that every 3-regular, 2-edge-connected graph contains a 1-factor.

All smallest connected r-regular graphs of even order without 1-factors have been studied and characterized for every $r \geqslant 3$ by Wallis [6]. This result has no connectivity condition (except that the graph be connected). Pila [4] introduced a connectivity contition to this problem when he determined, for integers $r(\geqslant 3)$ and $n(\geqslant 1)$, the smallest order of an r-regular, n-connected graph of even order having no 1-factor.

It is well-known that an r-regular, $(r-1)$-edge-connected graph $(r \geqslant 3)$ of even order contains a 1-factor (see [1], for example). This result is best possible in the sense that an r-regular, $(r-2)$-edge-connected graph, $r \geqslant 3$, of even order need not contain a 1-factor. Chartrand, Goldsmith and Schuster [1] determined the smallest order of an r-regular, $(r-2)$-edge-connected graph $(r \geqslant 3)$ of even order and containing no 1-factor.

In [2] the concept of a near 1-factor was introduced. A *near* 1-*factor* of a graph G of order $2n \geqslant 4$ is a factor of the type $(n-2)K_2 \cup P_3 \cup K_1$. (Note that a 1-factor of G is a factor of the type nK_2.) Therefore, a 1-factor and a near 1-factor differ in the placement of a single edge—more formally, the replacement

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of any edge uv in a 1-factor F of G by an edge uw in the complement of F produces a near 1-factor of G.

Chartrand, Kapoor, Lesniak and Schuster [2] showed that the smallest r-regular, $(r-2)$-edge-connected graphs of even order and not containing a 1-factor can, however, be isomorphically factored into near 1-factors.

It is rather easy to see that every r-regular, $(r-1)$-edge-connected graph of even order contains a near 1-factor. The following result was obtained in [2].

THEOREM C (Chartrand, Kapoor, Lesniak and Schuster [2]). *If G is an r-regular, $(r-2)$-edge-connected graph $(r \geqslant 3)$ of even order p containing less than $2r$ distinct edge cut sets of cardinality $r-2$, then G contains a near 1-factor.*

The following extension of Tutte's Theorem [2] was also obtained and will prove to be useful.

THEOREM D (Chartrand, Kapoor, Lesniak and Schuster [2]). *Let G be a graph of even order $2n \geqslant 4$ with at most one isolated vertex such that $G \not\cong nK_2$. Then G has a near 1-factor if and only if for every proper subset S of $V(G)$, the number of odd components of $G-S$ does not exceed $|S|+2$.*

Using Theorem D and following the general scheme employed by Wallis in [6], we now proceed to determine the smallest order of connected regular graphs without near 1-factors.

Smallest connected regular graphs without near 1-factors

The following elementary fact (which was also proved in [6]) will be useful in this and the subsequent section.

LEMMA 1. *If G is an r-regular graph, and if H is a subgraph of G induced by a nonempty subset T of vertices of G, where $|T| \leqslant r$, then the number of edges joining H and $G-H$ is at least r.*

PROOF. Let $t = |T|$. If v is a vertex of H, then $\deg_H v \leqslant t-1$ and $\deg_G v = r$; therefore, v is adjacent to at least $r-(t-1)$ vertices of $G-T$. Then the number of edges joining H and $G-T$ is at least $t(r-t+1)$. Since $1 \leqslant t \leqslant r$, we have

$$(t-1)(r-t) = rt - r - (t^2 - t) \geqslant 0,$$

which implies that $t(r-t+1) \geqslant r$, thereby proving the lemma.

LEMMA 1. *If G is a connected r-regular graph of even order p, and if G does not contain a near 1-factor, then $p \geqslant f(r)$, where*

$$f(r) = \begin{cases} 34 & \text{for } r = 3, \\ 44 & \text{for } r = 4 \text{ and } r = 6, \\ 56 & \text{for } r = 8, \\ 5r + 11 & \text{for } r \geqslant 5 \text{ and } r \text{ odd}, \\ 5r + 6 & \text{for } r \geqslant 10 \text{ and } r \text{ even}. \end{cases}$$

PROOF. If G does not contain a near 1-factor, then according to Theorem D, there is a set $S \subseteq V(G)$, with $|S| = k$, such that $G - S$ has n odd components and $n > k + 2$. The fact that p is even implies that n and k have the same parity; therefore, the former inequality can be improved to

$$n \geqslant k + 4. \tag{1}$$

Let $G_1, G_2, \ldots, G_n$ be the odd components of $G - S$ and let a_i denote the number of edges joining G_i and S. Further, let m be the number of odd components of $G - S$ with more than r vertices, so that $n - m$ is the number of odd components of $G - S$ with at most r vertices. Note that if r is odd and G_i is a component of $G - S$ with more than r vertices, then G_i has at least $r + 2$ vertices.

The next step will be to show that $m \geqslant 5$. This is done by finding upper and lower bounds on the number of edges joining S and the n odd components of $G - S$. In our setting, this number is equal to $\sum_{i=1}^{n} a_i$. Each vertex of S contributes at most r edges to this sum because G is r-regular; therefore,

$$\sum_{i=1}^{n} a_i \geqslant kr. \tag{2}$$

On the other hand, there is at least one edge joining each component to S because G is connected. But we can say more about those $n - m$ components of order at most r. By Lemma 1, each such component is joined to S by at least r edges; therefore,

$$\sum_{i=1}^{n} a_i \geqslant (n - m)r + m. \tag{3}$$

Comparing the inequalities (2) and (3), we obtain

$$k \geqslant (n - m) + m/r, \tag{4}$$

but $m/r \geqslant 0$, so that

$$k \geqslant n - m. \tag{5}$$

Using inequality (1), we have $k \geqslant (k+4) - m$ and, consequently, $m \geqslant 4$. Since $m > 0$, it follows by (4) that $k > n - m$. Using (1) again, we obtain $k > n - m \geqslant k + 4 - m$, which implies that

$$m \geqslant 5. \tag{6}$$

A lower bound for the order p of G is obtained by observing that (i) each of the $n - m$ components of $G - S$ having order at most r has at least one vertex, (ii) each of the m remaining components has at least $r + 1$ vertices if r is even and at least $r + 2$ vertices if r is odd, and (iii) the vertices of S are in none of these components. Therefore,

$$p \geqslant (n-m) + m(r+1) + k \quad \text{if } r \text{ is even,} \tag{7}$$

and

$$p \geqslant (n-m) + m(r+2) + k \quad \text{if } r \text{ is odd.} \tag{8}$$

However, $m \geqslant 5$, $n - m \geqslant 0$, and $k \geqslant 1$; thus

$$p \geqslant 5r + 6 \quad \text{if } r \text{ is even,} \tag{9}$$

and

$$p \geqslant 5r + 11 \quad \text{if } r \text{ is odd.} \tag{10}$$

The bounds indicated in (9) and (10) can be improved, though, for the cases $r = 3, 4, 6$ and 8. We study these cases separately.

Case 1. *Assume* $r = 3$.

We know by (6) that $m \geqslant 5$. Now $k \geqslant 2$, for if k were equal to 1, then the vertex in S cannot be joined to five or more components. From (2) it follows that $\sum_{i=1}^{n} a_i \leqslant 3k$, and from (3) we have

$$m \times 1 + (n-m) \times 3 \leqslant \sum_{i=1}^{n} a_i.$$

From (1) and the last two inequalities, we obtain

$$3n - 2m \leqslant 3k \leqslant 3(n-4).$$

Therefore $m \geqslant 6$.

Using the fact that $k \geqslant 2$ and inequality (8), we obtain $p \geqslant 32$. If $p = 32$, then $m = 6$, $n - m = 0$, and $k = 2$.

The graph G is 3-regular, S has two vertices, and $G - S$ has six components, from which it follows that G is disconnected, a contradiction. Thus $p \geqslant 34$.

Case 2. *Assume that* $r = 4, 6$ *or* 8.

We will show when r is even that

(11) $$m \geqslant 4 + \frac{8}{r-2} = \frac{4r}{r-2}$$

and

(12) $$k \geqslant \frac{8}{r-2} + (n-m).$$

If p' and q' are the order and size of an odd component G_i of $G - S$, then

$$\sum_{v \in V(G_i)} \deg_G v = 2q' + a_i = rp',$$

which implies that a_i is even. However $a_i \neq 0$ since G is connected, so that $a_i \geqslant 2$.

Now inequality (3) can be modified to produce

$$\sum_{i=1}^{n} a_i \geqslant (n-m)r + 2m.$$

From (2) we have

(13) $$\begin{aligned} kr &\geqslant (n-m)r + 2m \\ &= rn - (r-2)m \\ &\geqslant r(k+4) - (r-2)m \\ &= rk + 4r - (r-2)m. \end{aligned}$$

Hence $(r-2)m \geqslant 4r$, and finally

$$m \geqslant \frac{4r}{r-2} = 4 + \frac{8}{r-2},$$

thereby proving (11).

From (13) it follows that

$$\begin{aligned} k &\geqslant (n-m) + \frac{2m}{r} \\ &\geqslant (n-m) + \frac{2}{r} \times \frac{4r}{r-2} \\ &= (n-m) + \frac{8}{r-2}, \end{aligned}$$

which verifies (12).

Using (11), (12) and (7), we conclude that

(i) if $r = 4$, then $m \geqslant 8$, $k \geqslant 4$, and $p \geqslant 44$;

(ii) if $r = 6$, then $m \geqslant 6$, $k \geqslant 2$, and $p \geqslant 44$; and

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(iii) if $r = 8$, then $m \geqslant 6$, $k \geqslant 2$, and $p \geqslant 56$.

This completes the proof of Theorem 1.

In the remainder of this section it is shown that the bounds presented in Theorem 1 are best possible.

Theorem 2. *The smallest order p of a connected r-regular graph without a near 1-factor is $p = f(r)$, where*

$$f(r) = \begin{cases} 34 & \text{for } r = 3, \\ 44 & \text{for } r = 4, 6, \\ 56 & \text{for } r = 8, \\ 5r + 11 & \text{for } r \geqslant 5 \text{ and } r \text{ odd}, \\ 5r + 6 & \text{for } r \geqslant 10 \text{ and } r \text{ even}. \end{cases}$$

Proof. According to Theorem 1, it is enough to show the existence of connected r-regular graphs of order $f(r)$ having no near 1-factor. We now consider a number of cases.

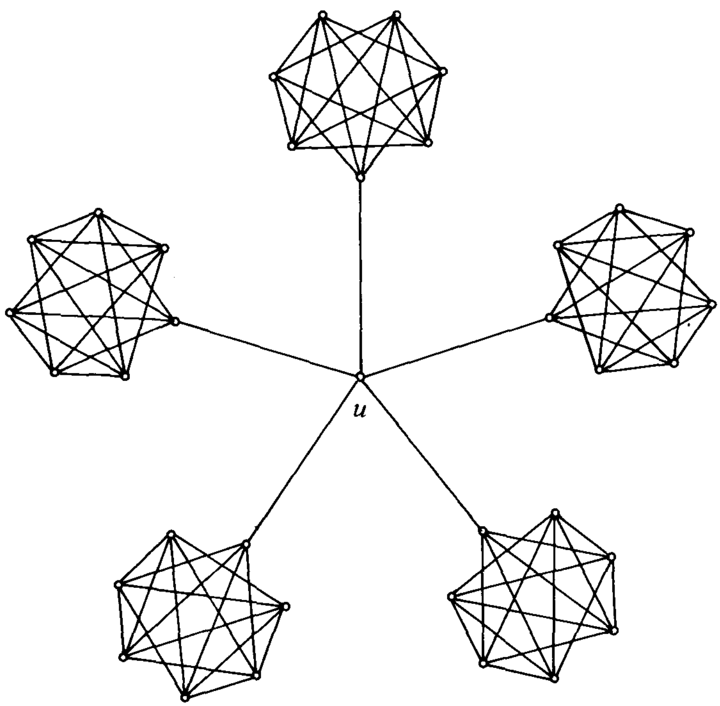

Figure 2. A smallest connected 5-regular graph without a near one-factor

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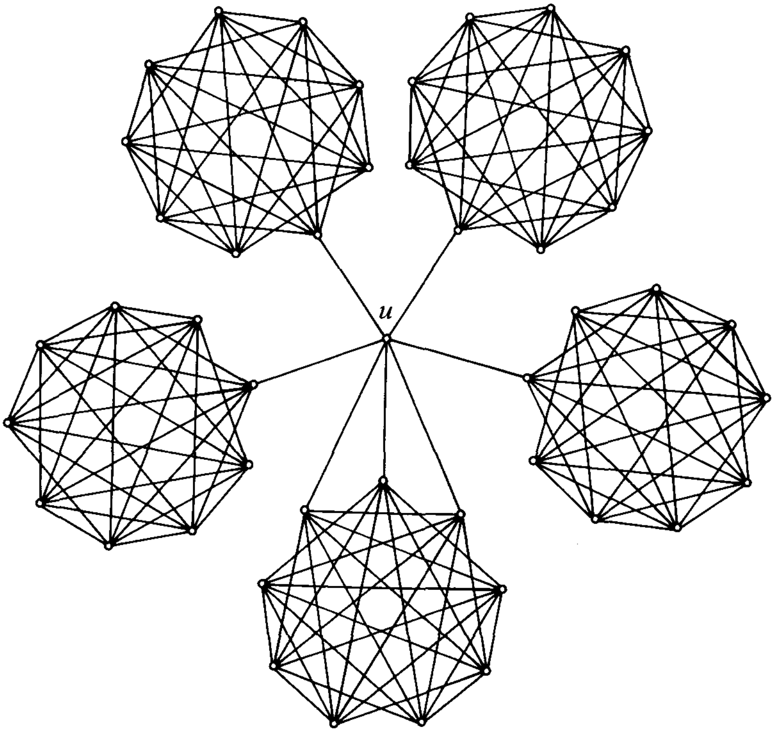

FIGURE 3. A smallest connected 7-regular graph without a near one-factor

Case 1. *Assume r is odd, $r \geqslant 5$ and $p = 5r + 11$.*

Using the terminology presented in the preceding theorem, we have in this case, that $k = 1$ and $m = 5$.

The five odd components are described as

$$G_i = \overline{((r-1)/2)K_2 \cup P_3} \quad \text{for } 1 \leqslant i \leqslant 4,$$

and

$$G_5 = \overline{P_{r-2} \cup 2K_2}.$$

Note that G_5 has $r - 4$ vertices of degree $r - 1$ and all others of degree r. The graphs G_i $(1 \leqslant i \leqslant 4)$ have exactly one vertex of degree $r - 1$, while all five graphs G_i $(1 \leqslant i \leqslant 5)$ have order $r + 2$. A graph G is constructed by adding an extra vertex u and joining u to each vertex of degree $r - 1$ in the graph $G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5$. In this case $S = \{u\}$, and $G - S$ has five odd components. Thus, by Theorem D, G has no near 1-factor. Figures 2 and 3 illustrate the construction for $r = 5$ and $r = 7$, respectively.

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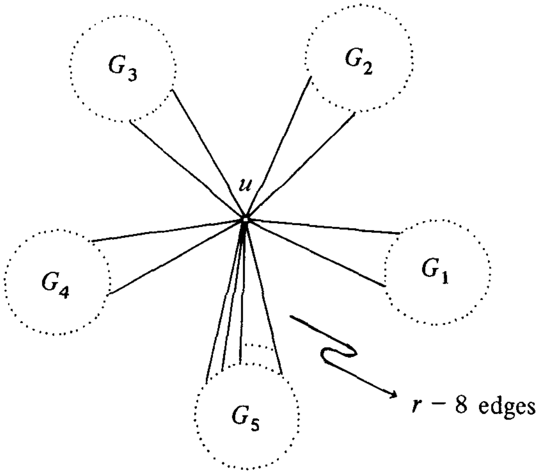

FIGURE 4. Construction of the smallest r-regular connected graph without a near one-factor for r even and $r \geqslant 10$

Case 2. *Assume r is even*, $r \geqslant 10$ *and* $p = 5r + 6$.

Now we have $k = 1$ and $m = 5$ as in the former case. Here S consists of a single vertex u and five components $G_1, G_2, \ldots, G_5$ of order $t + 1$, where

$$G_i = \overline{K_2 \cup (r-1)K_1} = K_{r+1} - e \quad \text{for } i = 1, 2, 3, 4$$

and

$$G = \overline{((r-8)/2)K_2 \cup 9K_1}.$$

Figure 4 illustrates the construction, and the circles represent the components.

Case 3. *Assume* $r = 3$.

For $k = 4$ and $m = 6$ we obtain the graph of order 34 in Figure 5. The set $S = \{a, b, c, d\}$ and the six odd components guarantee that this graph does not contain a near 1-factor by Theorem D.

Case 4. *Assume* $r = 4$.

For $r = 4$, we obtain $k = 4$, $m = 8$, and $p = 44$ (see Figure 6, where $S = \{a, b, c, d\}$).

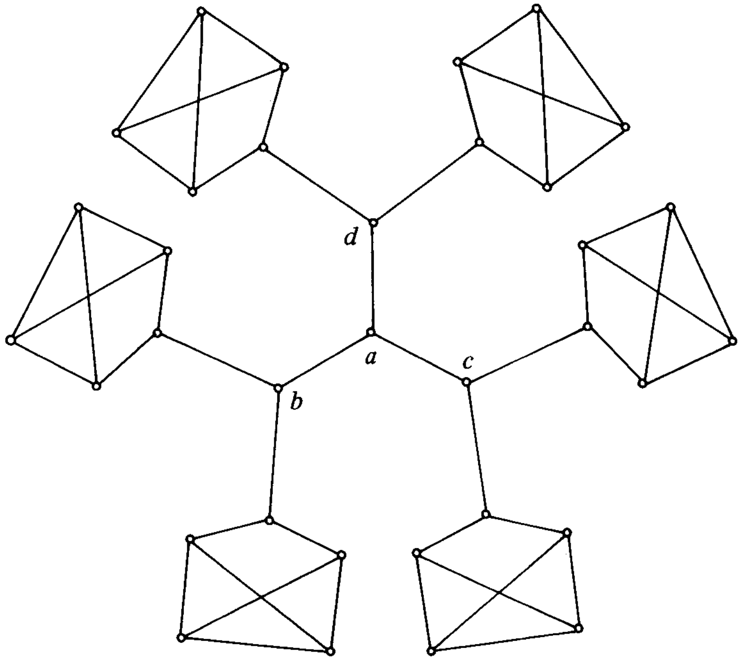

FIGURE 5. A smallest connected cubic graph without a near one-factor

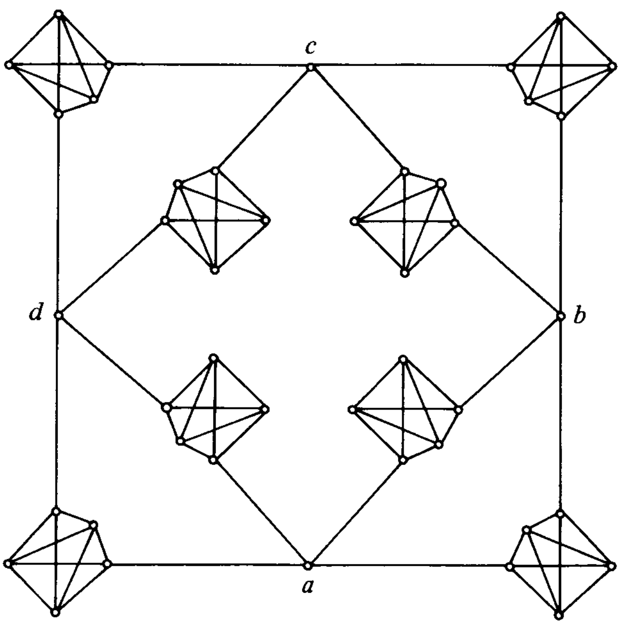

FIGURE 6. A smallest connected 4-regular graph without a near one-factor

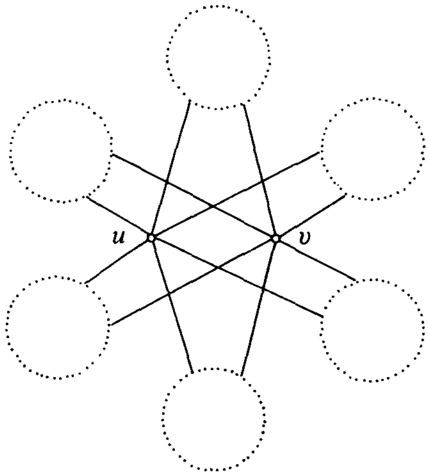

FIGURE 7. A smallest connected 6-regular graph without a near one-factor

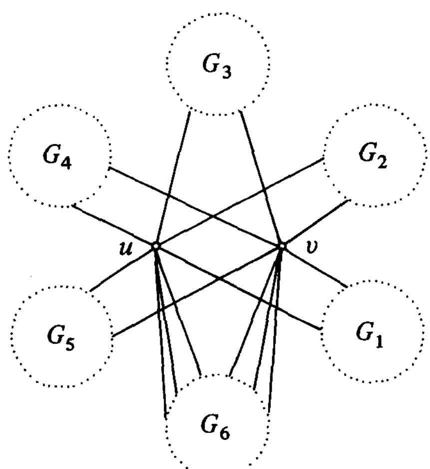

FIGURE 8. Smallest connected 8-regular graph without a near one-factor

Case 5. *Assume* $r = 6$.

Here we have $k = 2$, $m = 6$, and $p = 44$. Take six components isomorphic to $K_7 - e$, and let $S = \{u, v\}$. Join each vertex of degree 5 in the components to one of the vertices in S. The graph obtained is indicated in Figure 7, where each circle represents $K_7 - e$.

Case 6. *Assume* $r = 8$.

In this case we have, $k = 2$, $m = 6$, and $p = 56$. Let $S = \{u, v\}$. For $1 \leqslant i \leqslant 5$, let $G_i = K_9 - e$ and define G_6 to be K_9 with three independent edges removed. A graph G is obtained by joining the vertices of S to each vertex of degree 7 in the components G_i, $1 \leqslant i \leqslant 5$, and by joining u to three vertices of degree 7 in G_6 and v to the remaining three vertices of degree 7 in G_6, as shown in Figure 8. This completes the proof of the theorem.

Smallest r-regular, $(r - 2)$-edge-connected graphs without near 1-factors

In the next two results we will be concerned with the minimality problem, but now with restrictions on the edge-connectivity. As we saw in the introduction, an r-regular, $(r - 1)$-edge-connected graph must have a near 1-factor, but an r-regular, $(r - 2)$-edge-connected graph need not. However for such graphs of small order, they must contain a near 1-factor, as we verify.

THEOREM 3. *Let G be an r-regular*, $(r - 2)$*-edge-connected graph*, $r \geqslant 3$, *of even order p. If* $p < 4(r\lceil r/2\rceil + r - 1)$, *then G has a near* 1*-factor*.

PROOF. Assume, to the contrary, that G has no near 1-factor. Then, by Theorem D, there exists a proper subset S of $V(G)$ such that $G - S$ has odd components $G_1, G_2, \ldots, G_n$, where $|S| = k$ and $n > k + 2$. However, n and k have the same parity; therefore,

$$n \geqslant k + 4. \tag{14}$$

Let a_i be the number of edges joining G_fi and S. By Lemma 1, if $|V(G_i)| \leqslant r$, then $a_i \geqslant r$. Let m denote the number of odd components G_i of $G - S$ having more than r vertices.

Because G is $(r - 2)$-edge-connected, then $a_i \geqslant r - 2$ whenever $|V(G_i)| \geqslant r + 1$; therefore,

$$\begin{aligned} \sum_{i=1}^{n} a_i &= \sum_{|V(G_i)| \leqslant r} a_i + \sum_{|V(G_i)| \geqslant r+1} a_i \\ &\geqslant \sum_{|V(G_i)| \leqslant r} r + \sum_{|V(G_i)| \geqslant r+1} (r - 2) \\ &= (n - m)r + m(r - 2) \\ &= nr - 2m. \end{aligned} \tag{15}$$

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Further,

$$\sum_{i=1}^{n} a_i \leqslant \sum_{v \in S} \deg v = kr. \tag{16}$$

By (15) and (16), $nr - 2m \leqslant kr$, so that

$$2m \geqslant nr - kr = (n-k)r \geqslant 4r;$$

thus

$$m \geqslant 2r. \tag{17}$$

Using (16) and the fact that $n \geqslant k + 4$, we have

$$kr \geqslant \sum_{i=1}^{n} a_i \geqslant n(r-2) \geqslant (k+4)(r-2),$$

so that

$$k \geqslant 2r - 4. \tag{18}$$

Hence, for r even, we have by (17) and (18) that

$$\begin{aligned} p &\geqslant m(r+1) + k \geqslant 2r(r+1) + 2r - 4 \\ &= 2r^2 + 4r - 4 \\ &= 4(r\lceil r/2 \rceil + r - 1), \end{aligned}$$

which contradicts the hypothesis.

For r odd, we have

$$\begin{aligned} p &\geqslant m(r+2) + k \geqslant 2r(r+2) + 2r - 4 \\ &= 2r^2 + 6r - 4 \\ &= 4(r\lceil r/2 \rceil + r - 1), \end{aligned}$$

again producing a contradiction.

The smallest order of a connected cubic graph without a near 1-factor is 34, as was seen in Theorem 2. For $r = 3$, $4(r\lceil r/2 \rceil + r - 1) = 32$; therefore Theorem 3 does not give the best result for cubic graphs. For $r > 3$, the bound given in Theorem 3 is sharp.

THEOREM 4. *For every integer $r > 3$, there exists an r-regular, $(r-2)$-edge-connected graph of order $4(r\lceil r/2 \rceil + r - 1)$ containing no near 1-factor.*

PROOF. We present different constructions according to the parity of r.

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Case 1. *Assume* $r = 2n$ *for some* $n \geqslant 2$.

Let $H = \overline{(n-1)K_2 \cup 3K_1}$. Note that H has (odd) order $r + 1$, three of its vertices have degree r, and the remaining $r - 2$ vertices have degree $r - 1$. For $i = 1, 2, 3, 4$, let $A_i = (n-1)K_1$, and let $B_i = nH$. An r-regular graph G is produced by adding some edges among the graphs A_i and B_i, for $i = 1, 2, 3, 4$, so that the order of G will be $4|V(A_1) \cup V(B_1)| = 4|V((n-1)K_1) \cup V(nH)| = 4(n - 1 + n(r+1)) = 4(r\lceil r/2 \rceil + r - 1)$.

For $i = 1, 2, 3$, in each component H of B_i, join each of the $r - 2$ vertices of degree $r - 1$ to one vertex of $A_i \cup A_{i+1}$ in such a way that each vertex of $A_i \cup A_{i+1}$ is adjacent to exactly one vertex of each component of B_i. In a similar way, add the corresponding edges between B_4 and $A_4 \cup A_1$. The graph G obtained is r-regular, and it has the property that for every two nonadjacent vertices u and v, there exist at least $r - 2$ edge-disjoint $u - v$ paths. Therefore, G is $(r-2)$-edge-connected. By removing from G the $4n - 4$ vertices of $A_1 \cup A_2 \cup A_3 \cup A_4$, we obtain the union of $4n$ components isomorphic to H, where H has odd order $r + 1$. Therefore, by Theorem D, G has no near 1-factor. Figure 6 also illustrates this theorem for $r = 4$.

Case 2. *Assume* $r = 2n + 1$ *for some integer* $n \geqslant 2$.

Let $H = \overline{P_r \cup P_2}$, so that H has odd order $r + 2$, four vertices of degree r, and $r - 2$ vertices of degree $r - 1$. Let A_1 and A_3 be isomorphic to nK_1, and let A_2 and A_4 be isomorphic to $(n-1)K_1$. Further, let B_i be isomorphic to the union of $n + 1$ copies of H or n copaies of H, depending on whether $i \in \{1,3\}$ or $i \in \{2,4\}$, respectively. We assume that the graphs A_i and B_i, $i = 1, 2, 3, 4$, have pairwise disjoint vertex sets. The union of these sets will give the vertex set of the r-regular graph G without a near 1-factor, that is, G has

$$2(n+1)(r+2) + 2n(r+2) + 2n + 2(n-1) = 4r\lceil r/2 \rceil + r - 1$$

vertices.

For $i = 1, 2, 3$, and for each of those components in B_i isomorphic to H, join each of the $r - 2$ vertices of degree $r - 1$ to a vertex of $A_i \cup A_{i+1}$ in such a way that every vertex of $A_i \cup A_{i+1}$ is adjacent to exactly one vertex of each component of B_i. Analogously, add the appropriate edges between B_4 and $A_4 \cup A_1$.

The graph G so obtained is r-regular and $(r-2)$-edge-connected. By removing from G the $2n + 2(n-1) = 4n - 2$ vertices of $A_1 \cup A_2 \cup A_3 \cup A_4$, we obtain the union of $2(n+1) + 2n = 4n + 2$ odd components H; so by Theorem D, G has no near 1-factor. Moreover, it is easily seen that G is $(r-2)$-edge-connected.

Figure 9 illustrates the theorem for $r = 5$, where each dotted circle represents a copy of H.

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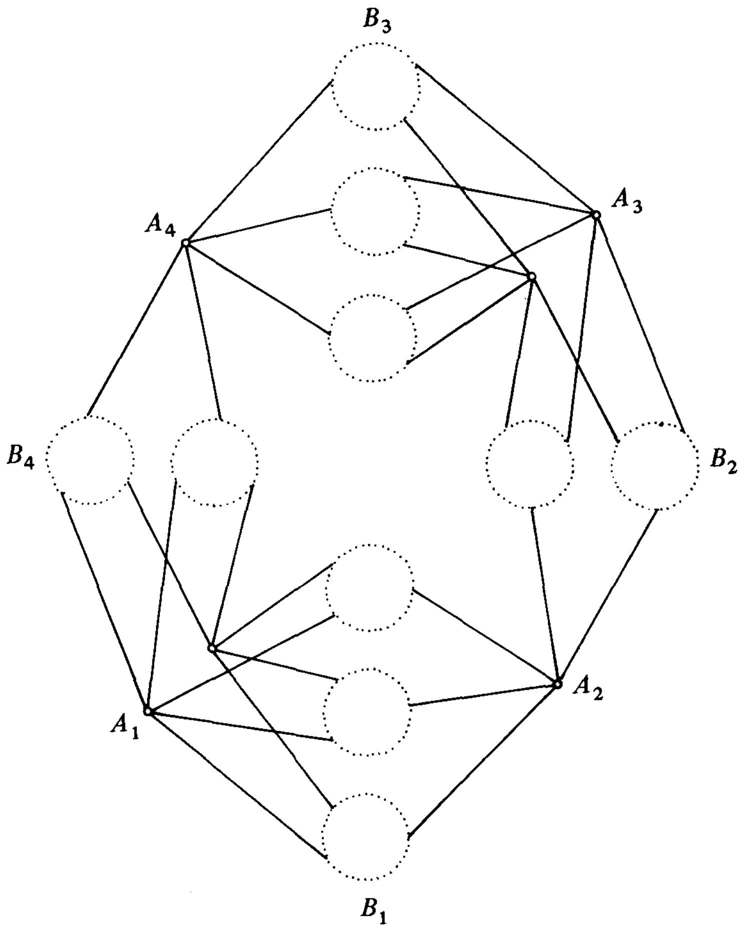

FIGURE 9. A smallest 3-edge-connected 5-regular graph without a near one-factor

A smallest connected cubic graph with six bridges and without a near 1-factor

If we apply Theorem C to cubic graphs, we see that every cubic graph with less than six bridges contains a near 1-factor. A cubic graph of order 44 with six bridges and without a near 1-factor was found by Chartrand, Kapoor, Lesniak and Schuster [2]. A natural question arises: What is the minimum order of a cubic graph G with six bridges and containing no near 1-factor? The answer to this question is provided in our next result.

THEOREM 5. *The smallest order of a connected cubic graph of even order with six bridges and containing no near* 1*-factor is* 40.

PROOF. Suppose, to the contrary, that there exists a graph G that is connected and cubic, has six bridges but no near 1-factor, and has even order p, where $p < 40$.

By Theorem D, there exists a proper subset S of $V(G)$ having cardinality k such that the number n of odd components of $G - S$ is at least $k + 4$.

From the proof of Theorem 1, we know that $k \geqslant 2$ and $p \geqslant (n - m) + 5m + k$, where $m(\geqslant 6)$ is the number of odd components with at least five vertices.

Suppose that $k = 2$. The removal of the two vertices of S will produce at most five components, a contradiction when $n \geqslant 6$. Therefore $k \geqslant 3$.

Suppose that $k = 3$, so that $n \geqslant 7$. The removal of the three vertices of S produces at most seven components, but since $n \geqslant 7$, we conclude that $n = 7$. Therefore each vertex of S must be incident with three bridges, necessarily all of which are distinct bridges of G; therefore G has at least nine bridges—a contradiction. Hence $k \geqslant 4$.

Suppose that $k = 4$, so that $n \geqslant 8$ and $p \geqslant 36$. First assume that $p = 36$. In this case $G - S$ must have six components of order 5 and two trivial components with vertices u and w. By a straightforward argument it follows that S is an independent set of vertices, and that exactly two vertices of S are joined to both u and w.

Suppose $v_1, v_2 \in S$ are joined to u and w. Then each of v_1 and v_2 is incident with exactly one bridge, while the other two vertices of S are incident with three bridges, which implies that G has eight bridges—a contradiction.

Suppose $k = 4$ but $p = 38$. Then $m = 6$ and $n - m = 4$. In this case the ten components of $G - S$ are joined to S with at least 18 edges, which is impossible since $|S| = 4$. Therefore, $k \geqslant 5$.

Suppose next that $k = 5$; then $p \geqslant 38$. Since $p < 40$, we have $p = 38$, which implies that $m = 6$ and $n = 9$. Hence $G - S$ has six components of order 5 and three trivial components. The number of edges joining $G - S$ with S is at least 15, but since $|S| = 5$, the number of edges joining S to $G - S$ is at most 15. This implies that the number of such edges is exactly 15. The three trivial components are joined to S with nine edges; so there are six edges between S and the six components of order 5. Therefore each component of $G - S$ of order 5 is joined to S by exactly one edge, necessarily each such edge being a bridge. Since G is connected, no vertex of S can be joined to three components of order 5. Therefore, there is a vertex of S that is joined to two components of $G - S$ having order 5, and to one trivial component. These three edges are necessarily bridges. Each of the other components of order 5 contributes one bridge. So G

has at least seven bridges—a contradiction. Therefore $k \geqslant 6$, but this implies that $p \geqslant 40$, producing the final contradiction.

Following a case by case argument, we were able to show that there are exactly two cubic graphs of order 40 with six bridges and having no near 1-factor. Both graphs have an isomorphic factorization into the graph H where $H = 16K_2 \cup 2P_3 \cup 2K_1$. Note that H differs from a near 1-factor only in the location of one edge, so that a near 1-factor has edge independence number equal to 19, while the corresponding value for H is 18. Figures 10 and 11 show both cubic graphs, with their factorizations indicated by the different ways in which their edges are represented. By studying the possible factorizations of the components of $G - S$ having order 5, one may conclude that neither graph of Figures 10 and 11 can be isomorphically factored into any other graph having edge independence number 18.

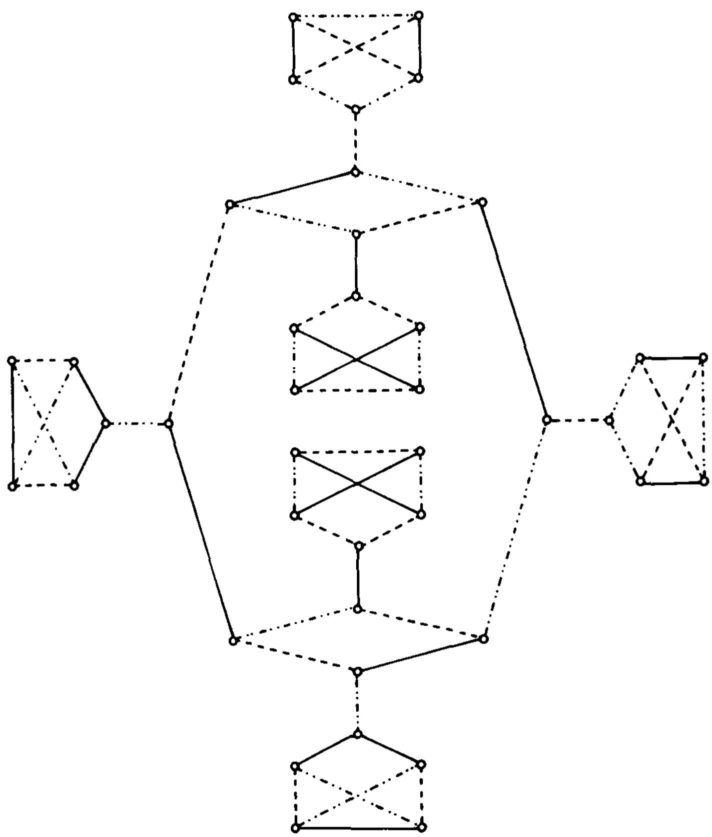

FIGURE 10. A cubic graph of order 40 with six bridges and no near one-factor

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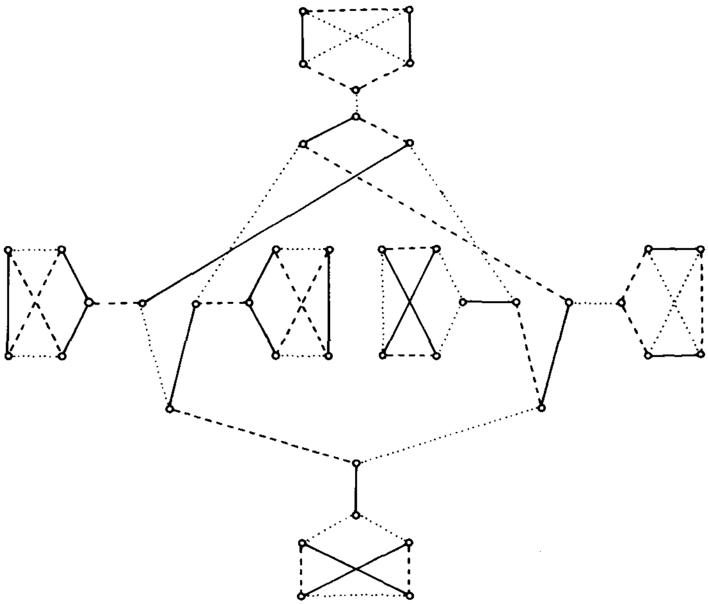

FIGURE 11. A cubic graph of order 40 with six bridges and no near one-factor

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