



On the Gras Conjecture for Imaginary Quadratic Fields

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Abstract. In this paper we extend K. Rubin's methods to prove the Gras conjecture for abelian extensions of a given imaginary quadratic field k and prime numbers p that divide the number of roots of unity in k .

1 Introduction

Let $k \subset \mathbb{C}$ be an imaginary quadratic field, and let \mathcal{O}_k be the ring of integers of k . Let $H \subset \mathbb{C}$ be the Hilbert class field of k . Let $K \subset \mathbb{C}$ be a finite abelian extension of k , and write G for the Galois group of K/k , $G := \text{Gal}(K/k)$. Let \mathcal{O}_K and \mathcal{O}_K^\times be the ring of integers of K and the group of units of \mathcal{O}_K , respectively. In [8, Theorem 3.3], K. Rubin applied the technique of Euler systems to prove the Gras conjecture for K , when $H \subset K$ and for all prime number p , $p \nmid w_H[K : k]$, where w_H is the number of roots of unity in H . Soon after he generalized his result in [9, Theorem 1] to all K (that is, without the assumption $H \subset K$) and all p , $p \nmid w_k[K : k]$, where w_k is the number of roots of unity in k . The Gras conjecture is a very subtle information about the ideal class group, used for example, in the proof of [8, Theorem 10.3] as the last step in the direction of the main conjecture.

In this paper we complete the result of Rubin. Indeed, we prove the Gras conjecture for those prime numbers $p \mid w_k$ and $p \nmid [K : k]$. As a first step, we prove a weak form of the Gras conjecture for every prime number $p \nmid [K : k]$. More precisely, let \mathcal{E}_K be the group of units of K defined in the following section (see the end of Section 4 for a description of the elements of \mathcal{E}_K as elliptic units, and also for a comparison with the group of elliptic units considered by Rubin in [8, 9]). Let $\text{Cl}(K)$ be the ideal class group of K , and let $\mathfrak{g} := [K : k]$. Then, by applying the elementary approach used in [13] (hence without using Euler systems), we prove the following formula for every nontrivial, irreducible, rational character Ψ of G :

$$(1.1) \quad \left[e_\Psi(\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \mathcal{O}_K^\times) : e_\Psi(\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \mathcal{E}_K) \right] = \# \left[e_\Psi(\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \text{Cl}(K)) \right],$$

where e_Ψ is the idempotent of $\mathbb{Z}[\mathfrak{g}^{-1}][G]$ associated with Ψ . By $\#X$ we mean the cardinality of the finite set X . Of course the formula (1.1) is already known if we replace \mathfrak{g} by $w_k \mathfrak{g}$. This is a consequence of [9, Theorem 1]. Let us remark that the product of (1.1) for all the nontrivial irreducible rational characters of G yields the

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equality

$$(1.2) \quad [\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \mathcal{O}_K^{\times} : \mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \mathcal{E}_K] = \frac{\#(\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \text{Cl}(K))}{\#(\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \text{Cl}(k))},$$

which has been known for a long time. Indeed, (1.2) is a straightforward consequence of [1, Théorème 5]. It was the ultimate ingredient used by Rubin to prove the Gras conjecture.

In Sections 4 and 5, we define our Euler systems for $p \mid w_k$ and establish all the results needed to apply them to the p -part of the ideal class group of K . Their application gives Theorem 5.4, which, with the help of (1.1) (in fact (1.2) is sufficient), implies the following theorem.

Theorem 1.1 *Let p be a prime number such that $p \mid w_k$ and $p \nmid [K : k]$. Let χ be a nontrivial irreducible \mathbb{Z}_p -character of G . Then*

$$(1.3) \quad [e_{\chi}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_K^{\times}) : e_{\chi}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{E}_K)] = \#[e_{\chi}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{Cl}(K))],$$

where e_{χ} is the idempotent of $\mathbb{Z}_p[G]$ associated with χ .

In a forthcoming paper we shall apply Theorem 1.1 to the main conjecture for prime numbers $p, p \mid w_k$.

2 The Group \mathcal{E}_K

It is well known that Stark conjectures are satisfied for abelian extensions of imaginary quadratic fields. Moreover, the Stark units are constructed by using appropriate elliptic units. On the other hand, the groups of elliptic units are generated by the norms of these Stark units.

For each nonzero ideal \mathfrak{m} of \mathcal{O}_k , we denote by $H_{\mathfrak{m}} \subset \mathbb{C}$ the ray class field of k modulo \mathfrak{m} . Suppose $\mathfrak{m} \notin \{(0), \mathcal{O}_k\}$, then Stark proved in [11] the existence of an element $\varepsilon = \varepsilon_{\mathfrak{m}} \in H_{\mathfrak{m}}$ characterized, up to a root of unity, by the following three properties

- (i) Let $w_{\mathfrak{m}}$ be the number of roots of unity in $H_{\mathfrak{m}}$. Then the extension $H_{\mathfrak{m}}(\varepsilon^{1/w_{\mathfrak{m}}})/k$ is abelian.
- (ii) If \mathfrak{m} is divisible by two prime ideals, then ε is a unit of $\mathcal{O}_{H_{\mathfrak{m}}}$. If $\mathfrak{m} = \mathfrak{q}^e$, where \mathfrak{q} is a prime ideal, then

$$\varepsilon \mathcal{O}_{H_{\mathfrak{m}}} = (\mathfrak{q})_{\mathfrak{m}}^{\frac{w_{\mathfrak{m}}}{w_k}},$$

where $(\mathfrak{q})_{\mathfrak{m}}$ is the product of the prime ideals of $\mathcal{O}_{H_{\mathfrak{m}}}$ that divide \mathfrak{q} .

- (iii) Let $|z| = z\bar{z}$ for any complex number z , where \bar{z} is the complex conjugate of z . Then

$$(2.1) \quad L'_{\mathfrak{m}}(0, \chi) = -\frac{1}{w_{\mathfrak{m}}} \sum_{\sigma \in \text{Gal}(H_{\mathfrak{m}}/k)} \chi(\sigma) \ln |\varepsilon^{\sigma}|$$

for all the complex irreducible characters χ of $\text{Gal}(H_{\mathfrak{m}}/k)$.

Here $s \mapsto L_m(s, \chi)$ is the L -function associated with χ , defined for the complex numbers s such that $Re(s) > 1$, by the Euler product

$$L_m(s, \chi) = \prod_{\mathfrak{p} \nmid m} (1 - \chi(\sigma_{\mathfrak{p}})N(\mathfrak{p})^{-s})^{-1},$$

where \mathfrak{p} runs through all prime ideals of \mathcal{O}_k not dividing m . For such ideals, $\sigma_{\mathfrak{p}}$ and $N(\mathfrak{p})$ are the Frobenius automorphism of H_m/k and the order of the field $\mathcal{O}_k/\mathfrak{p}$, respectively.

For any finite abelian extension L of k , we denote by μ_L the group of roots of unity in L , by w_L the order of μ_L , and by $\mathcal{F}_L \subset \mathbb{Z}[\text{Gal}(L/k)]$ the annihilator of μ_L . The description of \mathcal{F}_L given in [12, p. 82, Lemme 1.1] and property (i) of ε_m imply that for any $\eta \in \mathcal{F}_{H_m}$ there exists $\varepsilon_m(\eta) \in H_m$ such that

$$\varepsilon_m(\eta)^{w_m} = \varepsilon_m^\eta.$$

Definition 2.1 Let \mathcal{P}_K be the subgroup of K^\times generated by μ_K and by all the norms

$$N_{H_m/H_m \cap K}(\varepsilon_m(\eta)),$$

where \mathfrak{m} is any nonzero proper ideal of \mathcal{O}_k and η is any element of \mathcal{F}_{H_m} . By definition,

$$\mathcal{E}_K := \mathcal{P}_K \cap \mathcal{O}_K^\times.$$

At the end of section 4 we give another description of \mathcal{E}_K as a group of elliptic units.

3 The Weak Gras Conjecture for \mathcal{E}_K

Let $p \nmid g$ be a prime number, and let $\mathbb{Z}_{(p)}$ be the localization of \mathbb{Z} at p . Let \mathcal{O}_p be the integral closure of $\mathbb{Z}_{(p)}$ in $\mathbb{Q}(\mu_g)$. Remark that \mathcal{O}_p is a (semi-local) principal ring. Moreover, if $\zeta \in \mu_g$ is such that $\zeta \neq 1$, then $(1 - \zeta) \in \mathcal{O}_p^\times$. Let us set $v_\infty(z) := -\ln |z|$ for all $z \in \mathbb{C}^\times$. Let $\ell_K: K^\times \rightarrow \mathbb{R}[G]$ be the G -equivariant map defined by

$$\ell_K(x) = \sum_{\sigma \in G} v_\infty(x^\sigma) \sigma^{-1}.$$

Let \widehat{G} be the group of complex irreducible characters of G .

Lemma 3.1 Let $\chi \in \widehat{G}$ be such that $\chi \neq 1$. Let χ_{pr} be the character of $\text{Gal}(H_{\mathfrak{f}_\chi}/k)$ defined by χ , where \mathfrak{f}_χ is the conductor of the fixed field K_χ of $\ker(\chi)$. Then the following equality holds in $\mathbb{C}[G]$

$$(3.1) \quad \mathcal{O}_p \ell_K(\mathcal{E}_K) e_\chi = \mathcal{O}_p \mathcal{F}_K L'_{\mathfrak{f}_\chi}(0, \bar{\chi}_{\text{pr}}) e_\chi.$$

Proof Formula (3.1) is a direct consequence of (2.1) and can be proved in the same way as [13, formula (3.2)]. Indeed, let \mathfrak{m} be a nonzero proper ideal of \mathcal{O}_k , and let $\eta \in \mathcal{F}_{H_m}$. Let

$$\varepsilon_{K,m} := N_{H_m/H_m \cap K}(\varepsilon_m) \quad \text{and} \quad \varepsilon_{K,m}(\eta) := N_{H_m/H_m \cap K}(\varepsilon_m(\eta)).$$

Then

$$w_m \ell_K(\varepsilon_{K,m}(\eta)) = \text{res}_K^{H_m}(\eta) \ell_K(\varepsilon_{K,m}),$$

where $\text{res}_K^{H_m} : \mathcal{F}_{H_m} \rightarrow \mathcal{F}_K$ is the natural restriction map. Since this map is surjective we can proceed now exactly as in the proof of the [13, formula (3.2)]. ■

To prove formula (1.1) we shall use the generalized index of Sinnott; see [10, §1, page 187]. Let V be an \mathbb{F} -vector space of finite dimension d , where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$. Let M and N be two lattices of E , that is, two free \mathbb{Z} -submodules of E , of rank d such that $\mathbb{F}M = \mathbb{F}N = V$. Then we define the index $(M : N)$ by

$$(M : N) = |\det(\gamma)|,$$

where γ is any automorphism of the \mathbb{F} -vector space E such that $\gamma(M) = N$. If $N \subset M$, then $(M : N)$ coincides with the usual index $[M : N]$. We also have the following transitivity formula:

$$(M : P) = (M : N)(N : P).$$

This leads to the identity

$$(M : N) = \frac{[M + N : N]}{[M + N : M]},$$

which may be used as a definition of $(M : N)$. We refer the reader to [10] for more details about this generalized index.

Remark 3.2 By the Dirichlet Theorem we know that $\ell_K(\mathcal{O}_K^\times)$ is a lattice of $\mathbb{R}[G](1 - e_1)$, where e_1 is the idempotent associated with the trivial character of G . In particular, for every nontrivial, irreducible, rational character Ψ of G , the \mathbb{Z} -module $\ell_K(\mathcal{O}_K^\times)e_\Psi$ is a lattice of $\mathbb{R}[G]e_\Psi$. This implies that for every nontrivial $\chi \in \widehat{G}$, the \mathcal{O}_p -module $\mathcal{O}_p \ell_K(\mathcal{O}_K^\times)e_\chi$ is free of rank one. Thus, there exists $R_\chi \in \mathbb{C}^\times$ such that

$$\mathcal{O}_p \ell_K(\mathcal{O}_K^\times)e_\chi = R_\chi \mathcal{O}_p e_\chi.$$

Lemma 3.3 *Let Ψ be a nontrivial irreducible rational character of G . Then there exists $u \in \mathcal{O}_p^\times$ such that*

$$(\mathbb{Z}[G]e_\Psi : \ell_K(\mathcal{O}_K^\times)e_\Psi) = u \prod_{\chi | \Psi} R_\chi.$$

Proof Let γ be an automorphism of the \mathbb{R} -vector space $e_\Psi \mathbb{R}[G]$ such that $0 < \det(\gamma)$ and $\gamma(\mathbb{Z}[G]e_\Psi) = \ell_K(\mathcal{O}_K^\times)e_\Psi$. Let $(b_\chi)_{\chi|\Psi}$ be a \mathbb{Z} -basis of $\mathbb{Z}[G]e_\Psi$. Then $(b_\chi)_{\chi|\Psi}$ is an \mathcal{O}_p -basis of $\mathcal{O}_p[G]e_\Psi$. Since $(e_\chi)_{\chi|\Psi}$ is also an \mathcal{O}_p -basis of $\mathcal{O}_p[G]e_\Psi$, the automorphism β_1 of the \mathbb{C} -vector space $\mathbb{C}[G]e_\Psi$, defined by

$$\beta_1(b_\chi) = e_\chi, \quad \chi|\Psi,$$

is such that $\det(\beta_1) \in \mathcal{O}_p^\times$. In the same manner, since $(\gamma(b_\chi))_{\chi|\Psi}$ and $(R_\chi e_\chi)_{\chi|\Psi}$ both are \mathcal{O}_p -basis of $\mathcal{O}_p \ell_K(\mathcal{O}_K^\times)e_\Psi$, the automorphism β_2 of $\mathbb{C}[G]e_\Psi$, defined by

$$\beta_2(\gamma(b_\chi)) = R_\chi e_\chi, \quad \chi|\Psi,$$

is such that $\det(\beta_2) \in \mathcal{O}_p^\times$. Let γ' be the automorphism of $\mathbb{C}[G]e_\Psi$ defined by $\gamma'(e_\chi) = R_\chi e_\chi$ for all $\chi|\Psi$. Then $\det(\gamma') = \prod_{\chi|\Psi} R_\chi$. Let us extend u to $\mathbb{C}[G]e_\Psi$ by linearity. Since $(\mathbb{Z}[G]e_\Psi : \ell_K(\mathcal{O}_K^\times)e_\Psi) = \det(\gamma)$ and $\gamma \circ \beta_1^{-1} = \beta_2^{-1} \circ \gamma'$, the lemma follows. ■

Lemma 3.4 *Let $F \subseteq K$ be an extension of k , and let R_F be the regulator of F . We denote by Ξ_F the set of $\chi \in \widehat{G}$ such that χ is trivial on $\text{Gal}(K/F)$. Then there exists $v \in \mathcal{O}_p^\times$ such that*

$$(3.2) \quad R_F = v \prod_{\substack{\chi \in \Xi_F \\ \chi \neq 1}} R_\chi.$$

Proof It is easy to see that $R_F = (I_F : \ell_F(\mathcal{O}_F^\times))$, where I_F is the augmentation ideal of $\mathbb{Z}[\text{Gal}(F/k)]$. Let

$$D := \text{Gal}(K/F) \quad \text{and} \quad s(D) := \sum \sigma, \sigma \in D.$$

Let $\text{cor}_{K/F} : \mathbb{Z}[\text{Gal}(F/k)] \rightarrow \mathbb{Z}[G]$ be the corestriction map. Then we have

$$(I_F : \ell_F(\mathcal{O}_F^\times)) = (\text{cor}_{K/F}(I_F) : \text{cor}_{K/F}(\ell_F(\mathcal{O}_F^\times))) = (s(D)I_K : \ell_K(\mathcal{O}_F^\times)).$$

But the group $\ell_K(\mathcal{O}_F^\times)/s(D)\ell_K(\mathcal{O}_K^\times)$ is finite and is annihilated by $\#D$. Thus,

$$(I_F : \ell_F(\mathcal{O}_F^\times)) = w(s(D)I_K : s(D)\ell_K(\mathcal{O}_K^\times)),$$

for some unit $w \in \mathcal{O}_p^\times$. To get formula (3.2) we proceed now exactly as in the proof of Lemma 3.3. ■

Theorem 3.5 *Let Ψ be a nontrivial, irreducible, rational character of G . Then*

$$[e_\Psi(\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} \mathcal{O}_K^\times) : e_\Psi(\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} \mathcal{E}_K)] = \#(e_\Psi(\mathbb{Z}[g^{-1}] \otimes_{\mathbb{Z}} \text{Cl}(K))).$$

Proof Since K_χ does not depend on the choice of $\chi|\Psi$, let us set $K_\psi := K_\chi$. Let Ξ_Ψ be the set of $\chi \in \widehat{G}$ such that $\ker(\chi)$ strictly contains $\text{Gal}(K/K_\Psi)$. For any $I \subseteq \Xi_\Psi$, we define

$$K_I := \begin{cases} K_\Psi & \text{if } I = \emptyset, \\ \bigcap_{\chi \in I} F_\chi & \text{if } I \neq \emptyset. \end{cases}$$

For any $I \subseteq \Xi_\Psi$ let ζ_{K_I} (resp. ζ_k) be the Dedekind zeta function of K_I (resp. k), and let $\tilde{\zeta}_{K_I}(0)$ be the first nonzero coefficient of the Taylor expansion of $\zeta_{K_I}(s)$ at $s = 0$. We also set $\Xi_I := \Xi_{K_I}$. It is well known that ζ_{K_I} has a zero of order $[K_I : k] - 1$ at 0, and that for any nontrivial $\chi \in \widehat{G}$, $s \mapsto L_{\bar{f}_\chi}(s, \chi_{\text{pr}})$ has a zero of order 1 at 0. Then

$$(3.3) \quad -\frac{h_I R_I}{w_I} = \tilde{\zeta}_{K_I}(0) = \zeta_k(0) \prod_{\substack{\chi \in \Xi_I \\ \chi \neq 1}} L'_{\bar{f}_\chi}(0, \chi_{\text{pr}}),$$

where $h_I := \#(\text{Cl}(K_I))$, $w_I := \#(\mu_{K_I})$, and R_I is the regulator of K_I . For any $\chi \in \widehat{G}$, let $h_\chi \in \mathcal{O}_p$ and $w_\chi \in \mathcal{O}_p$ be such that

$$\mathcal{O}_p h_\chi = \text{Fitt}_{\mathcal{O}_p}(e_\chi(\mathcal{O}_p \otimes_Z \text{Cl}(K))) \quad \text{and} \quad \mathcal{O}_p w_\chi = \text{Fitt}_{\mathcal{O}_p}(e_\chi(\mathcal{O}_p \otimes_Z \mu_K)).$$

By the inclusion-exclusion principle, as in the proof of [13, Proposition 3.2], we obtain from (3.3) and Lemma 3.4 the formula

$$(3.4) \quad \mathcal{O}_p \prod_{\chi|\Psi} L'_{\bar{f}_\chi}(0, \bar{\chi}_{\text{pr}}) = \mathcal{O}_p \prod_{\chi|\Psi} h_\chi w_\chi^{-1} R_\chi.$$

Let us remark that

$$(3.5) \quad \mathcal{O}_p \mathcal{F}_K e_\chi = \mathcal{O}_p w_\chi e_\chi$$

for all $\chi \in \widehat{G}$. Indeed, since μ_K is a cyclic $\mathbb{Z}[G]$ -module, we have $\mathcal{F}_K = \text{Fitt}_{\mathbb{Z}[G]}(\mu_K)$, and then

$$\mathcal{O}_p \mathcal{F}_K = \text{Fitt}_{\mathcal{O}_p[G]}(\mathcal{O}_p \otimes_Z \mu_K) = \bigoplus_{\chi \in \widehat{G}} \text{Fitt}_{\mathcal{O}_p}(e_\chi(\mathcal{O}_p \otimes_Z \mu_K)) e_\chi,$$

which implies (3.5). Therefore, proceeding as in the proof of Lemma 3.3, one can show from Lemma 3.1 and (3.5) that

$$(3.6) \quad \mathcal{O}_p(\mathbb{Z}[G]e_\Psi : \ell_K(\mathcal{E}_K)e_\Psi) = \mathcal{O}_p \prod_{\chi|\Psi} w_\chi L'_{\bar{f}_\chi}(0, \bar{\chi}_{\text{pr}}).$$

From (3.6) and (3.4), we obtain

$$(3.7) \quad \mathcal{O}_p(\mathbb{Z}[G]e_\Psi : \ell_K(\mathcal{E}_K)e_\Psi) = \mathcal{O}_p \prod_{\chi|\Psi} h_\chi R_\chi = \mathcal{O}_p \#(e_\Psi(\mathbb{Z}_{(p)} \otimes_Z \text{Cl}(K))) \prod_{\chi|\Psi} R_\chi.$$

From (3.7) and Lemma 3.3, we have

$$(3.8) \quad \mathcal{O}_p[\ell_K(\mathcal{O}_K^\times)e_\Psi : \ell_K(\mathcal{E}_K)e_\Psi] = \mathcal{O}_p \#(e_\Psi(\mathbb{Z}_{(p)} \otimes_Z \text{Cl}(K))).$$

Since p is prime to \mathfrak{g} , (3.8) gives

$$\mathcal{O}_p[\mathbb{Z}[\mathfrak{g}^{-1}]\ell_K(\mathcal{O}_K^\times)e_\Psi : \mathbb{Z}[\mathfrak{g}^{-1}]\ell_K(\mathcal{E}_K)e_\Psi] = \mathcal{O}_p\#(e_\Psi(\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \text{Cl}(K))).$$

This being true for every prime $p \nmid \mathfrak{g}$, and since the integers we are comparing are prime to \mathfrak{g} , we have

$$[\mathbb{Z}[\mathfrak{g}^{-1}]\ell_K(\mathcal{O}_K^\times)e_\Psi : \mathbb{Z}[\mathfrak{g}^{-1}]\ell_K(\mathcal{E}_K)e_\Psi] = \#(e_\Psi(\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \text{Cl}(K))).$$

But $\mathcal{O}_K^\times/\mathcal{E}_K \simeq \ell_K(\mathcal{O}_K^\times)/\ell_K(\mathcal{E}_K)$, and the theorem follows. ■

4 The Euler System

For any finite abelian extension F of k , and any fractional ideal \mathfrak{a} of k prime to the conductor of F/k , we denote by $(\mathfrak{a}, F/k)$ the automorphism of F/k associated with \mathfrak{a} by the Artin map. If $\mathfrak{a} \subset \mathcal{O}_k$, then we denote by $N(\mathfrak{a})$ the cardinality of $\mathcal{O}_k/\mathfrak{a}$. Let \mathcal{J} be the group of fractional ideals of k , and let us consider its subgroup $\mathcal{P} := \{x\mathcal{O}_k, x \in k^\times\}$. Let $H := H_{(1)} \subset \mathbb{C}$ be the Hilbert class field of k . Then the Artin map gives an isomorphism from $\text{Cl}(k) := \mathcal{J}/\mathcal{P}$ into $\text{Gal}(H/k)$. Let p be a prime number such that $p \nmid w_k$, and let $\text{Cl}_p(k)$ be the p -part of $\text{Cl}(k)$. Then fix $\mathfrak{a}_1, \dots, \mathfrak{a}_s$, a finite set of ideals of \mathcal{O}_k such that

$$\text{Cl}_p(k) = \langle \bar{\mathfrak{a}}_1 \rangle \times \dots \times \langle \bar{\mathfrak{a}}_s \rangle,$$

where $\langle \bar{\mathfrak{a}}_i \rangle \neq 1$ is the group generated by the class $\bar{\mathfrak{a}}_i$ of \mathfrak{a}_i in $\text{Cl}(k)$. If n_i is the order of $\langle \bar{\mathfrak{a}}_i \rangle$, then $(\mathfrak{a}_i)^{n_i} = a_i\mathcal{O}_k$ with $a_i \in \mathcal{O}_k$. If $\text{Cl}_p(k) = 1$, then we set $s = 1, \mathfrak{a}_1 = \mathcal{O}_k$, and $a_1 = 1$.

Let $p \nmid w_k$ be a prime number as above, and let M be a power of p . Let μ_M be the group of M -th roots of unity in \mathbb{C} . Then we define

$$K_M := K(\mu_M, (\mathcal{O}_k^\times)^{1/M}).$$

Moreover, we denote by \mathcal{L} the set of prime ideals ℓ of \mathcal{O}_k such that ℓ splits completely in the Galois extension $K_M(a_1^{1/M}, \dots, a_s^{1/M})/k$. Exactly as in [9, Lemma 3] or in [2, Lemma 3.1], we have the following lemma.

Lemma 4.1 *For each prime $\ell \in \mathcal{L}$ there exists a cyclic extension $K(\ell)$ of K of degree M , contained in the compositum $K.H_\ell$, unramified outside ℓ , and such that $K(\ell)/K$ is totally ramified at all primes above ℓ .*

Proof See, for instance, the proof of [2, Lemma 3.1]. ■

Let \mathcal{S} be the set of squarefree ideals of \mathcal{O}_k divisible only by primes $\ell \in \mathcal{L}$. If $\mathfrak{a} = \ell_1 \cdots \ell_n \in \mathcal{S}$, then we set $K(\mathfrak{a}) := K(\ell_1) \cdots K(\ell_n)$ and $K(\mathcal{O}_k) := K$. If \mathfrak{g} is an ideal of \mathcal{O}_k , then we denote by $\mathcal{S}(\mathfrak{g})$ the set of ideals $\mathfrak{a} \in \mathcal{S}$ that are prime to \mathfrak{g} . Following Rubin we define an Euler system to be a function $\alpha: \mathcal{S}(\mathfrak{g}) \rightarrow \mathbb{C}^\times$, such that

- E1 $\alpha(\mathfrak{a}) \in K(\mathfrak{a})^\times$;
- E2 $\alpha(\mathfrak{a}) \in \mathcal{O}_{K(\mathfrak{a})}^\times$; if $\mathfrak{a} \neq \mathcal{O}_k$;
- E3 $N_{K(\mathfrak{a}\ell)/K(\mathfrak{a})}(\alpha(\mathfrak{a}\ell)) = \alpha(\mathfrak{a})^{1-\text{Fr}(\ell)^{-1}}$, where $\text{Fr}(\ell)$ is the Frobenius of ℓ in $\text{Gal}(K(\mathfrak{a})/k)$;
- E4 $\alpha(\mathfrak{a}\ell) \equiv \alpha(\mathfrak{a})^{\text{Frob}(\ell)^{-1(N(\ell)-1)/M}}$ modulo all primes above ℓ .

For the convenience of the reader we recall now the construction of Euler systems by using elliptic units. To this end we use the elliptic functions $\Psi(\cdot; L, L'): z \mapsto \Psi(z; L, L')$ introduced by G. Robert in [4, 6], where $L \subset L'$ are lattices of \mathbb{C} such that the index $[L' : L]$ is prime to 6. As proved by Robert, for instance in [3, 5], if \mathfrak{m} is a nonzero proper ideal of \mathcal{O}_k and \mathfrak{g} is an ideal of \mathcal{O}_k prime to $6\mathfrak{m}$, then $\Psi(1; \mathfrak{m}, \mathfrak{g}^{-1}\mathfrak{m}) \in H_{\mathfrak{m}}$. Let us denote by $r_{\mathfrak{m}}$ the order of the kernel of the natural map $\mu_k \rightarrow (\mathcal{O}_k/\mathfrak{m})^\times$, and let \mathfrak{m}' be a nonzero proper ideal of \mathcal{O}_k such that $\mathfrak{m}|\mathfrak{m}'$, \mathfrak{m}' is divisible by the same prime ideals that divide \mathfrak{m} and $r_{\mathfrak{m}'} = 1$, then

$$N_{H_{\mathfrak{m}'}/H_{\mathfrak{m}}}(\Psi(1; \mathfrak{m}', \mathfrak{g}^{-1}\mathfrak{m}'))^{w_{\mathfrak{m}}} = \varepsilon_{\mathfrak{m}}^{N(\mathfrak{g})-(\mathfrak{g}, H_{\mathfrak{m}}/k)}.$$

In particular, \mathcal{P}_K is generated as an abelian group by μ_K and by all the norms

$$\Psi_{\mathfrak{m}}(\mathfrak{g}) := N_{H_{\mathfrak{m}}/H_{\mathfrak{m}} \cap K}(\Psi(1; \mathfrak{m}, \mathfrak{g}^{-1}\mathfrak{m})),$$

where \mathfrak{m} and \mathfrak{g} are any nonzero ideals of \mathcal{O}_k such that $\mathfrak{m} \neq \mathcal{O}_k$ and \mathfrak{g} is prime to $6\mathfrak{m}$. If $\mathfrak{m} = \mathfrak{n}\mathfrak{q}$, where $\mathfrak{n} \neq \mathcal{O}_k$ and \mathfrak{q} is a prime ideal of \mathcal{O}_k , then

$$N_{H_{\mathfrak{m}}/H_{\mathfrak{n}}}(\Psi(1; \mathfrak{m}, \mathfrak{g}^{-1}\mathfrak{m}))^{r_{\mathfrak{m}}} = \begin{cases} \Psi(1; \mathfrak{n}, \mathfrak{g}^{-1}\mathfrak{n}) & \text{if } \mathfrak{q}|\mathfrak{n} \\ \Psi(1; \mathfrak{n}, \mathfrak{g}^{-1}\mathfrak{n})^{1-(\mathfrak{q}, H_{\mathfrak{n}}/k)^{-1}} & \text{if } \mathfrak{q} \nmid \mathfrak{n}. \end{cases}$$

Moreover, if $\mathfrak{q} \nmid \mathfrak{n}$, then $\Psi(1; \mathfrak{m}, \mathfrak{g}^{-1}\mathfrak{m})^{N(\mathfrak{q})} \equiv \Psi(1; \mathfrak{n}, \mathfrak{g}^{-1}\mathfrak{n})$ modulo all primes above \mathfrak{q} . Therefore, the map $\alpha: \mathcal{S}(\mathfrak{m}\mathfrak{g}) \rightarrow \mathbb{C}^\times$, defined by

$$\alpha(\mathfrak{a}) := N_{KH_{\mathfrak{m}\mathfrak{a}}/K(\mathfrak{a})}(\Psi(1; \mathfrak{m}\mathfrak{a}, \mathfrak{g}^{-1}\mathfrak{m}\mathfrak{a})),$$

is an Euler system satisfying $\alpha(1) = \Psi_{\mathfrak{m}}(\mathfrak{g})$. In particular, we have the following corollary.

Corollary 4.2 *If $u \in \mathcal{E}_K$, then there exists an ideal \mathfrak{f} of \mathcal{O}_k and an Euler system $\alpha: \mathcal{S}(\mathfrak{f}) \rightarrow \mathbb{C}^\times$ such that $\alpha(1) = u$*

Proof In view of the discussion above we only have to check the corollary for the roots of unity in K . We leave this as an exercise or direct the reader to the proof of [8, Proposition 1.2]. ■

Remark 4.3 We recall that the group of elliptic units considered by Rubin in [8, 9] is the subgroup of \mathcal{O}_K^\times generated by μ_K and by all $\Psi_{\mathfrak{m}}(\mathfrak{g})^{\sigma^{-1}}$, where $\sigma \in G$ and \mathfrak{m} and \mathfrak{g} are as above. Let us denote this group by C_K . It is clear that $C_K \subset \mathcal{E}_K$ and $(\mathcal{E}_K)^{\mathfrak{g}} \subset C_K$, and thus

$$\mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} \mathcal{E}_K = \mathbb{Z}[\mathfrak{g}^{-1}] \otimes_{\mathbb{Z}} C_K.$$

5 The Gras Conjecture

Exactly as in [8, Proposition 2.2], one can prove that for any Euler system $\alpha: \mathcal{S}(\mathfrak{g}) \rightarrow \mathbb{C}^\times$ there is a natural map

$$(5.1) \quad \kappa_\alpha: \mathcal{S}(\mathfrak{g}) \longrightarrow K^\times / (K^\times)^M, \quad \kappa_\alpha(\mathfrak{a}) \equiv \alpha(\mathfrak{a})^{D_\mathfrak{a}} \text{ modulo } (K^\times)^M.$$

Let $\mathcal{J} = \bigoplus_\lambda \mathbb{Z}\lambda$ be the group of fractional ideals of K written additively. If ℓ is a prime ideal of \mathcal{O}_k , then we define $\mathcal{J}_\ell := \bigoplus_{\lambda|\ell} \mathbb{Z}\lambda$. If $y \in K^\times$, then we denote by $(y)_\ell \in \mathcal{J}_\ell$, $[y] \in \mathcal{J}/M\mathcal{J}$ and $[y]_\ell \in \mathcal{J}_\ell/M\mathcal{J}_\ell$ the projections of the fractional ideal $(y) := y\mathcal{O}_K$. Let us suppose that $\ell \in \mathcal{L}$. Let λ' be a prime ideal of $\mathcal{O}_{K(\ell)}$ above ℓ , and let $\pi \in \lambda' - (\lambda')^2$. Then $\pi^{1-\sigma_\ell}$ has exact order M in the cyclic group $(\mathcal{O}_{K(\ell)}/\lambda')^\times$, because $K(\ell)/K$ is cyclic, totally ramified at $\lambda := \lambda' \cap \mathcal{O}_K$. In particular, using the isomorphism $\mathcal{O}_{K(\ell)}/\lambda' \simeq \mathcal{O}_K/\lambda$, there exists $x_\lambda \in (\mathcal{O}_K/\lambda)^\times$ such that the image of $\pi^{1-\sigma_\ell}$ in $(\mathcal{O}_K/\lambda)^\times$ is equal to $(x_\lambda)^d$, where $d := (N(\ell) - 1)/M$. Let us remark that the projection of x_λ in $(\mathcal{O}_K/\lambda)^\times / ((\mathcal{O}_K/\lambda)^\times)^M$ is well defined, does not depend on π , and, in fact, has exact order M . Thus, the isomorphism $\mathcal{O}_K/\ell\mathcal{O}_K \simeq \bigoplus_{\lambda|\ell} \mathcal{O}_K/\lambda$ allows us to define a G -equivariant isomorphism

$$\widehat{\varphi}_\ell: (\mathcal{O}_K/\ell\mathcal{O}_K)^\times / ((\mathcal{O}_K/\ell\mathcal{O}_K)^\times)^M \longrightarrow \mathcal{J}_\ell/M\mathcal{J}_\ell$$

such that the image of an element $x := \bigoplus_{\lambda|\ell} (x_\lambda)^{e_\lambda}$ is $\widehat{\varphi}_\ell(x) := \bigoplus_{\lambda|\ell} e_\lambda \lambda$. Let us consider the map

$$\psi_\ell: K(\ell)^\times \longrightarrow (\mathcal{O}_K/\ell\mathcal{O}_K)^\times / ((\mathcal{O}_K/\ell\mathcal{O}_K)^\times)^M,$$

which associates with z the sum $\bigoplus_{\lambda|\ell} z_\lambda$ such that the image of $z^{1-\sigma_\ell}$ in $(\mathcal{O}_K/\lambda)^\times$ is equal to $(z_\lambda)^d$. Let $\varphi_\ell := -\widehat{\varphi}_\ell$, then

$$(\varphi_\ell \circ \psi_\ell)(x) = [N_{K(\ell)/K}(x)]_\ell.$$

The map φ_ℓ induces a homomorphism $\{y \in K^\times / (K^\times)^M, [y]_\ell = 0\} \rightarrow \mathcal{J}_\ell/M\mathcal{J}_\ell$, which we also denote by φ_ℓ . Then, as in [8, Proposition 2.4], one can prove that for any Euler system $\alpha: \mathcal{S}(\mathfrak{g}) \rightarrow \mathbb{C}^\times$, and any $\mathfrak{a} \in \mathcal{S}(\mathfrak{g})$, such that $\mathfrak{a} \neq 1$

$$(5.2) \quad [\kappa_\alpha(\mathfrak{a})]_\ell = \begin{cases} 0 & \text{if } \ell \nmid \mathfrak{a}, \\ \varphi_\ell(\kappa_\alpha(\mathfrak{a}/\ell)) & \text{if } \ell | \mathfrak{a}. \end{cases}$$

In the sequel, if p is a prime number such that $p \nmid [K : k]$, χ a nontrivial irreducible \mathbb{Z}_p -character of G , and Π is a $\mathbb{Z}_p[G]$ -module, then we define $\Pi_\chi := e_\chi \Pi$. If Π is a $\mathbb{Z}[G]$ -module, then we define $\Pi_\chi := e_\chi(\mathbb{Z}_p \otimes_{\mathbb{Z}} \Pi)$. Before proving Theorem 1.1 we first need to prove the analogue of [9, Theorem 4] and [8, Theorem 3.1]. For this, if $p | w_k$ is a prime number and M is a p -power, then we set

$$K' := K_M(a_1^{1/M}, \dots, a_s^{1/M}).$$

Lemma 5.1 *Let p be a prime number such that $p \mid w_k$, and let $M \geq p$ be a power of p . Let us consider the natural map*

$$\Theta: K^\times / (K^\times)^M \longrightarrow K_M^\times / (K_M^\times)^M.$$

- (i) *If $p = 3$ or ($p = 2$ and $\mu_4 \subset K$) or $M = 2$, then $\ker(\Theta) = \mathcal{O}_k^\times / (\mathcal{O}_k^\times)^M$.*
- (ii) *If $4 \mid M$ and $w_k \in \{2, 6\}$, then $\ker(\Theta)$ is generated by the projections in $K^\times / (K^\times)^M$ of \mathcal{O}_k^\times and $2^{M/2}$*

In particular, $\ker(\Theta)$ is annihilated by $[K : k] - s(G)$, where $s(G) := \sum \sigma, \sigma \in G$. Furthermore, since K'/K_M is a Kummer extension and a_1, \dots, a_s are elements of k , the kernel of the natural map

$$K^\times / (K^\times)^M \longrightarrow K'^\times / (K'^\times)^M$$

is also annihilated by $[K : k] - s(G)$.

Proof Let $x \in K^\times \cap (K_M^\times)^M$. If $p = 3$, then $K_M = K(\mu_{3M})$. By [7, Lemma 5.7(i)] we have $x^3 \in (K^\times)^{3M}$, that is to say, $x \in \mu_3(K^\times)^M \subset \mu_k(K^\times)^M$. If $p = 2$ and $w_k = 4$, then $K_M = K(\mu_{4M})$. Again by [7, Lemma 5.7(i)] we deduce that $x \in \mu_k(K^\times)^M$. Suppose now that $p = 2$ and $w_k \in \{2, 6\}$. Then $K_M = K(\mu_{2M})$. By using the same arguments as before, we see that $x \in \mu_k(K(\mu_4)^\times)^M$. If $\mu_4 \subset K$ or $M = 2$, we are done. Let us assume that $\mu_4 \not\subset K$ and $4 \mid M$, and write $x = z^M \zeta$, for some $z \in K(\mu_4)$ and $\zeta \in \mu_k$. Let σ be the unique nontrivial automorphism of $K(\mu_4)/K$. If $z^{\sigma^{-1}} \in \mu_2$, then it is easy to check that $x \in \mu_k(K^\times)^M$. Suppose we have the case $z^{\sigma^{-1}} = i$, where $i^2 = -1$. Since $\sigma(i) = -i$ and $z = a + ib$, where $a, b \in K$, the equation $a - ib = \sigma(z) = i(a + ib)$ implies that $b = -a$, $z = a(1 - i)$ and $x = a^M 2^{M/2} (-1)^{M/4} \zeta$. The complex number $\zeta_8 := (1 + i)\sqrt{2}/2$ is a root of unity of order 8. An easy computation shows that we can not have $z^{\sigma^{-1}} = \zeta_8$. This proves assertions (i) and (ii). The rest of the lemma is straightforward. ■

Lemma 5.2 *Suppose p is a prime number such that $p \nmid [K : k]$ and $p \mid w_k$. Let M be a power of p . Let χ be a nontrivial irreducible \mathbb{Z}_p -character of G . Let H^χ be the abelian extension of K corresponding to the χ -part $\text{Cl}(K)_\chi$. Then $H^\chi \cap K' = K$.*

Proof The group G acts trivially on $\text{Gal}(H^\chi \cap K_M/K)$ because K_M is abelian over k . On the other hand, $\text{Gal}(H^\chi \cap K_M/K)$ is a G -quotient of $\text{Gal}(H^\chi/K) \simeq \text{Cl}(K)_\chi$. This implies that $H^\chi \cap K_M = K$, since $\chi \neq 1$. In addition, if $p \nmid [H : k]$, then $K' = K_M$. In particular we have proved that $H^\chi \cap K' = K$ in case $p \nmid [H : k]$. Let $E := K_M(H^\chi \cap K')$. By Kummer theory we deduce from the inclusion $E \subset K'$ that $E = K_M(V^{1/M})$, where V is a subgroup of the multiplicative group $\langle a_1, \dots, a_s \rangle \subset k^\times$. If $p \mid [H : k]$, then G acts on $\text{Gal}(E/K_M)$ via the trivial character. This implies that $\text{Gal}(E/K_M) = 1$ because this group is isomorphic to $\text{Gal}(H^\chi \cap K'/K)$ on which G acts via $\chi \neq 1$. The proof of the lemma is now complete. ■

Theorem 5.3 *Suppose p is a prime number such that $p \nmid [K : k]$ and $p \mid w_k$. Let M be a power of p . Let χ be a nontrivial irreducible \mathbb{Z}_p -character of G . Let $\beta \in (K^\times / (K^\times)^M)_\chi$ and A be a $\mathbb{Z}_p[G]$ -quotient of $\text{Cl}(K)_\chi$. Let m be the order of β in $K^\times / (K^\times)^M$, W the G -submodule of $K^\times / (K^\times)^M$ generated by β , H the abelian extension of K corresponding*

to A , and $L := H \cap K'(W^{1/M})$. Then there is a $\mathbb{Z}[G]$ generator c' of $\text{Gal}(L/K)$ such that for any $c \in A$ whose restriction to L is c' , there are infinitely many prime ideals λ of \mathcal{O}_K of degree one such that

- (i) the projection of the class of λ in A is c ;
- (ii) if $\ell := \lambda \cap \mathcal{O}_k$ then $\ell \in \mathcal{L}$;
- (iii) $[\beta]_\ell = 0$ and there is $u \in (\mathbb{Z}/m\mathbb{Z}[G])_\chi^\times$ such that $\varphi_\ell(\beta) = (M/m)u\lambda$.

Proof We follow [8, Theorem 3.1]. Since $W \subset (K^\times/(K^\times)^M)_\chi$ and $\chi \neq 1$, we deduce from Lemma 5.1 that the Galois group of the Kummer extension $K'(W^{1/M})/K'$ is isomorphic as a $\mathbb{Z}[\text{Gal}(K_M/k)]$ -module to $\text{Hom}(W, \mu_M)$. But $W \simeq (\mathbb{Z}/m\mathbb{Z}[G])_\chi$, which is a direct factor of $(\mathbb{Z}/m\mathbb{Z})[G]$. On the other hand, $\text{Hom}((\mathbb{Z}/m\mathbb{Z})[G], \mu_M)$ is $\mathbb{Z}[\text{Gal}(K_M/k)]$ -cyclic, generated for instance by the group homomorphism $\Psi: (\mathbb{Z}/m\mathbb{Z})[G] \rightarrow \mu_M$ defined by $\Psi(1) = \zeta$ and $\Psi(g) = 1$, for $g \neq 1$, where $\zeta \in \mu_M$ is a primitive m -th root of unity. Therefore, we can find $\tau \in \text{Gal}(K'(W^{1/M})/K')$, which generates $\text{Gal}(K'(W^{1/M})/K')$ over $\mathbb{Z}[\text{Gal}(K_M/k)]$. The restriction c' of τ to L is a $\mathbb{Z}[G]$ -generator of $\text{Gal}(L/K) \simeq \text{Gal}(LK'/K')$ by Lemma 5.2. Let $c \in \text{Gal}(H/K) = A$ be any extension of c' to H . Then one can find $\sigma \in \text{Gal}(HK'(W^{1/M})/K)$ such that

$$\sigma|_H = c \quad \text{and} \quad \sigma|_{K'(W^{1/M})} = \tau.$$

By the Chebotarev density theorem there exist infinitely many primes λ of \mathcal{O}_K whose Frobenius in $\text{Gal}(HK'(W^{1/M})/K)$ is the conjugacy class of σ , and such that $\ell := \lambda \cap \mathcal{O}_k$ is unramified in $K'(W^{1/M})/k$. Now it is immediate that (i) and (ii) are satisfied. The rest of the proof is exactly the same as the proof of [8, Theorem 3.1]. ■

Theorem 5.4 Suppose p is a prime number such that $p \nmid [K : k]$ and $p \mid w_k$. Let χ be a nontrivial irreducible \mathbb{Z}_p -character of G . Then we have

$$(5.3) \quad \#\text{Cl}(K)_\chi \mid \#(\mathcal{O}_K^\times/\mathcal{E}_K)_\chi.$$

Proof We proceed exactly as in the proofs of [8, Theorem 3.2] or [2, Theorem 4.4]. Let $\widehat{\chi}$ be a $\overline{\mathbb{Q}}_p$ -irreducible character of G such that $\widehat{\chi}|_\chi$, and let $\widehat{\chi}(G) := \{\widehat{\chi}(\sigma), \sigma \in G\}$. Then the ring $R := \mathbb{Z}_p[G]_\chi$ is isomorphic to $\mathbb{Z}_p[\widehat{\chi}(G)]$, which is the ring of integers of the unramified extension $\mathbb{Q}_p[\widehat{\chi}(G)]$ of \mathbb{Q}_p . Thus, R is a discrete valuation ring. Moreover, the R -torsion of any R -module is equal to its \mathbb{Z}_p -torsion. Since $\chi \neq 1$, Dirichlet unit theorem implies that the quotient $(\mathcal{O}_K^\times)_\chi/(\mu_K)_\chi$ is a free R -module of rank 1. Let us define

$$M := p\#(\mathcal{O}_K^\times/\mathcal{E}_K)_\chi \# \text{Cl}(\mathcal{O}_K)_\chi.$$

Let μ , U , and V be the images of μ_K , \mathcal{O}_K^\times , and \mathcal{E}_K in $K^\times/(K^\times)^M$, respectively. We deduce from the above, that U_χ/μ_χ is a free R/MR -module of rank 1. But since

$$(5.4) \quad U_\chi/V_\chi \simeq (\mathcal{O}_K^\times)_\chi/(\mathcal{E}_K)_\chi \simeq R/tR$$

for some divisor t of M , there exists $\xi \in U_\chi$ giving an R -basis of U_χ/μ_χ and such that $\xi^t \in (\mathcal{E}_K)_\chi$. In particular, ξ has order M in $K^\times/(K^\times)^M$. By Corollary 4.2

there exists an ideal \mathfrak{g} of \mathcal{O}_k and an Euler system $\alpha: \mathcal{S}(\mathfrak{g}) \rightarrow \mathbb{C}^\times$, such that the map $\kappa := \kappa_\alpha$ defined by (5.1) satisfies $\kappa(1) = \xi^t$. We define inductively classes $c_0, \dots, c_i \in \text{Cl}(\mathcal{O}_K)_\chi$, prime ideals $\lambda_1, \dots, \lambda_i$ of \mathcal{O}_K , coprime with \mathfrak{g} , and ideals $\mathfrak{a}_0, \dots, \mathfrak{a}_i$ of \mathcal{O}_k such that $c_0 = 1$ and $\mathfrak{a}_0 = 1$. Let $i \geq 0$, and suppose that c_0, \dots, c_i and $\lambda_1, \dots, \lambda_i$ (if $i \geq 1$) are defined. Then we set $\mathfrak{a}_i = \prod_{1 \leq n \leq i} \ell_n$ (if $i \geq 1$), where $\ell_n := \lambda_n \cap \mathcal{O}_k$. Moreover:

- If $\text{Cl}(\mathcal{O}_K)_\chi \neq \langle c_0, \dots, c_i \rangle_G$, where $\langle c_0, \dots, c_i \rangle_G$ is the G -module generated by c_0, \dots, c_i , then we define c_{i+1} to be any element of $\text{Cl}(\mathcal{O}_K)_\chi$ whose image in $\text{Cl}(\mathcal{O}_K)_\chi / \langle c_0, \dots, c_i \rangle_G$ is nontrivial and is equal to a class c which restricts to the generator c' of $\text{Gal}(L/K)$ in Theorem 5.3 applied to $\beta := \kappa(\mathfrak{a}_i)_\chi$, the image of $\kappa(\mathfrak{a}_i)$ in $(K^\times / (K^\times)^M)_\chi$, and to $A := \text{Cl}(\mathcal{O}_K)_\chi / \langle c_0, \dots, c_i \rangle_G$. Also we let λ_{i+1} be any prime ideal of \mathcal{O}_K prime to \mathfrak{g} and satisfying Theorem 5.3 with the same conditions.
- If $\text{Cl}(\mathcal{O}_K)_\chi = \langle c_0, \dots, c_i \rangle_G$, then we stop.

This construction of our classes c_j implies that the ideals $\ell_j := \lambda_j \cap \mathcal{O}_k \in \mathcal{S}(\mathfrak{g})$. Let m_i be the order of $\kappa(\mathfrak{a}_i)_\chi$ in $K^\times / (K^\times)^M$, and let $t_i := M/m_i$. By Theorem 5.3(iii) we have $\varphi_{\ell_{i+1}}(\kappa(\mathfrak{a}_i)_\chi) = ut_i \lambda_{i+1}$ for some $u \in \mathbb{Z}/M\mathbb{Z}[G]_\chi^\times$. But $\mathfrak{a}_{i+1} = \mathfrak{a}_i \ell_{i+1}$. Thus

$$(5.5) \quad [\kappa(\mathfrak{a}_{i+1})_\chi]_{\ell_{i+1}} = ut_i \lambda_{i+1},$$

thanks to (5.2). Now, by the definition of t_{i+1} , the fractional ideal of \mathcal{O}_K generated by $\kappa(\mathfrak{a}_{i+1})_\chi$ is a t_{i+1} -th power. Thus, we must have $t_{i+1} | t_i$. Actually, we can say more. Indeed, there exist $\zeta \in \mu_K$ and $z \in K^\times$ such that $\kappa(\mathfrak{a}_{i+1})_\chi = \zeta z^{t_{i+1}}$. Therefore, (5.2) and (5.5) imply

$$(5.6) \quad z\mathcal{O}_K = (\lambda_{i+1})^{ut_i/t_{i+1}} \left(\prod_{j=1}^i \lambda_j^{u_j} \right) \mathfrak{b}^{M/t_{i+1}},$$

where $u_j \in \mathbb{Z}_p[G]$ for all $j \in \{1, \dots, i\}$ and \mathfrak{b} is a fractional ideal of \mathcal{O}_K . But we see from (5.4) that $t_0 \#(\mathcal{O}_K^\times / \mathcal{E}_K)_\chi$, and since $t_{i+1} | t_0$, the integer M/t_{i+1} annihilates $\text{Cl}(\mathcal{O}_K)_\chi$. The identity (5.6) then implies

$$(5.7) \quad (t_i/t_{i+1})c_{i+1} \in \langle c_0, \dots, c_i \rangle_G.$$

Let $\dim(\chi) := [\mathbb{Q}_p[\widehat{\chi}(G)] : \mathbb{Q}_p]$, then (5.7) implies

$$\#\text{Cl}(\mathcal{O}_K)_\chi \left| \prod_{j=1}^i (t_{j-1}/t_j)^{\dim(\chi)} \right| t_0^{\dim(\chi)} = \#(\mathcal{O}_K^\times / \mathcal{E}_K)_\chi. \quad \blacksquare$$

Proof of Theorem 1.1 Let the hypotheses and notation be as in Theorem 1.1. Let Ψ be the irreducible rational character of G such that $\chi | \Psi$. Formula (1.1) may be written as follows

$$\prod_{\chi' | \Psi} \#\text{Cl}(K)_{\chi'} = \prod_{\chi' | \Psi} \#(\mathcal{O}_K^\times / \mathcal{E}_K)_{\chi'},$$

where χ' runs over the irreducible \mathbb{Z}_p -characters of G such that $\chi' | \Psi$. Moreover, the formula (5.3) is satisfied for such characters χ' , since $\chi \neq 1$. This implies (1.3). \blacksquare

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