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Hölder Continuous Solutions of Degenerate Differential Equations with Finite Delay

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Abstract. Using known operator-valued Fourier multiplier results on vector-valued Hölder continuous function spaces $C^\alpha(\mathbb{R};X)$, we completely characterize the C^α-well-posedness of the first order degenerate differential equations with finite delay $(Mu)'(t) = Au(t) + Fu_t + f(t)$ for $t \in \mathbb{R}$ by the boundedness of the (M,F)-resolvent of A under suitable assumption on the delay operator F, where A, M are closed linear operators on a Banach space X satisfying $D(A) \cap D(M) \neq \{0\}$, the delay operator F is a bounded linear operator from $C([-r,0];X)$ to X, and $r > 0$ is fixed.

1 Introduction

In this paper, we study the C^α-well-posedness of the first order degenerate differential equations with finite delay

$$(P) : (Mu)'(t) = Au(t) + Fu_t + f(t), \quad (t \in \mathbb{R}),$$

where $A: D(A) \to X$ and $M: D(M) \to X$ are closed linear operators on a complex Banach space X satisfying $D(A) \cap D(M) \neq \{0\}$, $0 < \alpha < 1$, F is a bounded linear operator from $C([-r,0];X)$ into X for some fixed $r > 0$, and u_t is defined by $u_t(s) = u(t+s)$ when $t \in \mathbb{R}$ and $s \in [-r,0]$. The typical model of (P) is the case when A is the Laplacian and M is the multiplication operator by a non-negative function m on an appropriate function space defined on a bounded domain $\Omega \subset \mathbb{R}^n$, which was first considered by Carroll and Showalter [6], and was later studied by Marinoschi [10]. This model describes the filtration of water in unsaturated porous media in which saturation might occur.

Using known $\dot{C}^\alpha$-Fourier multiplier results by Arendt, Batty and Bu [1], we are able to characterize the C^α-well-posedness of (P) by the boundedness of the (M,F)-resolvent of A when $0 < \alpha < 1$. Precisely, we show that, under suitable assumption $(\mathbf{H}_2)$ on the delay operator F, the problem (P) is C^α-well-posed if and only if $i\mathbb{R} \subset \rho_{M,F}(A)$

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and

$$\sup_{s\in\mathbb{R}} \|(isM - A - F_{is})^{-1}\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \|A(isM - A - F_{is})^{-1}\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \|sM(isM - A - F_{is})^{-1}\| < \infty,$$

(see Theorem 2.3), where $F_{is} \in \mathcal{L}(X)$ is defined by $F_{is}x := F(e^{is\cdot}x)$ when $x \in X$ and $s \in \mathbb{R}$, $\rho_{M,F}(A)$ is the (M,F)-resolvent of A (see the precise definition in the next section). Since this characterization of the C^α-well-posedness of (P) does not depend on the space parameter $0 < \alpha < 1$, we deduce that when (P) is C^α-well-posed for some $0 < \alpha < 1$, then it is C^α-well-posed for all $0 < \alpha < 1$ (see Corollary 2.4). We also show that when the underlying Banach space X is B-convex, then the same characterization of the C^α-well-posedness of (P) remains true under a weaker assumption $(\mathbf{H}_1)$ on F (see Corollary 2.5).

When $F = 0$, the corresponding degenerate problems (P) have been studied independently by Bu [4] and Ponce [12] (Ponce has studied the corresponding problems in the case $D(A) \cap D(M) \neq \{0\}$, while Bu has considered the same problems in the simpler case $D(A) \subset D(M)$), and in the case when $F = 0$ and $M = I_X$, the corresponding problems (P) have been treated by Arendt, Batty, and Bu [1]. Our results concerning the C^α-well-posedness of (P) can be regarded as generalization of known results obtained in [1, 4, 12].

We notice that similar first order degenerate differential equations with finite delay

$$(P_{\text{per}}) : (Mu)'(t) = Au(t) + Fu_t + f(t), \quad (t \in [0, 2\pi])$$

with periodic boundary condition $Mu(0) = Mu(2\pi)$ were studied by Bu [5], under suitable assumption on F, he gave necessary and sufficient conditions for (P_{per}) to be L^p-well-posed. The simpler case of (P_{per}) when $F = 0$ was first studied by Lizama and Ponce [8] (see also [9] for the study of a similar degenerate equation with infinite delay). See [7] for a more detailed study of linear abstract degenerate differential equations.

This paper is organized as follows. In the next section we give some preliminaries and necessary and sufficient conditions for the C^α-well-posedness of (P). In the third section, we give some concrete examples to which our abstract results can be applied.

2 Characterizations of the C^α-Well-Posedness

Let X be a complex Banach space with norm $\|\cdot\|$ and let $0 < \alpha < 1$. We denote by $C^\alpha(\mathbb{R}; X)$ the space of all X-valued functions u defined on $\mathbb{R}$ satisfying

$$\|u\|_\alpha := \sup_{s\neq t} \frac{\|u(s) - u(t)\|}{|s-t|^\alpha} < \infty.$$

Define

$$\|u\|_{C^\alpha} := \|u(0)\| + \|u\|_\alpha.$$

Then $C^\alpha(\mathbb{R}, X)$ equipped with norm $\|\cdot\|_{C^\alpha}$ becomes a Banach space. In order to define Fourier multipliers on $C^\alpha(\mathbb{R}, X)$, it is necessary to operate modulo constant functions. The kernel of the seminorm $\|\cdot\|_\alpha$ on $C^\alpha(\mathbb{R};X)$ is the space of all constant functions. The corresponding quotient space $\dot{C}^\alpha(\mathbb{R};X)$ is a Banach space under the quotient norm. We will frequently identify a function $u \in C^\alpha(\mathbb{R};X)$ with its equivalent class in $\dot{C}^\alpha(\mathbb{R};X)$, that is,

$$\dot{u} := \left\{ v \in C^\alpha(\mathbb{R};X) : u - v \equiv \text{constant} \right\}.$$

If X, Y are Banach spaces, we let $\mathcal{L}(X,Y)$ be the space of all bounded linear operators from X to Y. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. The notion of $\dot{C}^\alpha$-Fourier multiplier was introduced in [1], which is crucial for our investigation on the C^α-well-posedness of (P).

Definition 2.1 Let X, Y be complex Banach spaces, $m: \mathbb{R} \smallsetminus \{0\} \to \mathcal{L}(X,Y)$ be continuous. We say that m is a $\dot{C}^\alpha$-Fourier multiplier if there exists a mapping $L: \dot{C}^\alpha(\mathbb{R};X) \to \dot{C}^\alpha(\mathbb{R};Y)$ such that

$$\int_{\mathbb{R}} \mathcal{F}\varphi(s)(Lf)(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\varphi m)(s) f(s)\,ds \tag{2.1}$$

for all $f \in C^\alpha(\mathbb{R};X)$ and all $\varphi \in \mathcal{D}(\mathbb{R} \smallsetminus \{0\})$, where $\mathcal{D}(\mathbb{R} \smallsetminus \{0\})$ is the space of all C^∞-functions on $\mathbb{R} \smallsetminus \{0\}$ with compact support containing in $\mathbb{R} \smallsetminus \{0\}$, $\mathcal{F}$ is the Fourier transform given by

$$(\mathcal{F}h)(s) := \int_{\mathbb{R}} h(t)e^{-ist}\,dt, \quad (s \in \mathbb{R})$$

when $h \in L^1(\mathbb{R},X)$.

By [1, Lemma 5.1], the right-hand side of (2.1) does not depend on the representative of $\dot{f}$ as $\int_{\mathbb{R}} \mathcal{F}(\varphi m)(s)ds = 2\pi(\varphi m)(0) = 0$. Moreover, identity (2.1) defines $Lf \in C^\alpha(\mathbb{R};X)$ uniquely up to an additive constant by [1, Lemma 5.1].

We will use the following result due to Arendt, Batty, and Bu [1], which gives a sufficient condition for a C^2-function $M: \mathbb{R} \smallsetminus \{0\} \to \mathcal{L}(X,Y)$ to be a $\dot{C}^\alpha$-Fourier multiplier.

Theorem 2.2 *Let X, Y be Banach spaces and let $m: \mathbb{R} \smallsetminus \{0\} \to \mathcal{L}(X,Y)$ be a C^2-function satisfying*

$$\sup_{s \neq 0} \left(\|m(s)\| + \|sm'(s)\| + \|s^2 m''(s)\| \right) < \infty. \tag{2.2}$$

Then m is a $\dot{C}^\alpha$-Fourier multiplier. If X, Y are B-convex, then the first order condition

$$\sup_{s \neq 0} \left(\|m(s)\| + \|sm'(s)\| \right) < \infty \tag{2.3}$$

is already sufficient for m to be a $\dot{C}^\alpha$-Fourier multiplier.

Recall that a Banach space X is B-convex if it does not contain l_1^n uniformly [11]. This is equivalent to saying that X has Fourier type $1 < p \le 2$; *i.e.,* the Fourier transform $\mathcal{F}$ is a bounded linear operator from $L^p(\mathbb{R};X)$ to $L^q(\mathbb{R};X)$, where $1/p + 1/q = 1$ [3]. It is well known that when $1 < p < \infty$, $L^p(\mu)$ has Fourier type $\min\{p, \frac{p}{p-1}\}$. Each Hilbert space has Fourier type 2.

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Let $u \in L^1_{loc}(\mathbb{R}; X)$. We say that u is of subexponential growth if for all $\epsilon > 0$,

$$\int_{-\infty}^{\infty} e^{-\epsilon|t|}\left\| u(t)\right\| dt < \infty.$$

For such function u, we define its Carleman transform on $\mathbb{C} \setminus i\mathbb{R}$ by

$$\widehat{u}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t}u(t)\,dt, & Re\lambda > 0,\\ -\int_0^\infty e^{\lambda t}u(-t)\,dt, & Re\lambda < 0,\end{cases}$$

[2, p. 292]. Then $\widehat{u}$ is a holomorphic function on $\mathbb{C} \setminus i\mathbb{R}$. A point $i\eta \in i\mathbb{R}$ is called *regular* for $\widehat{u}$, if $\widehat{u}$ has a holomorphic extension to a neighbourhood of $i\eta$. The Carleman spectrum $sp_C(u)$ of u is defined by

$$sp_C(u) := \left\{ \eta \in \mathbb{R} : i\eta \text{ is not regular for } \widehat{u} \right\}.$$

It is known that if $sp_C(u) = \varnothing$, then $u = 0$ [2].

Let $0 < \alpha < 1$; we denote by $C^{1+\alpha}(\mathbb{R}; X)$ the space of all X-valued functions u defined on $\mathbb{R}$, such that $u \in C^1(\mathbb{R}; X)$ and $u' \in C^\alpha(\mathbb{R}; X)$. The space $C^{1+\alpha}(\mathbb{R}; X)$ is equipped with the norm

$$\|u\|_{C^{1+\alpha}(\mathbb{R};X)} := \left\| u(0)\right\| + \|u'\|_\alpha,$$

and it is a Banach space.

Let $A: D(A) \to X$ and $M: D(M) \to X$ be closed linear operators on X satisfying $D(A) \cap D(M) \neq \{0\}$, $F \in \mathcal{L}(C([-r, 0]; X), X)$ for some fixed $r > 0$, where $C([-r, 0]; X)$ is the Banach space of all X-valued continuous functions defined on $[-r, 0]$ equipped with the norm

$$\|u\| := \max_{-r\le s\le 0}\left\| u(s)\right\|.$$

We consider the first order degenerate differential equations with finite delay

$$(P) : (Mu)'(t) = Au(t) + Fu_t + f(t), \quad (t \in \mathbb{R}),$$

where u_t is defined by $u_t(s) = u(t + s)$ for $t \in \mathbb{R}$ and $s \in [-r, 0]$. If $0 < \alpha < 1$, we say that (P) is C^α-well-posed, if for all $f \in C^\alpha(\mathbb{R}; X)$, there exists a unique $u \in C^\alpha(\mathbb{R}; D(A) \cap D(M))$, such that $Mu \in C^{1+\alpha}(\mathbb{R}; X)$ and (P) is satisfied for all $t \in \mathbb{R}$. Here we consider $D(A) \cap D(M)$ as a Banach space equipped with the norm

$$\|x\|_{D(A)\cap D(M)} := \|x\| + \|Ax\| + \|Mx\|.$$

We notice that $Fu. \in C^\alpha(\mathbb{R}; X)$ when $u \in C^\alpha(\mathbb{R}; X)$. Indeed, for all $s, t \in \mathbb{R}$,

$$\|Fu_s - Fu_t\| \le \|F\|\|u_s - u_t\|_{C([-r,0];X)} \le \|F\|\|u\|_\alpha|s - t|^\alpha.$$

For fixed $z \in \mathbb{C}$, we define

$$F_z(x) := F(e^{z\cdot}x)$$

when $x \in X$. It is clear that F_z is a bounded linear operator on X.

Let $\phi_{s,1}(t) = ite^{ist}$ and $\phi_{s,2}(t) = -t^2e^{ist}$ when $s \in \mathbb{R}$ and $t \in [-r, 0]$. Then it is easy to show that

$$\left(\frac{d}{ds}F_{is}\right)x = F(\phi_{s,1}x), \quad \left(\frac{d^2}{ds^2}F_{is}\right)x = F(\phi_{s,2}x).$$

Thus,

$$\sup_{s\in\mathbb{R}} \left\| \frac{d}{ds} F_{is} \right\| < \infty, \quad \sup_{s\in\mathbb{R}} \left\| \frac{d^2}{ds^2} F_{is} \right\| < \infty. \tag{2.4}$$

We say that F satisfies $(\mathbf{H}_1)$ if

$$\sup_{s\in\mathbb{R}} \left\| s\frac{d}{ds} F_{is} \right\| < \infty.$$

We say that F satisfies $(\mathbf{H}_2)$ if F satisfies $(\mathbf{H}_1)$ and

$$\sup_{s\in\mathbb{R}} \left\| s^2\frac{d^2}{ds^2} F_{is} \right\| < \infty.$$

We define the (M,F)-resolvent of A by

$$\rho_{M,F}(A) := \big\{ z \in \mathbb{C} : \ zM - A - F_z : D(A) \cap D(M) \to X \text{ is a bijection and } (zM - A - F_z)^{-1} \in \mathcal{L}(X) \big\}.$$

Let $z \in \rho_M(A)$. Then $(zM - A - F_z)^{-1} \in \mathcal{L}(X)$ is a bijection from X onto $D(A)$ by definition. The operators $M(zM - A - F_z)^{-1}$, $A(zM - A - F_z)^{-1}$ make sense, and they belong to $\mathcal{L}(X)$ by the Closed Graph Theorem and the closedness of M and A.

The following is the main result of this paper, which gives a necessary and sufficient condition for (P) to be C^α-well-posed.

Theorem 2.3 *Let X be a complex Banach space, $0 < \alpha < 1$ and let A, M be closed linear operators on X satisfying $D(A) \cap D(M) \neq \{0\}$, $F \in \mathcal{L}(C([-r,0];X), X)$ for some fixed $r > 0$, we assume that F satisfies $(\mathbf{H}_2)$. Then (P) is C^α-well-posed if and only if $i\mathbb{R} \subset \rho_{M,F}(A)$, and*

$$\sup_{s\in\mathbb{R}} \left\| (isM - A - F_{is})^{-1} \right\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \left\| A(isM - A - F_{is})^{-1} \right\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \left\| sM(isM - A - F_{is})^{-1} \right\| < \infty.$$

Proof Assume that $i\mathbb{R} \subset \rho_{M,F}(A)$, $\sup_{s\in\mathbb{R}} \|m(s)\| < \infty$, $\sup_{s\in\mathbb{R}} \|Am(s)\| < \infty$, and $\sup_{s\in\mathbb{R}} \|sMm(s)\| < \infty$, where $m(s) := (isM - A - F_{is})^{-1}$ when $s \in \mathbb{R}$. Let

$$g(s) = Mm(s), h(s) = isMm(s)$$

when $s \in \mathbb{R}$. Then h is uniformly bounded on $\mathbb{R}$ by assumption. Consequently,

$$\sup_{s\in\mathbb{R}} \|g(s)\| < \infty$$

by continuity. We have

$$\sup_{s\in\mathbb{R}} \left\| m(s) \right\| < \infty, \ \sup_{s\in\mathbb{R}} \left\| Am(s) \right\| < \infty \tag{2.5}$$

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by assumption. For $s \in \mathbb{R}$,

$$m'(s) = m(s)\Big(\frac{d}{ds}F_{is} - iM\Big)m(s)$$

and

$$m''(s) = 2m(s)\Big(\frac{d}{dt}F_{is} - iM\Big)m(s)\Big(\frac{d}{ds}F_{is} - iM\Big)m(s) + m(s)\frac{d^2}{ds^2}F_{is}m(s)$$

by [4, Lemma 1]. This implies that

$$\sup_{s\in\mathbb{R}}\big\|sm'(s)\big\| < \infty, \qquad \sup_{s\in\mathbb{R}}\big\|sAm'(s)\big\| < \infty,$$
$$\sup_{s\in\mathbb{R}}\big\|s^2m''(s)\big\| < \infty, \qquad \sup_{s\in\mathbb{R}}\big\|s^2Am''(s)\big\| < \infty$$

by (2.4), (2.5) and the assumption $(\mathbf{H}_2)$ on F. These estimations together with (2.5) imply that m considered as a function with values in $\mathcal{L}(X, D(A))$ satisfies (2.2). Here we consider $D(A)$ as a Banach space equipped with its graph norm. Thus, m is a $\dot{C}^\alpha$-Fourier multiplier by Theorem 2.2. In particular, Am is a $\dot{C}^\alpha$-Fourier multiplier. A similar argument shows that, considering m as a function with values in $\mathcal{L}(X, D(M))$, m is also a $\dot{C}^\alpha$-Fourier multiplier. Consequently, considering m as a function with values in $\mathcal{L}(X, D(A)\cap D(M))$, m is also a $\dot{C}^\alpha$-Fourier multiplier. In particular, Mm is a $\dot{C}^\alpha$-Fourier multiplier.

Next we show that h is also a $\dot{C}^\alpha$-Fourier multiplier. We have

$$\begin{aligned}
h'(s) &= iMm(s) + isMm(s)\Big(\frac{d}{ds}F_{is} - iM\Big)m(s),\\
h''(s) &= 2iMm(s)\Big(\frac{d}{ds}F_{is} - iM\Big)m(s)\\
&\quad + 2isMm(s)\Big(\frac{d}{ds}F_{is} - iM\Big)m(s)\Big(\frac{d}{ds}F_{is} - iM\Big)m(s)\\
&\quad + isMm(s)\frac{d^2}{ds^2}F_{is}m(s).
\end{aligned}$$

We deduce that h satisfies (2.2) by (2.5) and the assumption $(\mathbf{H}_2)$ on F. Hence, h is a $\dot{C}^\alpha$-Fourier multiplier by Theorem 2.2. Using the identity $isMm(s) - Am(s) - F_{is}m(s) = I_X$, one deduces that $F_{i\cdot}m$ is a $\dot{C}^\alpha$-Fourier multiplier.

Let $f \in C^\alpha(\mathbb{R}; X)$. Then there exist $u \in C^\alpha(\mathbb{R}; D(A)\cap D(M))$ and $v, w \in C^\alpha(\mathbb{R}; X)$, such that

$$\int_{\mathbb{R}} \mathcal{F}\phi_1(s)u(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi_1 m)(s)f(s)\,ds, \tag{2.6}$$

$$\int_{\mathbb{R}} \mathcal{F}\phi_2(s)v(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi_2 h)(s)f(s)\,ds, \tag{2.7}$$

$$\int_{\mathbb{R}} \mathcal{F}\phi_3(s)w(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi_3 F_{i\cdot}m)(s)f(s)\,ds$$

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for all $\phi_1, \phi_2, \phi_3 \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. Using the closedness of A and M, we deduce from (2.6) that

$$\int_{\mathbb{R}} \mathcal{F}\phi_1(s)Au(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi_1 Am)(s)f(s)\,ds, \tag{2.8}$$

$$\int_{\mathbb{R}} \mathcal{F}\phi_1(s)Mu(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi_1 Mm)(s)f(s)\,ds \tag{2.9}$$

for all $\phi_1 \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. Letting $\phi_1 = \mathrm{id} \cdot \phi_2$ in (2.9), where $\mathrm{id}(s) := is$ when $s \in \mathbb{R}$, we obtain

$$\int_{\mathbb{R}} \mathcal{F}(\mathrm{id} \cdot \phi_2)(s)Mu(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi_2 h)(s)f(s)\,ds$$

for all $\phi_2 \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. This equality together with (2.7) implies that

$$\int_{\mathbb{R}} \mathcal{F}(\mathrm{id} \cdot \phi_2)(s)Mu(s)\,ds = \int_{\mathbb{R}} \mathcal{F}\phi_2(s)v(s)\,ds$$

for all $\phi_2 \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. Thus, $Mu \in C^{1+\alpha}(\mathbb{R}; X)$ and $(Mu)' = v + x$ for some $x \in X$ by [1, Lemma 6.2].

On the other hand, by (2.6) and the boundedness of F on $C([-r, 0]; X)$,

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{F}(\phi_1 F_{i\cdot} m)(s)f(s)\,ds &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-its}\phi_1(t)F(e^{it\cdot}m(t)f(s))\,dt\,ds \\ &= F\int_{\mathbb{R}} e^{its}[e^{it\cdot}\phi_1(t)]m(t)\,dt f(s)\,ds \\ &= F\int_{\mathbb{R}} u(s)\mathcal{F}\phi_1(s - \cdot)\,ds \\ &= \int_{\mathbb{R}} Fu_s\mathcal{F}\phi_1(s)\,ds \end{aligned} \tag{2.10}$$

for all $\phi_1 \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. Here the integrals on the second and third lines are understood as integrals for function defined on $\mathbb{R}$ with values in $C([-r, 0]; X)$.

Using the fact that $(Mu)' = v + x$ and combining (2.7), (2.8), and (2.10), one deduces that

$$\int_{\mathbb{R}} \mathcal{F}\phi(s)\big[(Mu)'(t) - Au(t) - Fu_t - f(s)\big]\,ds = 0$$

for all $\phi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. This implies that there exists $y \in X$ satisfying

$$(Mu)'(t) = Au(t) + Fu_t + f(t) + y$$

when $t \in \mathbb{R}$ by [1, Lemma 5.1]. The assumption $0 \in \rho_{M,F}(A)$ means that $A + F_0$ is a bijection from $D(A) \cap D(M)$ onto X, and $(A + F_0)^{-1} \in \mathcal{L}(X)$. Then $u - (A + F_0)^{-1}y$ solves (P). We have shown the existence.

To show the uniqueness, we let $u \in C^\alpha(\mathbb{R}; D(A) \cap D(M))$ be such that $Mu \in C^{1+\alpha}(\mathbb{R}; X)$ and $(Mu)'(t) = Au(t) + Fu_t$ when $t \in \mathbb{R}$. Taking the Carleman transform $\widehat{u}$ of u [2, (4.25), p. 292], one has $\widehat{u}(\lambda) \in D(A)$,

$$\widehat{Au}(\lambda) = A\widehat{u}(\lambda),\ \widehat{(Mu)'}(\lambda) = \lambda M\widehat{u}(\lambda) - Mu(0),$$

$$\widehat{Fu_\cdot}(\lambda) = \begin{cases} F_\lambda\widehat{u}(\lambda) + F[e^{\lambda\cdot}\int_\cdot^0 e^{-\lambda t}u(t)\,dt], & Re\lambda > 0, \\ F_\lambda\widehat{u}(\lambda) - F[e^{\lambda\cdot}\int_\cdot^0 e^{\lambda t}u(-t)\,dt], & Re\lambda < 0 \end{cases}$$

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for all $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. It follows that

$$(\lambda M - A - F_\lambda)\widehat{u}(\lambda) = \begin{cases} Mu(0) + F[e^{\lambda\cdot}\int_\cdot^0 e^{-\lambda t}u(t)\,dt], & Re\lambda > 0,\\ Mu(0) - F[e^{\lambda\cdot}\int_\cdot^0 e^{\lambda t}u(-t)\,dt], & Re\lambda < 0\end{cases}$$

for all $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. Since $i\mathbb{R} \subset \rho_{M,F}(A)$, the operator $(isM - A - F_{is})^{-1} \in \mathcal{L}(X)$ when $s \in \mathbb{R}$. This implies that the Carleman spectrum $sp_C(u)$ of u is empty. Hence, $u = 0$ by [2, Theorem 4.8.2]. We have shown the uniqueness. Hence, (P) is C^α-well-posed.

Conversely, assume that (P) is C^α-well-posed. Let $L\colon C^\alpha(\mathbb{R};X) \to S(\mathbb{R};X)$ be the solution operator of (P) that associates the solution with each $f \in C^\alpha(\mathbb{R};X)$, where

$$S(\mathbb{R};X) := \left\{ u \in C^\alpha(\mathbb{R}; D(A) \cap D(M)) : Mu \in C^{1+\alpha}(\mathbb{R};X)\right\}$$

is the solution space of (P). It is easy to show that $S(\mathbb{R};X)$ equipped with the norm

$$\|u\|_{S(\mathbb{R};X)} := \|u\|_{C^\alpha(\mathbb{R};D(A)\cap D(M))} + \|Mu\|_{C^{1+\alpha}(\mathbb{R};X)}$$

is a Banach space. Then L is linear and bounded, which follows easily from the Closed Graph Theorem.

Let $s \in \mathbb{R}$ be fixed, we are going to show that $is \in \rho_{M,F}(A)$. Let $x \in D(A) \cap D(M)$ be such that $isMx = Ax + F_{is}x$ and $u = e_s \otimes x$, where $(e_s \otimes x)(t) = e^{\mathrm{ist}}x$ when $t \in \mathbb{R}$. It is clear that $u \in C^\alpha(\mathbb{R}; D(A) \cap D(M))$ and $Mu = e_s \otimes Mx \in C^{1+\alpha}(\mathbb{R};X)$ and $(Mu)'(t) = Au(t) + Fu_t$ for all $t \in \mathbb{R}$. This means that $u \in S(\mathbb{R};X)$ solves (P) when taking $f = 0$. Hence $u = 0$ by the uniqueness of the solution of (P). Consequently $x = 0$. We have shown that $isM - A - F_{is}$ is injective.

To show that $isM - A - F_{is}$ is also surjective, we let $y \in X$ and consider $f = e_s \otimes y$. Then $f \in C^\alpha(\mathbb{R};X)$. Let $u \in S(\mathbb{R};X)$ be the unique solution of (P), *i.e.*,

$$(Mu)'(t) = Au(t) + Fu_t + f(t)$$

for all $t \in \mathbb{R}$. For fixed $\xi \in \mathbb{R}$, we consider the function u_ξ given by $u_\xi(t) = u(t + \xi)$ when $t \in \mathbb{R}$. Then both functions u_ξ and $e^{i\xi s}u$ solve the problem

$$(Mv)'(t) = Av(t) + Fv_t + e^{i\xi s}f(t).$$

We deduce from the uniqueness that $u_\xi = e^{i\xi s}u$, that is, $u(t + \xi) = e^{i\xi s}u(t)$ for $t, \xi \in \mathbb{R}$. Let $x = u(0) \in D(A) \cap D(M)$. Then $u = e_s \otimes x$. Since u solves $(Mu)'(t) = Au(t) + Fu_t + f(t)$, we have $ise_s \otimes Mx = Ae_s \otimes x + e_s \otimes y$. Letting $t = 0$, we have $isMx - Ax - F_{is}x = y$. We have shown that $isM - A - F_{is}$ is surjective. Thus, $isM - A - F_{is}$ is a bijection from $D(A) \cap D(M)$ onto X and $x = (isM - A - F_{is})^{-1}y$. Therefore, $u = e_s \otimes (isM - A - F_{is})^{-1}y$.

When $s = 0$, the function f is the constant function y and u is the constant function $-(A + F_0)^{-1}y$. By the boundedness of the linear operator L, we have

$$\|(A + F_0)^{-1}y\| = \|u\|_{C^\alpha} \le \|L\|\|f\|_{C^\alpha} = \|L\|\|y\|.$$

Therefore, $(A + F_0)^{-1} \in \mathcal{L}(X)$, that is, $0 \in \rho_{M,F}(A)$. This implies that there exists $\epsilon > 0$ such that $i[-\epsilon, \epsilon] \subset \rho_{M,F}(A)$ as $\rho_{M,F}(A)$ is an open subset of $\mathbb{C}$ [4]. When $|s| > \epsilon$,

$$\begin{aligned} \gamma_\alpha|s|^\alpha\|(isM - A - F_{is})^{-1}y\| &= \|e_s \otimes (isM - A - F_{is})^{-1}y\|_\alpha \\ &= \|u\|_\alpha \le \|L\|\,\|f\|_{C^\alpha} = \|L\|(\|f\|_\alpha + \|f(0)\|) = \|L\|(\gamma_\alpha|s|^\alpha\|y\| + \|y\|). \end{aligned} \tag{2.11}$$

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Consequently, $is \in \rho_{M,F}(A)$. We have shown that $i\mathbb{R} \subset \rho_{M,F}(A)$. It follows from (2.11) and the fact $0 \in \rho_{M,F}(A)$ that

$$\sup_{s\in\mathbb{R}} \left\| (isM - A - F_{is})^{-1} \right\| < \infty$$

by continuity. Using the facts $Au, (Mu)' \in C^\alpha(\mathbb{R}; X)$, an argument similar to the one used in (2.11) gives

$$\sup_{s\in\mathbb{R}} \left\| A(isM - A - F_{is})^{-1} \right\| < \infty, \quad \sup_{s\in\mathbb{R}} \left\| sM(isM - A - F_{is})^{-1} \right\| < \infty.$$

This completes the proof. ∎

Since the necessary and sufficient condition given in Theorem 2.3 for the problem (P) to be C^α-well-posed does not depend on the parameter $0 < \alpha < 1$, we actually have the following corollary.

Corollary 2.4 *Let X be a complex Banach space, $0 < \alpha < 1$ and let A, M be closed linear operators on X satisfying $D(A) \cap D(M) \neq \{0\}$, $F \in \mathcal{L}(C([-r, 0]; X), X)$ for some fixed $r > 0$, we assume that F satisfies $(\mathbf{H}_2)$ and the problem (P) is C^α-well-posed for some $0 < \alpha < 1$, then it is C^α-well-posed for all $0 < \alpha < 1$.*

When the underlying Banach spaces X, Y are B-convex, the first order condition (2.3) is already sufficient for a function m to be a $\dot{C}^\alpha$-Fourier multiplier. This together with the proof of Theorem 2.3 gives the following result, which gives a necessary and sufficient condition for (P) to be C^α-well-posed under a weaker assumption on F when the underlying Banach space is B-convex. We should notice that when X has Fourier type $1 < p \leq 2$, then $D(A)$ equipped with its graph norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$ also has Fourier type $1 < p \leq 2$, where $D(A)$ is the domain of a closed linear operator A on a Banach space X. Thus, when X is B-convex, $D(A)$ equipped with its graph norm is also B-convex.

Corollary 2.5 *Let X be a B-convex complex Banach space, $0 < \alpha < 1$ and let A, M be closed linear operators on X satisfying $D(A) \cap D(M) \neq \{0\}$, $F \in \mathcal{L}(C([-r, 0]; X), X)$ for some fixed $r > 0$, we assume that F satisfies $(\mathbf{H}_1)$. Then (P) is C^α-well-posed if and only if $i\mathbb{R} \subset \rho_{M,F}(A)$, and*

$$\sup_{s\in\mathbb{R}} \left\| (isM - A - F_{is})^{-1} \right\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \left\| A(isM - A - F_{is})^{-1} \right\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \left\| sM(isM - A - F_{is})^{-1} \right\| < \infty.$$

3 Applications

In the last section, we give some examples to which our abstract results can be applied.

Example 3.1 Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and m be a non-negative bounded measurable function defined on Ω. Let X be the Hilbert

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space $H^{-1}(\Omega)$. We consider the following first order degenerate differential equations with finite delay:

$$(P_1)\begin{cases}\frac{d}{dt}(m(x)u(t,x)) = \Delta u(t,x) + Fu(t+\cdot,x) + f(t,x), & (t,x)\in\mathbb{R}\times\Omega,\\ u(t,x)=0, & (t,x)\in\mathbb{R}\times\partial\Omega,\end{cases}$$

where F is a bounded linear operator from $C([-r,0];H^{-1}(\Omega))$ to $H^{-1}(\Omega)$ for some fixed $r>0$, the Laplacian Δ acts on the second variable $x\in\Omega$.

Let M be the multiplication operator by m on $H^{-1}(\Omega)$ with domain $D(M)$. We assume that $D(\Delta)\cap D(M)\neq\{0\}$, where Δ is the Laplacian on $H^{-1}(\Omega)$ with Dirichlet boundary condition. Then it follows from [7, Section 3.7] that there exists a constant $C\geq 0$ such that

$$\|M(zM-\Delta)^{-1}\|\leq\frac{C}{1+|z|} \tag{3.1}$$

when $\mathrm{Re}(z)\geq-\beta(1+|\mathrm{Im}(z)|)$ for some positive constant β depending only on m. This implies that

$$\sup_{s\in\mathbb{R}}\|sM(isM-\Delta)^{-1}\|<\infty. \tag{3.2}$$

We deduce that

$$\sup_{s\in\mathbb{R}}\|\Delta(isM-\Delta)^{-1}\|<\infty. \tag{3.3}$$

If we assume furthermore that the function m is regular enough so that the corresponding operator M on $H^{-1}(\Omega)$ has a bounded inverse M^{-1}, then we deduce from (3.1) that

$$\sup_{s\in\mathbb{R}}\|s(isM-\Delta)^{-1}\|<\infty. \tag{3.4}$$

In particular,

$$\sup_{s\in\mathbb{R}}\|(isM-\Delta)^{-1}\|<\infty. \tag{3.5}$$

We assume that the delay operator F satisfies $(\mathbf{H}_1)$ and $i\mathbb{R}\subset\rho_{M,F}(\Delta)$. We are going to show that (P_1) is C^α-well-posed for all $0<\alpha<1$. Indeed, for all $s\in\mathbb{R}$, the operator $isM-\Delta-F_{is}$ is a bijection from $D(\Delta)\cap D(M)$ onto $H^{-1}(\Omega)$ and its inverse $(isM-\Delta-F_{is})^{-1}\in\mathcal{L}(H^{-1}(\Omega))$ as $i\mathbb{R}\subset\rho_{M,F}(\Delta)$. For $s\in\mathbb{R}$,

$$isM-\Delta-F_{is}=\big[I_X-F_{is}(isM-\Delta)^{-1}\big](isM-\Delta).$$

Thus,

$$(isM-\Delta-F_{is})^{-1}=(isM-\Delta)^{-1}\big[I_X-F_{is}(isM-\Delta)^{-1}\big]^{-1}.$$

Hence,

$$\begin{gathered}\sup_{s\in\mathbb{R}}\|(isM-\Delta-F_{is})^{-1}\|<\infty,\\ \sup_{s\in\mathbb{R}}\|\Delta(isM-\Delta-F_{is})^{-1}\|<\infty,\\ \sup_{s\in\mathbb{R}}\|sM(isM-\Delta-F_{is})^{-1}\|<\infty\end{gathered} \tag{3.6}$$

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by (3.2), (3.3), and (3.5) as

$$\lim_{s\to\infty} \| F_{is}(isM - \Delta)^{-1} \| = 0.$$

Here we have used (3.4) and the uniform boundedness of the family $(F_{is})_{s\in\mathbb{R}}$, which is easy to verify from the definition of F_{is}. We deduce from (3.6) and Corollary 2.5 that (P_1) is C^α-well-posed. Here we have used the fact that every Hilbert space is B-convex.

Example 3.2 Let H be a complex Hilbert space and let P be a densely defined positive selfadjoint operator on H with $P \geq \delta > 0$. Let $M = P - \epsilon$ with $\epsilon < \delta$, and let $A = -\sum_{i=0}^{k} a_i P^i$ with $a_i \geq 0$, $a_k > 0$. Then there exists a constant $C > 0$, such that

$$\| M(zM - A)^{-1} \| \leq \frac{C}{1 + |z|}$$

whenever $Re(z) \geq -\beta(1 + |Im(z)|)$ for some positive constant β depending only on A and M by [7, p. 73]. This implies in particular that

$$\text{(3.7)} \qquad \sup_{s\in\mathbb{R}} \| sM(isM - A)^{-1} \| < \infty, \quad \sup_{s\in\mathbb{R}} \| A(isM - A)^{-1} \| < \infty.$$

If we assume that $0 \in \rho(M)$, then

$$\text{(3.8)} \qquad \sup_{s\in\mathbb{R}} \| s(isM - A)^{-1} \| < \infty.$$

Assume that the delay operator $F \in \mathcal{L}(C([-r, 0]; H), H)$ satisfies $(\mathbf{H}_1)$ and $i\mathbb{R} \subset \rho_{M,F}(A)$. Then the same argument used in Example 3.1 shows that the corresponding first order degenerate differential equations with finite delay

$$(Mu)'(t) = Au(t) + Fu_t + f(t), \quad (t \in \mathbb{R})$$

is C^α-well-posed for all $0 < \alpha < 1$ by Corollary 2.5.

We can also give a concrete application of this abstract observation. Let $X = L^2(\Omega)$, where $\Omega = (0, 1)$. Consider the problem

$$(P_2) \begin{cases} \frac{d}{dt}(1 - \frac{\partial^2}{\partial x^2})u(t, x) = -\frac{\partial^4}{\partial x^4}u(t, x) + Fu(t + \cdot, x) + f(t, x), & (t, x) \in \mathbb{R} \times \Omega, \\ u(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial\Omega. \end{cases}$$

Let $P = -\frac{\partial^2}{\partial x^2}$ with domain $D(P) = H^2(\Omega) \cap H_0^1(\Omega)$. Then P is a positive self adjoint operator on X [2, p. 153]. Let $M = P + I_X$ and $A = -P^2$. Then $0 \in \rho(M)$, (3.7), and (3.8) hold true. If we assume furthermore that the delay operator

$$F \in \mathcal{L}\big(C([-r, 0]; L^2(\Omega)), L^2(\Omega) \big)$$

satisfies $(\mathbf{H}_1)$ and $i\mathbb{R} \subset \rho_{M,F}(A)$, then (P_2) is C^α-well-posed for all $0 < \alpha < 1$.

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