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A PROPERTY OF SUBGROUPS OF FREE GROUPS

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Dedicated to B.H. Neumann for his 80th Birthday

INTRODUCTION

Let F be a free group on $a_1, \ldots, a_p$ $(p \geqslant 1)$, and X a finitely generated subgroup in F. Suppose either that X contains some non-trivial power of $a_1^2 \ldots a_p^2$, or that p is even and X contains some nontrivial power of $[a_1, a_2] \ldots [a_{p-1}, a_p]$. We discuss some properties of X which we can derive from this assumption.

In this note we use the terminology and notation of [2, 4, 6]; here $\langle \ldots \mid \ldots \rangle$ denotes a presentation of a group in terms of generators and relations. By $\langle b_1, \ldots, b_n \rangle$ we denote the group generated by $b_1, \ldots, b_n$; $[a, b] = aba^{-1}b^{-1}$ is the *commutator* of $a, b \in G$ (G a group). Frequently we obtain from one system $\{x_1, \ldots, x_m\}$ a new one by Nielsen transformations, and then denote the latter by the same symbols.

AN EQUATION IN A FREE GROUP

THEOREM 1. *Let F be a free group on $a_1, \ldots, a_p$ $(p \geqslant 1,\ p$ even) and $P(a_1, \ldots, a_p) = [a_1, a_2] \ldots [a_{p-1}, a_p] \in F$. Let $(x_1, \ldots, x_m)$ $(m \geqslant 1)$ be any finite sequence of elements of F and X be the subgroup of F generated by $x_1, \ldots, x_m$. Suppose that X contains some conjugate of some non-trivial power of $P(a_1, \ldots, a_p)$, and let β be the smallest positive integer such that some conjugate of $P^\beta(a_1, \ldots, a_p)$ lies in X. Then*

(a) *$(x_1, \ldots, x_m)$ is Nielsen equivalent to a free basis for X which contains a conjugate of $P^\beta(a_1, \ldots, a_p)$; or*
(b) *the index of X in F is β (and $\{1, P(a_1, \ldots, a_p), \ldots, P^{\beta-1}(a_1, \ldots, a_p)\}$ is a set of coset representatives for X in F).*

PROOF: In the following let α be the smallest positive number for which $y^{-1}P^\alpha(a_1, \ldots, a_p)y \in X$ for some $y \in F$. If $\alpha = 1$ then the statement holds by Theorem (2.2) of [5]. Hence, from now on let $\alpha \geqslant 2$. We may assume that $y = 1$ (by replacing the x_i by yx_iy^{-1} if necessary).

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By means of the classical Nielsen reduction method we obtain from $(x_1, \ldots, x_m)$ a system $(y_1, \ldots, y_k)$, $1 \leqslant k \leqslant m$, which generates X freely and has the Nielsen property, so that in particular, as a freely reduced word in $a_1, \ldots, a_p$, each y_i contains an uncancellable symbol a_j^{ε}, $\varepsilon = \pm 1$, in the sense that in reducing any product $y_r^{\varepsilon_r} y_s^{\varepsilon_s}$, ε_r, $\varepsilon_s = \pm 1$, these symbols cancel if and only if $r = s$ and $\varepsilon_r = -\varepsilon_s$. For each y_i we choose such a symbol and take the inverse symbol for y_i^{-1}.

We have an equation

$$\prod_{i=1}^{q} y_{\nu_i}^{\varepsilon_i} = P^{\alpha}(a_1, \ldots, a_p), \quad \varepsilon_i = \pm 1, \qquad \varepsilon_i = \varepsilon_{i+1} \text{ if } \nu_i = \nu_{i+1}. \tag{1}$$

If a factor $y_{\nu_t}^{\varepsilon_t}$ occurs twice in (1) then we have a partial product $\prod_{i=t}^{s} y_{\nu_i}^{\varepsilon_i}$ with $\nu_t = \nu_{s+1}$, $\varepsilon_t = \varepsilon_{s+1}$ and no $y_{\nu_i}^{\varepsilon_i}$ occurring twice in $y_{\nu_t}^{\varepsilon_t} \ldots y_{\nu_s}^{v\varepsilon_s}$. The symbol of $y_{\nu_t}^{\varepsilon_t}$ occurs twice in $\prod_{i=1}^{q} y_{\nu_i}^{\varepsilon_i}$, and also in $P^{\alpha}(a_1, \ldots, a_p)$. Hence $\prod_{i=t}^{s} y_{\nu_i}^{\varepsilon_i}$ is conjugate to a power of $P(a_1, \ldots, a_p)$. Therefore, after a suitable conjugation, we obtain an equation which may be assumed to have the form

$$\prod_{i=1}^{r} y_{\nu_i}^{\varepsilon_i} = P^{\beta}(a_1, \ldots, a_p), 1 \leqslant \beta \leqslant \alpha, \varepsilon_i = \pm 1, \varepsilon_i = \varepsilon_{i+1} \text{ if } \nu_i = \nu_{i+1}, \tag{2}$$

with no factor $y_{\nu_i}^{\varepsilon_i}$ occurring twice (or course we must then in fact have $\alpha = \beta$ because α is minimal). If some y_i occurs just once in (2) either with exponent $+1$ or with exponent -1 then case (a) occurs. Now, let in (2) each y_i occur exactly once with exponent $+1$ and once with exponent -1. (It can be shown — we omit the details — that every y_i must occur in (2).) This implies in particular that X is not cyclic, since $\beta \geqslant 1$. Hence $k \geqslant 2$, and by [6, 5.2] we may apply Nielsen transformations for quadratic words to get from $(y_1, \ldots, y_k)$ a system $(z_1, \ldots, z_k)$ such that

$$\prod_{i=1}^{r} y_{\nu_i}^{\varepsilon_i} = [z_1, z_2] \ldots [z_{\ell-1}, z_\ell] z_{\ell+1} z_{\ell+1}^{-1} \ldots z_k z_k^{-1}, 2 \leqslant \ell \leqslant k, \quad \ell \text{ even}. \tag{3}$$

Without loss of generality, we may assume that $\ell = k$ and, hence, that k is even and $(z_1, \ldots, z_k)$ is a free generating system of X.

We want to show that X has the finite index $[F : X] = \beta$. From Theorem (2.2) of [5] we know that $F = \langle z_1, \ldots, z_k, P(a_1, \ldots, a_p) \rangle$. Now we regard F as a cycloidal Fuchsian group of the first kind such that

(i) $a_1, \ldots, a_p$ are hyperbolic elements and $p/2$ is the genus of F;
(ii) $a = P(a_1, \ldots, a_p)^{-1}$ is a parabolic element; and
(iii) $\{a, a_1, \ldots, a_p\}$ is a canonical generating system of F.

Now X is, as a subgroup of F, also discrete. Moreover, X has finite index in F. The essential reasons for this are as follows. Since F is residually finite, for a suitable large integer γ, the image $X(n)$ of X has finite index $[F(n) : X(n)] \leqslant N$ in the factor group $F(n) = \langle a_1, \dots, a_p \mid (P(a_1, \dots, a_p))^{\beta\gamma n} = 1 \rangle$, $(n \in \mathbb{N})$, of F, where N is an integer which does not depend on n, and the set of coset representatives for $X(n)$ in $F(n)$ can be chosen independently of n by [**6**, 4.14] and since $F = \langle z_1, \dots, z_k, P(a_1, \dots, a_p) \rangle$. Recalling that a finitely generated subgroup of infinite index in $F(n)$ is automatically a free product of cyclic groups ([**1**]), it follows by letting n increase that X has finite index $[F : X] \leqslant N$ in F. (See also [**3**, Theorem 2.12].)

In a finitely generated free group no power of a primitive element is contained in its commutator subgroup. Hence, since the index $[F : X]$ is finite, no primitive element of X is conjugate in F to a power of $P(a_1, \dots, a_p)$, and X is a cycloidal Fuchsian group of the first kind such that

(i) $z_1, \dots, z_k$ are hyperbolic elements and $k/2$ is the genus of X,
(ii) $b = a^\beta = P(a_1, \dots, a_p)^{-\beta}$ is a parabolic element, and
(iii) $\{b, z_1, \dots, z_k\}$ is a canonical generating system of X.

If $[F : X] = 1$ then $\beta = 1$ and $F = X$. Now let $[F : X] > 1$, $g \in F \backslash X$, and consider $g\, P(a_1, \dots, a_p) g^{-1}$. There exists a natural number γ such that $g\, P^\gamma(a_1, \dots, a_p) g^{-1} \in X$ since $[F : X]$ is finite. Hence, there exists $h \in X$ and an integer $\delta \neq 0$ such that $hg\, p^\gamma(a_1, \dots, a_p) g^{-1} h^{-1} = p^\delta(a_1, \dots, a_p)$, because X is cycloidal. Therefore $\gamma = \delta$ and $hg \in \langle P(a_1, \dots, a_p) \rangle$ since F is free. Now $g = d\, P^\varphi(a_1, \dots, a_p)$, (whence $Xg = X\, P^\varphi(a_1, \dots, a_p)$) for some $d \in X$ and some natural number φ with $1 \leqslant \varphi < \beta$, since $h \in X$ and $P^\beta(a_1, \dots, a_p) \in X$. ∎

COROLLARY 1. *Let F be a free group on $a_1, \dots, a_p$ ($p \geqslant 1$, p even) and $P(a_1, \dots, a_p) = [a_1, a_2] \dots [a_{p-1}, a_p] \in F$. Let $\alpha \in \mathbb{N}$, $\alpha \geqslant 2$, and $x_1, y_1, \dots, x_q, y_q \in F$ ($q \geqslant 1$) be such that $P^\alpha(a_1, \dots, a_p) = \prod_{i=1}^{q} [x_i, y_i]$. Suppose that $P^\alpha(a_1, \dots, a_p)$ is not a proper power in the subgroup X generated by $x_1, y_1, \dots, x_q, y_q$. Then α is odd and $q \geqslant (\alpha(p-1)+1)/2$.*

PROOF: Note that X has finite index α in F by Theorem 1. If we regard F as a cycloidal Fuchsian group of the first kind such that $P(a_1, \dots, a_p)$ is parabolic then the Riemann–Hurwitz relation yields that α is odd and $\alpha(p-1) \leqslant 2q-1$, giving the result. ∎

If we regard a free group F on $a_1, \dots, a_p$ ($p \geqslant 1$) as a plane discontinuous group $F = \langle a, a_1, \dots, a_p \mid a\, a_1^2 \dots a_p^2 = 1 \rangle$ we get analogously the following,

THEOREM 2. *Let F be a free group on $a_1, \dots, a_p$ ($p \geqslant 1$) and let $Q(a_1, \dots, a_p) =$*

$a_1^2 \dots a_p^2 \in F$. Let $(x_1, \dots, x_m)$ $(m \geqslant 1)$ be any finite sequence of elements of F and X be the subgroup of F generated by $x_1, \dots, x_m$. Suppose that X contains some conjugate of some nontrivial power of $Q(a_1, \dots, a_p)$, and let β be the smallest positive integer such that some conjugate of $Q^\beta(a_1, \dots, a_p)$ lies in X. Then:

(a) *$\{x_1, \dots, x_m\}$ is Nielsen equivalent to a free basis for X which contains a conjugate of $Q^\beta(a_1, \dots, a_p)$; or*

(b) *the index of X in F is β, (and $\{1, Q(a_1, \dots, a_p), \dots, Q^{\beta-1}(a_1, \dots, a_p)\}$ is a set of coset representatives for X in F).*

COROLLARY 2. *Let F be a free group on $a_1, \dots, a_p$ $(p \geqslant 1)$ and $Q(a_1, \dots, a_p) = a_1^2 \dots a_p^2 \in F$. Let $\alpha \in \mathbb{N}$, $\alpha \geqslant 2$, and $x_1, \dots, x_q \in F$ $(q \geqslant 1)$ be such that*

$$Q^\alpha(a_1, \dots, a_p) = x_1^2 \dots x_q^2.$$

Then if $Q^\alpha(a_1, \dots, a_p)$ is not a proper power in the subgroup X generated by $x_1, \dots, x_q$, we have $q \geqslant \alpha(p-1)+1$.

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