A NOTE ON ASYMPTOTIC SERIES

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Introduction. We extend some observations of Popken **(2)** on the algebraic foundations of the theory of asymptotic series. The main result is the theorem in §5 which characterizes, for a particular function space, a class of linear functionals defined in §4. In §3 we discuss another class of linear functionals related to asymptotic series. In the first two paragraphs we give definitions which render this note self-contained.

This note grew out of a department seminar led by T. E. Hull, to whom I am indebted for many stimulating discussions.

1. Asymptotic sums. Let $c_0 + c_1x + c_2x^2 + \ldots$ be a formal power series with real coefficients. By an *asymptotic sum* (as $x \to 0+$) of the series, is meant a real-valued function f, having as domain the positive reals, such that for all nonnegative integers n,

$$f(x) - c_0 - c_1x - \ldots - c_nx^n = o(x^n), \qquad x \to 0+.$$

If f has this property, so also has, for example, $f(x) + e^{-1/x}$; the asymptotic sum of a formal series is therefore not unique. It is known that every such series has an asymptotic sum, which can be constructed in the following way. Let S denote a nested sequence

$$N_0 \supset N_1 \supset N_2 \supset \ldots$$

of neighborhoods of $x = 0$ with characteristic functions $u_0, u_1, u_2, \ldots$. Assume further that any positive x is in one and at most finitely many of these neighborhoods, so the series

$$c_0u_0(x) + c_1u_1(x) + c_2u_2(x)x^2 + \ldots$$

converges for all positive x and defines a function $f(S, x)$. Then for a suitable nest S, whose choice depends on the power series, the function $f(S, x)$ is an asymptotic sum of the formal series. If we select a fixed S, and associate with each formal power series the corresponding $f(S, x)$, we obtain an isomorphism between the ring of all formal power series and a ring of functions. Since it is impossible to find a nest S that gives an asymptotic sum for all series, this isomorphism pairs each series with an asymptotic sum of the series only over a subring. We do not know if, by some other method, it is possible to obtain a correspondence between the ring of all formal power series and a ring of functions, which is not only a ring isomorphism but also pairs each series with a function that is one of its asymptotic sums.

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2. A class of linear functionals. Denote by C the collection of all functions which are asymptotic sums of formal power series (we do not allow negative exponents). If f is an asymptotic sum of the series

$$c_0 + c_1x + c_2x^2 + \ldots,$$

let $L_nf = c_n$. Then L_n is a linear functional on C. If m is the first nonnegative integer such that L_mf is not zero, define $\phi(f) = e^{-m}$; if $L_mf = 0$ for all m, define $\phi(f) = 0$. Then $d(f, g) = \phi(f - g)$ is a pseudo-metric on C; two functions f and g are *asymptotically equal* (as $x \to 0+$) if $d(f, g) = 0$. These definitions are specializations to C of those given by Popken in **(2)**.

3. Continuous linear functionals. It is natural to consider linear functionals on C which are continuous in the sense that $Lf_n \to Lf$ whenever $f_n \to f$, i.e. whenever $d(f_n, f) \to 0$. Such a functional L is constant on the classes of asymptotically equal functions.

LEMMA 1. *Every continuous linear functional on C is a (finite) linear combination of functionals of the type described in* §2.

Proof. Let L be a continuous linear functional, and let $L(x^m) = a_m$. If $a_m \neq 0$, let $c_m = 1/a_m$; otherwise let $c_m = 0$. Let

$$f_n(x) = c_0 + c_1x + \ldots + c_nx^n.$$

Let f be any asymptotic sum of the formal series $c_0 + c_1x + c_2x^2 + \ldots$ (f exists, as remarked in §1). Then $d(f_n, f) \to 0$, so by continuity $L(f_n) \to L(f)$. Since $L(f_{n+1}) = L(f_n) + 1$ whenever $a_{n+1} \neq 0$, all but a finite number of the a_m must equal zero.

Now let $d_0 + d_1x + d_2x^2 + \ldots$ be any formal series, and let g be any one of its asymptotic sums. Let

$$g_m(x) = d_0 + d_1x + \ldots + d_nx^n.$$

Then $d(g_n, g) \to 0$, so $L(g_n) \to L(g)$, and since all a_m are zero for $m > N$ (say), we have

$$L(g_n) = d_0a_0 + d_1a_1 + \ldots + d_Na_N, \qquad n > N.$$

Therefore $L(g) = L(g_N)$. Recalling the definition of L_n, we prove that

$$L = a_0L_0 + a_1L_1 + \ldots + a_NL_N,$$

completing the proof of the Lemma.

Conversely, any linear combination of the L_n is a continuous linear functional on C. Since this class of linear functionals has such a transparent structure, it seems to be of limited interest.

4. Asymptotic continuity.

DEFINITION. A linear functional L is *asymptotically continuous* on a subspace of C if, whenever h is in this subspace, and $h(x) = o(x^n)$ as $x \to 0+$, for some nonnegative integer n, it follows that $h^*(t) = o(t^n)$ as $t \to 0+$, where h^* is defined by $h^*(t) = L(h(xt))$. Here L acts on functions of the positive variable x, t being a positive parameter. The transform h^* is not required to be an element of C.

It is easily verified that the functionals L_n of §2 are asymptotically continuous on C; it follows from Lemma 1 that *any continuous linear functional on C is also asymptotically continuous.* We do not know if there exist any asymptotically continuous linear functionals on C that are not also continuous; perhaps the definitions are equivalent for functionals defined and linear on all of C. On certain subspaces the two definitions are not equivalent, and we now consider one of these subspaces. Certain other subspaces can be treated by essentially identical methods, but we give the details for only one of them here.

Let H denote the smallest subspace of C containing the functions $\{x^m\}$ $(m = 0, 1, 2, \ldots)$, and $\{e^{-tx}\}$ for all nonnegative real numbers t. A general element of H is simply a finite linear combination of such functions. It is easily seen that the following properties are equivalent in H; a function possessing any one of them possesses them all:

(1) $f(x) = o(x^n)$ as $x \to 0+$,
(2) $f(xt) = o(x^n)$ as $x \to 0+$, for all fixed positive t,
(3) for some positive t, $f(xt) = o(x^n)$ as $x \to 0+$,
(4) $L_m f = 0$ $(m = 0, 1, \ldots, n)$, where L_m is as defined in §2.

LEMMA 2. *If L is an asymptotically continuous linear functional on H and if $L(x^m) = (-1)^m c_m m!$ for all nonnegative integers m, then $L(e^{-tx})$ is an asymptotic sum of the formal series $c_0 + c_1 t + c_2 t^2 + \ldots$.*

Proof. For each n, and each t,

$$e^{-xt} - \sum_{m=0}^{n} \frac{(-1)^m t^m x^m}{m!} = o(x^n), \qquad x \to 0+;$$

thus by definition of asymptotic continuity (using the linearity of L) we have

$$L(e^{-xt}) - c_0 - c_1 t - \ldots - c_n t^n = o(t^n), \qquad t \to 0+.$$

The following lemma and the theorem in §5 both show the existence of a large class of such linear functionals.

LEMMA 3. *If $a(x)$ is a function of bounded variation possessing all moments, i.e.*

$$\int_0^\infty x^n da(x) < \infty, \qquad n = 0, 1, 2, \ldots,$$

then the linear functional

$$L(f) = \int_0^\infty f(x) da(x)$$

is asymptotically continuous on H.

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Proof. If $h(x) = o(x^n)$, by Taylor's theorem

$$h(x) = h^{(n+1)}(\theta x)x^{n+1}/(n+1)!$$

where $0 \leqslant \theta \leqslant 1$, θ depending on x. Since $h^{(n+1)}(x)$ is a linear combination of terms, each of which is a power of x or a function of the form e^{-cx} for non-negative c, each term of $h^{(n+1)}(\theta xt)$, for $0 \leqslant \theta \leqslant 1$, $0 \leqslant t \leqslant 1$, is dominated in magnitude either by the corresponding term of $h^{(n+1)}(x)$ or by a constant (for $x \geqslant 1$). The hypothesis that $a(x)$ possesses all moments then ensures that the integral

$$\int_0^\infty h^{(n+1)}(\theta xt)da(x)$$

is finite and bounded for $0 \leqslant t \leqslant 1$.

We then have

$$\frac{h^*(t)}{t^n} = \frac{L(h(xt))}{t^n} = \int_0^\infty \frac{h(xt)}{t^n}\,da(x)$$
$$= \frac{t}{(n+1)!}\int_0^\infty h^{(n+1)}(\theta xt)\,da(x) \to 0, \qquad t \to 0+,$$

proving that $h^*(t) = o(t^n)$, $t \to 0+$.

In §1, one method of constructing asymptotic sums was described. We now describe another method, based on the theory of the moment problem. This generalizes one discussed by E. Borel in **(1)**. Given any sequence a_0, a_1, $a_2, \ldots$ of real numbers, there exists **(3**, p. 139**)** an L of the form of Lemma 3 such that $L(x^m) = a_m$. Given a formal power series

$$c_0 + c_1t + c_2t^2 + \ldots$$

we take $a_m = (-1)^m c_m m!$; by Lemma 3 this L is asymptotically continuous on H. Since $L(x^m) = (-1)^m c_m m!$ the function $L(e^{-tx})$ is an asymptotic sum of the given series, by Lemma 2.

One of the simplest examples is provided by the convergent series

$$1 - t + t^2/2! - t^3/3! + \ldots;$$

here $a_m = 1$ for all m, and we obtain $L(x^m) = 1$ for all m by taking $a(x)$ in Lemma 3 to be constant except for a unit jump at $x = 1$. Then $L(e^{-tx}) = e^{-t}$; this is actually the sum of the series. In less trivial cases, the determination of $a(x)$ may be a formidable task. Thus this method is difficult to apply; in a later paper we shall give more convenient methods based on the observation that any formal power series is the series expansion of a Schwartz distribution.

Despite these difficulties, the existence of this method shows that *any linear functional L_1 on the space of all polynomials can be extended to an asymptotically continuous linear functional on H.* The extension is not unique. Moreover, the following remark shows that the extension may not even be possible, if we demand continuity in the sense of §3, and this is one justification for introducing the definition of asymptotic continuity.

There exist asymptotically continuous linear functionals on H that are not continuous. For if all asymptotically continuous L were continuous, we would have

$$L(e^{-xt}) = \sum_{n=0}^{\infty} \frac{(-1)^n L(x^n) t^n}{n!} = \sum_{n=0}^{\infty} c_n t^n,$$

the left-hand side being a function of t, and the sum being necessarily convergent. The theory of the moment problem implies that, by proper choice of such L, we can obtain *any* given formal power series on the right side, an obvious contradiction.

5. The main theorem. We prove a theorem which characterizes all asymptotically continuous linear functionals on H. Since L is determined completely by its values on the basis functions, let L_1 be the function defined for all nonnegative integers m by $L_1(m) = L(x^m)$, and let L_2 be a function on the positive reals defined by $L_2(t) = L(e^{-tx})$. Then L can be identified with the pair $\{L_1, L_2\}$, since any arbitrary pair of such functions determines uniquely a linear functional L (not necessarily asymptotically continuous). Since $L_2(t)$ may be discontinuous, it is an easy corollary of this theorem that not every asymptotically continuous linear functional on H is of the type described in Lemma 3.

THEOREM. *$L = \{L_1, L_2\}$ is asymptotically continuous on H if and only if $L_2(t)$ is an asymptotic sum of the formal power series*

$$\sum_{m=0}^{\infty} \frac{(-1)^m L_1(m) t^m}{m!}.$$

Proof. The necessity is a rewording of Lemma 2, so we only need prove the sufficiency. Any element of H may be written in the form

$$h(x) = \sum_{m=0}^{p} a_m x^m + \sum_{m=1}^{r} b_m e^{-t_m x},$$

and by the remark preceding Lemma 2, if $h(x) = o(x^n)$ as $x \to 0+$, we compute

$$a_m = -\sum_{q=1}^{r} \frac{(-1)^m t_q^m b_q}{m!}, \qquad m = 0, 1, \ldots, n.$$

Thus we have

$$L(h(xt)) = \sum_{q=1}^{r} b_q \left[L_2(t_q t) - \sum_{m=0}^{n} \frac{(-1)^m t_q^m L_1(m) t^m}{m!} \right] + \sum_{m=n+1}^{p} a_m L_1(m) t^m.$$

If $L_2(t_q t)$ is an asymptotic sum of

$$\sum_{m=0}^{\infty} \frac{(-1)^m L_1(m)(t_q t)^m}{m!},$$

each term in the equation is $o(t^n)$, $t \to 0+$, proving that L is asymptotically continuous.

References

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