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On Fiber Cones of $\mathfrak{m}$-Primary Ideals

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Abstract. Two formulas for the multiplicity of the fiber cone $F(I) = \bigoplus_{n=0}^{\infty} I^n/\mathfrak{m}I^n$ of an $\mathfrak{m}$-primary ideal of a d-dimensional Cohen–Macaulay local ring $(R, \mathfrak{m})$ are derived in terms of the mixed multiplicity $e_{d-1}(\mathfrak{m}|I)$, the multiplicity $e(I)$, and superficial elements. As a consequence, the Cohen–Macaulay property of $F(I)$ when I has minimal mixed multiplicity or almost minimal mixed multiplicity is characterized in terms of the reduction number of I and lengths of certain ideals. We also characterize the Cohen–Macaulay and Gorenstein properties of fiber cones of $\mathfrak{m}$-primary ideals with a d-generated minimal reduction J satisfying $\ell(I^2/JI) = 1$ or $\ell(I\mathfrak{m}/J\mathfrak{m}) = 1$.

1 Introduction

The objective of this paper is to study Cohen–Macaulay and Gorenstein properties of the fiber cone $F(I) = \bigoplus_{n=0}^{\infty} I^n/\mathfrak{m}I^n$ of an $\mathfrak{m}$-primary ideal I of a d-dimensional Cohen–Macaulay local ring $(R, \mathfrak{m})$ in terms of invariants such as the multiplicity $e(I)$, the mixed multiplicity $e_{d-1}(\mathfrak{m}|I)$ and reduction number of I.

In order to state the main results, we recall necessary definitions first. Let I be an $\mathfrak{m}$-primary ideal of a d-dimensional local ring $(R, \mathfrak{m})$. The Hilbert function $HF(F(I), n)$ of the fiber cone $F(I)$ is defined as $HF(F(I), n) = \ell(I^n/\mathfrak{m}I^n)$, where ℓ denotes the length function. The function $HF(F(I), n)$ is a polynomial $HP(F(I), n)$ in n of degree $d - 1$ for all large n. We write this polynomial as

$$HP(F(I), n) = f_0(I)\binom{n+d-1}{d-1} - f_1(I)\binom{n+d-2}{d-2} + \cdots + (-1)^{d-1} f_{d-1}(I),$$

for certain integers $f_0(I), f_1(I), \ldots, f_{d-1}(I)$. The number $f_0(I)$ is called the *multiplicity* of $F(I)$.

Multiplicities and Reductions

For an $\mathfrak{m}$-primary ideal I in a Noetherian local ring R of dimension d, let $HF(I, n) := \ell(R/I^n)$ denote the *Hilbert–Samuel function* of I. It is well known that this function coincides with a polynomial $HP(I, n)$ of degree d. Write the polynomial as:

$$HP(I, n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

The coefficient $e_0(I)$, also denoted as $e(I)$, is called the multiplicity of I.

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Now we recall some basic facts about *reductions* from [16]. An ideal $K \subseteq I$ is called a *reduction* of I if there exists a nonnegative integer n such that $KI^n = I^{n+1}$. If K is minimal with respect to inclusion among reductions of I, then it is called a *minimal reduction* of I. The *reduction number* $r(I)$ of I is the least integer n such that $JI^n = I^{n+1}$, where J varies over all minimal reductions of I. If $R/\mathfrak{m}$ is infinite, then all minimal reductions of I are generated by the same number of elements called the *analytic spread* of I. The analytic spread of I, is the Krull dimension of the fiber cone $F(I)$. It is easy to see that if J is a reduction of I, then $e(I) = e(J)$.

Mixed Multiplicities and Joint Reductions

Mixed multiplicities and joint reductions of ideals are analogues of reductions and multiplicities of ideals. Let $I_1, I_2, \ldots, I_r$ be $\mathfrak{m}$-primary ideals. The *Bhattacharya function* of $I_1, I_2, \ldots, I_r$ is the numerical function $BF(n_1, n_2, \ldots, n_r)\colon \mathbb{N}^r \to \mathbb{N}$, defined by $BF(n_1, n_2, \ldots, n_r) = \ell(R/I_1^{n_1} I_2^{n_2} \cdots I_r^{n_r})$. By [28], for all $n_1, n_2, \ldots, n_r$, large, the Bhattacharya function is given by a polynomial $BP(n_1, n_2, \ldots, n_r)$ of total degree d in $n_1, n_2, \ldots, n_r$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{N}^r$ we put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$. We write the Bhattacharya polynomial in the form

$$BP(n_1, n_2, \ldots, n_r) = \sum_{|\alpha| \leq d} e_\alpha \binom{n_1 + \alpha_1}{\alpha_1} \binom{n_2 + \alpha_2}{\alpha_2} \ldots \binom{n_r + \alpha_r}{\alpha_r},$$

where e_α are certain integers. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{N}^r$ and $|\alpha| = d$. A multiset consisting of α_1 copies of I_1, α_2 copies of $I_2, \ldots, \alpha_r$ copies of I_r, will be denoted by $(I_1^{[\alpha_1]} | I_2^{[\alpha_2]} | \cdots | I_r^{[\alpha_r]})$. In case $|\alpha| = d$, we write $e_\alpha = e_\alpha(I_1^{[\alpha_1]} | I_2^{[\alpha_2]} | \cdots | I_r^{[\alpha_r]})$. These integers are positive and are called *mixed multiplicities* of the ideals $I_1, I_2, \ldots, I_r$. When $r = 2$, and $i + j = d$, we adopt the simpler notation $e_{(i,j)}(I^{[i]} | J^{[j]}) = e_j(I|J)$. D. Rees proved that $e_0(I|J) = e(I)$ and $e_d(I|J) = e(J)$ [18].

Rees introduced joint reductions for calculating mixed multiplicities [19]. Let $(R, \mathfrak{m})$ be a d-dimensional local ring. Let $I_1, I_2, \ldots, I_d$ be $\mathfrak{m}$-primary ideals. We say that $a_1 \in I_1, a_2 \in I_2, \ldots, a_d \in I_d$ is a *joint reduction* of $I_1, I_2, \ldots, I_d$ if $a_1 I_2 I_3 \cdots I_d + a_2 I_1 I_3 \cdots I_d + \cdots + a_d I_1 I_2 \cdots I_{d-1}$ is a reduction of $I_1 I_2 \cdots I_d$. D. Rees showed [19, Theorem 2.4] that if $(a_1, a_2, \ldots, a_d)$ is a joint reduction of the multiset

$$(I_1^{[\alpha_1]} | I_2^{[\alpha_2]} | \cdots | I_r^{[\alpha_r]}),$$

where $|\alpha| = d$, then $e((a_1, a_2, \ldots, a_d)) = e_\alpha(I_1^{[\alpha_1]} | I_2^{[\alpha_2]} | \cdots | I_r^{[\alpha_r]})$.

Rings and Ideals of Minimal and Almost Minimal Multiplicity

Let $\mu(I)$ denote the minimum number of elements required to generate an ideal I. For a Cohen–Macaulay local ring $(R, \mathfrak{m})$ of dimension d, $e(\mathfrak{m}) \geq \mu(\mathfrak{m}) - d + 1$. A Cohen–Macaulay local ring is said to have *minimal multiplicity* (resp., *almost minimal multiplicity*) if $e(\mathfrak{m}) = \mu(\mathfrak{m}) - d + 1$ (resp., $e(\mathfrak{m}) = \mu(\mathfrak{m}) - d + 2$). J. D. Sally studied Cohen–Macaulay local rings of minimal and almost minimal multiplicity. She proved that the associated graded ring $G(\mathfrak{m}) := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$ is Cohen–Macaulay when R is Cohen–Macaulay with minimal multiplicity [22]. She conjectured that if the ring has almost minimal multiplicity, then $G(\mathfrak{m})$ has depth at least $d - 1$ [24]. This conjecture was proved independently by Wang [29] and by M. E. Rossi and G. Valla [21]. Later on M. E. Rossi generalized the conjecture of J. D. Sally to the case of $\mathfrak{m}$-primary ideals. She proved that if $e(I) = \ell(I/I^2) + (1 - d)\ell(R/I) + 1$, then depth $G(I) \geq d - 1$ [20]. It is easy to see that $e(I) = \ell(I/I^2) + (1 - d)\ell(R/I) + 1$ if and only if for any minimal reduction J of I, $\ell(I^2/JI) = 1$.

Definition 1.1 An $\mathfrak{m}$-primary ideal I of a Cohen–Macaulay local ring satisfying the condition $\ell(I^2/JI) = 1$ for any minimal reduction J is called a *Sally ideal.*

The notions of minimal multiplicity and almost minimal multiplicity have been generalized in many directions. It was proved in [6] that for an $\mathfrak{m}$-primary ideal I of a Cohen–Macaulay local ring $(R, \mathfrak{m})$, $e_{d-1}(\mathfrak{m}|I) \geq \mu(I) - d + 1$. We say that I has *minimal mixed multiplicity* if $e_{d-1}(\mathfrak{m}|I) = \mu(I) - d + 1$ and I has *almost minimal mixed multiplicity* if $e_{d-1}(\mathfrak{m}|I) = \mu(I) - d + 2$. The Cohen–Macaulay property of fiber cones of ideals with minimal and almost minimal mixed multiplicities was studied in [6, 7]. J. Chuai [3] proved that for an $\mathfrak{m}$-primary ideal I in a Cohen–Macaulay local ring $(R, \mathfrak{m})$, $e(I) \geq \mu(I) - d + \ell(R/I)$. In [9], S. Goto termed an ideal to have minimal multiplicity if $e(I) = \mu(I) - d + \ell(R/I)$. He studied many properties of the associated graded ring, the fiber cone and the Rees algebra of ideals with minimal multiplicity. In [14], fiber cones of ideals having almost minimal multiplicity are studied, *i.e.,* ideals with the property $e(I) = \mu(I) - d + \ell(R/I) + 1$.

Main results

In this paper, we consider the Cohen–Macaulay and Gorenstein properties of fiber cones of Sally ideals, ideals with minimal and almost minimal (mixed) multiplicity. We assume, in the rest of this section, that $(R, \mathfrak{m})$ *is a d-dimensional Cohen–Macaulay local ring with infinite residue field.*

In Section 2, we obtain two formulas for $f_0(I)$ in terms of $e_{d-1}(\mathfrak{m}|I), e(I)$ and superficial elements for $\mathfrak{m}$ and I in the sense of Rees.

In Section 3, as a consequence of these formulas we recover one of the main results of [6] to the effect that for an ideal I of minimal mixed multiplicity, $F(I)$ is Cohen–Macaulay if and only if $r(I) \leq 1$. For an ideal I of almost minimal mixed multiplicity, we show that either $f_0(I) = e_{d-1}(\mathfrak{m}|I) - 1$ or $f_0(I) = e_{d-1}(\mathfrak{m}|I)$. In the former case, $F(I)$ is Cohen–Macaulay if and only if $r(I) \leq 1$, and in the latter case, $F(I)$ is Cohen–Macaulay if and only if $r(I) = 2$ and $\ell(I^2/JI + \mathfrak{m}I^2) = 1$. This result was proved in [7] under depth assumptions on $G(I)$. We improve it by carefully using the multiplicity formula for the fiber cone. If I is a Sally ideal with a minimal reduction J, then we show that $F(I)$ is Cohen–Macaulay if and only if $\mathfrak{m}I^2 = \mathfrak{m}JI$ if and only if the Hilbert series of $F(I)$ is

$$HS(F(I), t) = \frac{1 + (\mu(I) - d)t + t^2 + \cdots + t^r}{(1 - t)^d}.$$

In Section 4, we study the Gorenstein property of the Cohen–Macaulay fiber cones. For this purpose we use Macaulay's theorem about symmetry of the h-vector in the

Hilbert series of $F(I)$ and certain bilinear forms. It is fairly easy to show that for ideals of reduction number 1, $F(I)$ is Gorenstein if and only if $\mu(I) = d + 1$. We show that if $r(I) = 2$, then $F(I)$ is Gorenstein if and only if

$$(I^2\mathfrak{m} + JI : I) \cap I = \mathfrak{m}I + J \quad \text{and} \quad \ell(I^2/JI + \mathfrak{m}I^2) = 1.$$

For Sally ideals of reduction number at least 3, we show that $F(I)$ is Gorenstein if and only if $\mu(I) = d + 1$.

In Section 5, we characterize the Gorenstein property of fiber cones of ideals of almost minimal multiplicity when $G(I)$ is Cohen–Macaulay. We show that for such ideals $F(I)$ is Gorenstein if and only if $I \cap (\mathfrak{m}J : I) = \mathfrak{m}I + J$ and when this is the case, $\mu(I) \leq \mu(\mathfrak{m}) + d$.

In Section 6, we illustrate our results with a few examples.

2 Multiplicity Formulas for Fiber Cones

Throughout this section $(R, \mathfrak{m})$ will denote a local ring. In this section we derive two formulas for the multiplicity of the fiber cone of an $\mathfrak{m}$-primary ideal I in R in terms of the mixed multiplicity $e_{d-1}(m|I)$ and the multiplicity $e(I)$. These formulas are in terms of superficial sequences for a set of ideals in the sense of Rees. We begin with a discussion of superficial sequences and their relevance to joint reductions and hence mixed multiplicities. As we need these for $\mathfrak{m}$-primary ideals, we restrict our discussion to only such ideals. We begin by recalling for the reader's convenience the following definitions and results from I. Swanson's thesis [27].

Definition 2.1 ([27, Definition 1.14]) Let $(R, \mathfrak{m})$ be a local ring. Let $I_1, I_2, \dots, I_r$ be $\mathfrak{m}$-primary R-ideals. An element $a \in I_1$ is called *superficial* for the ideals $I_1, I_2, \dots, I_r$ if $\dim(R/(a)) = \dim(R) - 1$ and for some nonnegative integer c and for all $n_1 > c, n_2, \dots, n_r \geq 0$,

$$(I_1^{n_1} I_2^{n_2} \cdots I_r^{n_r} : a) \cap I_1^c I_2^{n_2} \cdots I_r^{n_r} = I_1^{n_1-1} I_2^{n_2} \cdots I_r^{n_r}.$$

Definition 2.2 ([27, p. 20]) A sequence $a_1, a_2, \dots, a_r$ of elements in R is called a *superficial sequence* for the ideals $I_1, I_2, \dots, I_r$ if $a_i \in I_i$ and the image of a_i in $R_{i-1} = R/(a_1, a_2, \dots, a_{i-1})$ is superficial for the images of the ideals $I_i, I_{i+1}, \dots, I_r$ in R_{i-1} for $i = 1, 2, \dots, r$.

Theorem 2.3 ([27, Theorem 1.16]) *Let $(R, \mathfrak{m})$ be of positive dimension d with $R/\mathfrak{m}$ infinite. Then superficial elements exist. Moreover, if $\mathbf{a} = a_1, a_2, \dots, a_d$ is a superficial sequence for the $\mathfrak{m}$-primary ideals $\mathbf{I} = I_1, I_2, \dots, I_d$, then $\mathbf{a}$ is a joint reduction of $\mathbf{I}$.*

Inspired by Rees' construction of joint reductions in his fundamental paper [19] on joint reductions and mixed multiplicities, we introduce the following:

Definition 2.4 An element $a \in I_1$ is called *Rees-superficial* for the $\mathfrak{m}$-primary ideals $I_1, I_2, \dots, I_r$ if for all large n_1 and all nonnegative integers $n_2, n_3, \dots, n_r$,

$$(a) \cap I_1^{n_1} I_2^{n_2} \cdots I_r^{n_r} = (a) I_1^{n_1-1} I_2^{n_2} \cdots I_r^{n_r}.$$

Definition 2.5 A sequence $a_1, a_2, \dots, a_r$ is called *Rees-superficial* for the ideals $I_1, I_2, \dots, I_d$, if the image of a_i in $R_{i-1} = R/(a_1, a_2, \dots, a_{i-1})$ is Rees-superficial for the images of $I_i, \dots, I_r$ in R_{i-1} for $i = 1, \dots, r$.

Lemma 2.6 (Rees' Basic Lemma [19, Lemma 1.2]) *Let $I_1, I_2, \dots, I_r$ be ideals of R where $R/\mathfrak{m}$ is infinite. Let $\mathcal{P}$ be a finite set of prime ideals of R so that no prime ideal in $\mathcal{P}$ contains the product $I_1 I_2 \cdots I_r$. Then there exists a Rees-superficial element $a \in I_1$ for the ideals $I_1, I_2, \dots, I_r$ so that a is not in any of the prime ideals in $\mathcal{P}$.*

Remark 2.7 It is clear that a nonzerodivisor in $I_1 \setminus I_1^2$ that is Rees-superficial for a set of ideals $(I_1, I_2, \dots, I_r)$ is also superficial. Moreover in a Cohen–Macaulay local ring with infinite residue field, maximal Rees-superficial sequences that are also regular sequences exist for a set of $\mathfrak{m}$-primary ideals, by Rees's basic lemma.

For a function $f \colon \mathbb{Z} \to \mathbb{N}$, put $\triangle f(n) = f(n) - f(n-1)$.

Proposition 2.8 *Let $(R, \mathfrak{m})$ be a local ring and I an $\mathfrak{m}$-primary ideal. Let a be a nonzerodivisor in R which is Rees-superficial for I and $\mathfrak{m}$. Let "$^-$" denote residue classes in $\overline{R} = R/aR$. Then for large n, $HF(F(\overline{I}), n) = \triangle\, HF(F(I), n)$.*

Proof We have the exact sequence

$$O \longrightarrow K_n \longrightarrow I^n/\mathfrak{m}I^n \xrightarrow{\mu_a} I^{n+1}/\mathfrak{m}I^{n+1} \longrightarrow C_n \longrightarrow 0,$$

where $\mu_a(x + \mathfrak{m}I^n) = ax + \mathfrak{m}I^{n+1}$, $K_n = (\mathfrak{m}I^{n+1} : a) \cap I^n/\mathfrak{m}I^n$ and $C_n = I^{n+1}/(aI^n + \mathfrak{m}I^{n+1})$. Since a is Rees-superficial for $\mathfrak{m}$ and I, $K_n = 0$ for all large n. Hence for large n, $\triangle\, HF(F(I), n+1) = \mu(I^{n+1}) - \mu(I^n) = \ell(C_n)$. For all large n,

$$\begin{aligned} HF(F(\bar{I}), n+1) &= \ell(I^{n+1} + aR/(\mathfrak{m}I^{n+1} + aR)) \\ &= \ell(I^{n+1}/(\mathfrak{m}I^{n+1} + aR \cap I^{n+1})) \\ &= \ell(I^{n+1}/(\mathfrak{m}I^{n+1} + aI^n)) \\ &= \mu(I^{n+1}) - \mu(I^n) \\ &= \triangle\, HF(F(I), n+1). \end{aligned}$$

■

Theorem 2.9 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of positive dimension d. Let I be an $\mathfrak{m}$-primary ideal.*

(i) *Let $a_1, a_2, \dots, a_{d-1} \in I, x \in \mathfrak{m}$ be a regular sequence in R which is a Rees-superficial sequence for the multiset $(I^{[d-1]} | \mathfrak{m}^{[1]})$. Then*

$$f_0(I) = e_{d-1}(\mathfrak{m}|I) - \lim_{n\to\infty} \ell\Big(\frac{\mathfrak{m}I^n}{xI^n + (a_1, a_2, \dots, a_{d-1})\mathfrak{m}I^{n-1}} \Big).$$

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(ii) *If $a_1, a_2, \dots, a_d \in I$ is a regular sequence in R which is a Rees-superficial sequence for the multiset $(I^{[d-1]}|\mathfrak{m}^{[1]})$. Then*

$$f_0(I) = e(I) - \lim_{n\to\infty} \ell\Big(\frac{\mathfrak{m}I^n}{a_dI^n + (a_1, a_2, \dots, a_{d-1})\mathfrak{m}I^{n-1}}\Big).$$

Proof (i) We induct on d. Let $d = 1$. We need to prove that

$$f_0(I) = e(\mathfrak{m}) - \lim_{n\to\infty} \ell\Big(\frac{\mathfrak{m}I^n}{xI^n}\Big).$$

For all $n \in \mathbb{N}$, $\ell(R/xR) + \ell(xR/xI^n) = \ell(R/\mathfrak{m}I^n) + \ell(\mathfrak{m}I^n/xI^n)$. Since $x \in \mathfrak{m}$ is superficial for $\mathfrak{m}$, xR is a minimal reduction of $\mathfrak{m}$. Therefore we get $\mu(I^n) = e(\mathfrak{m}) - \ell(\mathfrak{m}I^n/xI^n)$. Hence by taking limits we get the desired formula.

Now suppose $d = 2$. Let (a, x) be a regular sequence which is a Rees-superficial sequence for $I, \mathfrak{m}$. Then (a, x) is a joint reduction of the set $(I, \mathfrak{m})$, by Theorem 2.3. And $e_1(\mathfrak{m}|I) = \ell(R/(a, x))$ by [19, Theorem 2.4(ii)]. By the proof of [7, Lemma 4.2], we have for all $n \geq 1$,

$$\triangle\, HF(F(I), n) = e_1(\mathfrak{m}|I) - \ell\Big(\frac{\mathfrak{m}I^n}{xI^n + a\mathfrak{m}I^{n-1}}\Big) + \ell\Big(\frac{(\mathfrak{m}I^{n-1} : x) \cap (I^n : a)}{I^{n-1}}\Big).$$

Since (a, x) is superficial for $(I, \mathfrak{m})$, a is superficial for I and is regular in R, we have, for large n, $I^n : a = I^{n-1}$. Since $HP(F(I), n)$ is a degree one polynomial, $\triangle\, HF(F(I), n) = f_0(I)$ for large n. This establishes the formula for $d = 2$.

Now suppose $d \geq 3$. Put $\bar{R} = R/(a_1)$ and $L = (a_2, a_3, \dots, a_{d-1})$. By induction hypothesis and Proposition 2.8,

$$\begin{aligned}
f_0(I) &= f_0(\bar{I}) \\
&= e_{d-2}(\bar{\mathfrak{m}}|\bar{I}) - \lim_{n\to\infty} \ell\Big(\frac{\mathfrak{m}I^n + a_1R}{xI^n + L\mathfrak{m}I^{n-1} + a_1R}\Big) \\
&= e_{d-1}(\mathfrak{m}|I) - \lim_{n\to\infty} \ell\Big(\frac{\mathfrak{m}I^n + a_1R}{xI^n + L\mathfrak{m}I^{n-1} + a_1R}\Big) \\
&= e_{d-1}(\mathfrak{m}|I) - \lim_{n\to\infty} \ell\Big(\frac{\mathfrak{m}I^n}{xI^n + L\mathfrak{m}I^{n-1} + \mathfrak{m}I^n \cap a_1R}\Big) \\
&= e_{d-1}(\mathfrak{m}|I) - \lim_{n\to\infty} \ell\Big(\frac{\mathfrak{m}I^n}{xI^n + (a_1, \dots, a_{d-1})\mathfrak{m}I^{n-1}}\Big).
\end{aligned}$$

In the above equations we have used the fact that if a_1 is superficial for $\mathfrak{m}$ and I, then $e_{d-2}(\bar{\mathfrak{m}}|\bar{I}) = e_{d-1}(\mathfrak{m}|I)$ by [15, p. 118, line 3]. This establishes the formula.

(ii) Replace x by a_d in the above argument. ■

We now obtain a sufficient condition for $f_0(I) = e_{d-1}(\mathfrak{m}|I)$.

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Theorem 2.10 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and I an $\mathfrak{m}$-primary ideal. If $BF(r, s) = \ell(R/\mathfrak{m}^r I^s) = BP(r, s)$ for all $r, s \geq 0$, then*

$$HS(F(I), t) = \frac{\sum_{j=0}^{d-1} (1-t)^{d-j-1} g(j)}{(1-t)^d},$$

where $g(j) = \sum_{i=1}^{d-j} i e_{(i,j)}$. In particular $f_0(I) = e_{d-1}(\mathfrak{m}|I)$.

Proof For convenience, write $e_{(i,j)} = e(i, j)$. Then

$$\begin{aligned}
\mu(I^s) &= \ell(R/\mathfrak{m}I^s) - \ell(R/I^s) \\
&= \sum_{i+j \leq d} e(i, j) \binom{1+i}{i} \binom{s+j}{j} - \sum_{i+j \leq d} e(i, j) \binom{i}{i} \binom{s+j}{j} \\
&= \sum_{i+j \leq d} ie(i, j) \binom{s+j}{j} \\
&= \sum_{j=0}^{d-1} \Big[\sum_{i=1}^{d-j} ie(i, j) \Big] \binom{s+j}{j} \\
&= \sum_{j=0}^{d-1} g(j) \binom{s+j}{j}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
HS(F(I), t) &= \sum_{s \geq 0} \mu(I^s) t^s = \sum_{s \geq 0} \Big[\sum_{j=0}^{d-1} g(j) \binom{s+j}{j} \Big] t^s \\
&= \sum_{j=0}^{d-1} g(j) \Big[\sum_{s \geq 0} \binom{s+j}{j} t^s \Big] \\
&= \sum_{j=0}^{d-1} \frac{g(j)}{(1-t)^{j+1}} \\
&= \frac{\sum_{j=0}^{d-1} (1-t)^{d-j-1} g(j)}{(1-t)^d}.
\end{aligned}$$

Now put $t = 1$ in the numerator of $HS(F(I), t)$ to get $f_0(I) = e_{d-1}(\mathfrak{m}|I)$. ∎

The next result was communicated to us by E. Hyry.

Corollary 2.11 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring. Let the multi-Rees algebra $\mathcal{R} := R[\mathfrak{m}t_1, It_2]$ be Cohen–Macaulay. Then $f_0(I) = e_{d-1}(\mathfrak{m}|I)$. If $d = 2$, then R and $F(I)$ are Cohen–Macaulay with minimal multiplicity.*

Proof If $\mathcal{R}$ is Cohen–Macaulay, then by [12, proof of Theorem 6.1], $\ell(R/\mathfrak{m}^r I^s) = BP(r,s)$ for all $r, s \geq 0$. Therefore $f_0(I) = e_{d-1}(\mathfrak{m}|I)$, by Theorem 2.10. When $d = 2$ and $\mathcal{R}$ is Cohen–Macaulay, then, by [11, Corollary 3.5], $R[\mathfrak{m}t_1]$ and $R[It_2]$ are Cohen–Macaulay and hence $r(I) \leq 1$ and $r(\mathfrak{m}) \leq 1$ by [10, Remark 3.10]. Thus R is Cohen–Macaulay with minimal multiplicity. Since $r(I) \leq 1$, by [25, Theorems 1 and 7] $F(I)$ is Cohen–Macaulay and $f_0(I) = \mu(I) - 1$. Hence $F(I)$ has minimal multiplicity. ∎

3 Cohen–Macaulay Fiber Cones

In this section we use the multiplicity formula for fiber cones to detect their Cohen–Macaulay property. We begin by recovering Corollary 2.5 of [6] in a simpler way.

Proposition 3.1 *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen–Macaulay local ring and I an $\mathfrak{m}$-primary ideal of minimal mixed multiplicity. Then $F(I)$ is Cohen–Macaulay if and only if $r(I) \leq 1$.*

Proof Let J be any minimal reduction of I. Then $F(I)$ is Cohen–Macaulay if and only if $f_0(I) = \ell(F(I)/JF(I))$. Since

$$\frac{F(I)}{JF(I)} = \frac{R}{\mathfrak{m}} \oplus \frac{I}{J + \mathfrak{m}I} \oplus \Big(\bigoplus_{n=2}^{\infty} \frac{I^n}{JI^{n-1} + \mathfrak{m}I^n}\Big),$$

and $\ell(I/(J + \mathfrak{m}I)) = \mu(I) - d$, we have

$$\ell(F(I)/JF(I))) = 1 + \mu(I) - d + \sum_{n=2}^{\infty} \ell\Big(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}}\Big).$$

Thus $F(I)$ is Cohen–Macaulay if and only if for any Rees-superficial sequence $a_1, a_2, \dots, a_{d-1}, x$, where $a_1, a_2, \dots, a_{d-1} \in I, x \in \mathfrak{m}$

$$\begin{aligned} f_0(I) &= e_{d-1}(\mathfrak{m}|I) - \lim_{n\to\infty} \ell\Big(\frac{\mathfrak{m}I^n}{xI^n + (a_1, a_2, \dots, a_{d-1})\mathfrak{m}I^{n-1}}\Big) \\ &= \mu(I) - d + 1 + \sum_{n=2}^{\infty} \ell\Big(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}}\Big). \end{aligned}$$

By the proof of [6, Proposition 2.4], I has minimal mixed multiplicity if and only if $I^n\mathfrak{m} = xI^n + (a_1, a_2, \dots, a_{d-1})\mathfrak{m}I^{n-1}$ for all $n \geq 1$. Thus $F(I)$ is Cohen–Macaulay if and only if $I^2 = JI$. ∎

In the next result we improve Corollary 1.4 of [7] by removing the hypothesis of almost maximal depth for the associated graded ring of I.

Proposition 3.2 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal with almost minimal mixed multiplicity. Then*

(i) *Either $f_0(I) = e_{d-1}(\mathfrak{m}|I)$ or $f_0(I) = e_{d-1}(\mathfrak{m}|I) - 1$.*
(ii) *Let $f_0(I) = e_{d-1}(\mathfrak{m}|I)$. Then $F(I)$ is Cohen–Macaulay if and only if $r(I) = 2$ and $\ell(I^2/(JI + \mathfrak{m}I^2)) = 1$.*
(iii) *Let $f_0(I) = e_{d-1}(\mathfrak{m}|I) - 1$. Then $F(I)$ is Cohen–Macaulay if and only if $r(I) \le 1$.*

Proof (i) Let $a_1, a_2, \dots, a_{d-1} \in I, x \in \mathfrak{m}$ be a Rees-superficial sequence for I and $\mathfrak{m}$. Put $L = (a_1, a_2, \dots, a_{d-1})$ and $\alpha = \lim_{n\to\infty} \ell(\mathfrak{m}I^n/(xI^n + L\mathfrak{m}I^{n-1}))$. Since I has almost minimal mixed multiplicity, $\ell(\mathfrak{m}I^n/(xI^n + L\mathfrak{m}I^{n-1})) \le 1$ for all n by [7, Lemma 2.2]. Hence $\alpha = 0$ or 1. This proves (i).

(ii) By the computations in the above result, $F(I)$ is Cohen–Macaulay if and only if

$$e_{d-1}(\mathfrak{m}|I) - \alpha = \mu(I) - d + 1 + \sum_{n=2}^{\infty} \ell\Big(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}}\Big),$$

if and only if

$$1 - \alpha = \sum_{n=2}^{\infty} \ell\Big(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}}\Big).$$

Let $f_0(I) = e_{d-1}(\mathfrak{m}|I)$. Then $\alpha = 0$. Thus $F(I)$ is Cohen–Macaulay if and only if

$$\sum_{n=2}^{\infty} \ell(I^n/(\mathfrak{m}I^n + JI^{n-1})) = 1,$$

if and only if $r(I) = 2$ and $\ell(I^2/(JI + \mathfrak{m}I^2)) = 1$.

(iii) Let $f_0(I) = e_{d-1}(\mathfrak{m}|I) - 1$. Hence $\alpha = 1$. Thus $F(I)$ is Cohen–Macaulay if and only if

$$\sum_{n=2}^{\infty} \ell(I^n/\mathfrak{m}I^n + JI^{n-1}) = 0.$$

This holds if and only if $I^2 = JI$. ∎

In a personal communication, G. Valla raised a question regarding the Cohen–Macaulay property of fiber cones of Sally ideals. In Example 6.1 we show that $F(I)$ need not be Cohen–Macaulay, even if $G(I)$ is Cohen–Macaulay. First we characterize Cohen–Macaulay fiber cones of Sally ideals in dimension one.

Theorem 3.3 *Let $(R, \mathfrak{m})$ be a 1-dimensional Cohen–Macaulay local ring, I a Sally ideal and $J = (x)$ a minimal reduction of I, with reduction number r. Then the following are equivalent:*

(i) *$F(I)$ is Cohen–Macaulay;*
(ii) *$HS(F(I), t) = (1 + (\mu(I) - 1)t + t^2 + t^3 + \cdots + t^r)/(1 - t)$;*
(iii) *$\mu(I^k) = \mu(I) + k - 1$, for $2 \le k \le r$;*
(iv) *$\mu(I^2) = \mu(I) + 1$;*
(v) *$\mathfrak{m}I^2 = \mathfrak{m}JI$.*

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Proof (i) $\Rightarrow$ (ii). Let $F(I)$ be Cohen–Macaulay. Then by [6, Theorem 2.1],

$$HS(F(I),t) = \frac{1 + (\mu(I)-1)t + \sum_{i=2}^{r} \ell(I^i/(JI^{i-1} + \mathfrak{m}I^i)t^i}{(1-t)}.$$

Since I is a Sally ideal $\mathfrak{m}I^n \subset JI^{n-1}$ for all $n \geq 2$ and $\ell(I^n/JI^{n-1}) = 1$ for all $n = 2, 3, \dots, r$. Hence (ii) follows.

(ii) $\Rightarrow$ (iii). From the formula for the Hilbert series of $F(I)$, we obtain the equation $\mu(I^k) = \mu(I) + k - 1$ for $k = 1, 2, \dots, r$.

(iii) $\Rightarrow$ (iv). Put $k = 2$.

(iv) $\Rightarrow$ (v). For $n \geq 1$ we have following exact sequence.

$$(1) \qquad 0 \longrightarrow \frac{(\mathfrak{m}I^{n+1} : x) \cap I^n}{\mathfrak{m}I^n} \longrightarrow \frac{I^n}{\mathfrak{m}I^n} \xrightarrow{\phi_x} \frac{I^{n+1}}{\mathfrak{m}I^{n+1}} \longrightarrow \frac{I^{n+1}}{xI^n} \longrightarrow 0,$$

and the isomorphism:

$$(2) \qquad \frac{(\mathfrak{m}I^{n+1} : x) \cap I^n}{\mathfrak{m}I^n} \cong \frac{xI^n \cap \mathfrak{m}I^{n+1}}{x\mathfrak{m}I^n}.$$

Assume $\mu(I^2) = \mu(I) + 1$. Then, from the exact sequence (1) and the isomorphism (2) for $n = 1$, we get $\mathfrak{m}I^2 = x\mathfrak{m}I$.

(v) $\Rightarrow$ (i). Consider the function $H_{\mathfrak{m}}(I,n) := \ell(R/\mathfrak{m}I^n)$ and write the corresponding polynomial as:

$$P_{\mathfrak{m}}(I,n) = \sum_{i=0}^{d} (-1)^i g_i(I) \binom{n+d-i-1}{d}.$$

Then by [13, Theorem 5.3],

$$g_1(I) = \sum_{n \geq 1} \ell(\mathfrak{m}I^n / x\mathfrak{m}I^{n-1}) - 1.$$

Since $\mathfrak{m}I^2 = x\mathfrak{m}I$ we get

$$(3) \qquad g_1(I) = \ell(\mathfrak{m}I/x\mathfrak{m}) - 1.$$

We know $F(I)$ is Cohen–Macaulay if and only if

$$g_1(I) = \sum_{n \geq 1} \ell(\mathfrak{m}I^n + xI^{n-1}/xI^{n-1}) - 1$$

by [13, Theorem 4.3]. Since $\ell(I^2/xI) = 1$, $\mathfrak{m}I^n \subset xI^{n-1}$ for all $n \geq 2$. Therefore, by (3),

$$\sum_{n \geq 1} \ell(\mathfrak{m}I^n + xI^{n-1}/xI^{n-1}) - 1 = \ell(\mathfrak{m}I + xR/xR) - 1 = \ell(\mathfrak{m}I/x\mathfrak{m}) - 1 = g_1(I).$$

Hence $F(I)$ is Cohen–Macaulay. ∎

Now we characterize the Cohen–Macaulayness of $F(I)$ in higher dimensions.

Theorem 3.4 *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d \geq 1$, I *a Sally ideal with a minimal reduction* J*. Then the following are equivalent:*

(i) $F(I)$ *is Cohen–Macaulay.*

(ii) $\mathfrak{m}I^2 = \mathfrak{m}JI$.

(iii) *The Hilbert series of* $F(I)$ *is given by*

$$HS(F(I), t) = \frac{1 + (\mu(I) - d)t + t^2 + \cdots + t^r}{(1-t)^d}.$$

(iv) $f_0(I) = \mu(I) - d + r$.

Proof We apply induction on d. We have proved the theorem for $d = 1$. Now let $d \geq 2$.

(i) $\Rightarrow$ (ii). Since I is a Sally ideal, by [20, Corollary 1.7], depth $G(I) \geq d - 1$. Hence we can choose an $x \in J$ such that x^* is regular in $G(I)$ and x^o is regular in $F(I)$. Then $F(I/(x)) \cong F(I)/(x^o)$ is Cohen–Macaulay. By induction, $\bar{\mathfrak{m}}\bar{I}^2 = \bar{\mathfrak{m}}\bar{J}\bar{I}$. Therefore $\mathfrak{m}I^2 = \mathfrak{m}JI + (x) \cap \mathfrak{m}I^2$. Since x^* is regular in $G(I)$ and x^o regular in $F(I)$, $(x) \cap \mathfrak{m}I^2 = x\mathfrak{m}I$ by [5, Theorem 1.1]. Therefore $\mathfrak{m}I^2 = \mathfrak{m}JI$.

(ii) $\Rightarrow$ (i). For $x \in J$, such that x^* is regular in $G(I)$ and x^o superficial for $F(I)$, let "$-$" (overbar) denote "modulo (x)". Then $\overline{\mathfrak{m}}\bar{I}^2 = \overline{\mathfrak{m}}\bar{J}\bar{I}$. By induction, $F(\bar{I})$ is Cohen–Macaulay. By "Sally machine" [13, Lemma 2.7], x^o is regular in $F(I)$ and hence $F(I)$ is Cohen–Macaulay.

(i) $\Rightarrow$ (iii). Since $F(I)$ is Cohen–Macaulay,

$$HS(F(I), t) = \frac{HS(F(I)/JF(I), t)}{(1-t)^d}.$$

Since $\mathfrak{m}I^2 \subset JI$, we have $\ell(I^n/\mathfrak{m}I^n + JI^{n-1}) = \ell(I^n/JI^{n-1}) = 1$ for all $n = 2, \ldots, r$. Therefore the Hilbert series of $F(I)$ is

$$HS(F(I), t) = \frac{1 + (\mu(I) - d)t + t^2 + \cdots + t^r}{(1-t)^d}.$$

(iii) $\Rightarrow$ (iv). The assertion follows directly from the fact that if $HS(F(I), t) = h(t)/(1-t)^d$, then $f_0(I) = h(1)$.

(iv) $\Rightarrow$ (i). Since $f_0(I) = \mu(I) - d + r$, we have

$$\begin{aligned} 1 + \sum_{n=1}^{r} \ell\Big(\frac{I^n}{JI^{n-1} + \mathfrak{m}I^n}\Big) &= 1 + \ell(I/\mathfrak{m}I + J) + \sum_{n=2}^{r} \ell(I^n/JI^{n-1}) \\ &= 1 + \mu(I) - \ell(\mathfrak{m}I + J/\mathfrak{m}I) + r - 1 \\ &= \mu(I) - \ell(J/\mathfrak{m}J) + r \\ &= \mu(I) - d + r \\ &= f_0(I). \end{aligned}$$

Therefore, by [6, Theorem 2.1], $F(I)$ is Cohen–Macaulay. ■

4 Gorenstein Fiber Cones

Throughout this section and the next we will assume, unless otherwise stated, $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension d with infinite residue field, I is an $\mathfrak{m}$-primary ideal and $F(I)$ is Cohen–Macaulay.

In this section we study the Gorenstein property of $F(I)$ for several classes of ideals. We do this by keeping reduction numbers in mind. It is clear that if $r(I) = 0$, then $F(I)$ is a polynomial ring. Thus we may begin with the case when $r(I) = 1$. In this case $F(I)$ is Cohen–Macaulay [25, Theorem 1].

Proposition 4.1 *Assume $r(I) = 1$. If $F(I)$ is Gorenstein, then $\mu(I) = d + 1$.*

Proof The Hilbert series of $F(I)$ is $(1 + (\mu(I) - d)t)/(1 - t)^d$ by [6, Theorem 2.1]. By a theorem of Macaulay [26, Theorem 4.1], the h-vector of a standard Gorenstein graded k-algebra, where k is a field, is symmetric. Hence $\mu(I) - d = 1$. ■

Remark 4.2 The symmetry of the h-vector does not imply that the fiber cone is Gorenstein even when it is Cohen–Macaulay; see Example 6.2.

In general we have the following:

Proposition 4.3 *If $\mu(I) = d + 1$, then $F(I)$ is a hypersurface.*

Proof Set $I = (u_1, \dots, u_{d+1})$. Consider the map $\phi\colon k[X_1, \dots, X_{d+1}] \to F(I)$ given by $\phi(X_i) = u_i + \mathfrak{m}I$ for $i = 1, \dots, d + 1$. Clearly ϕ is surjective and $\ker(\phi)$ is a height one ideal of $S = k[X_1, \dots, X_{d+1}]$. Since $S/\ker(\phi) \cong F(I)$ is Cohen–Macaulay, $\ker(\phi)$ is a height one unmixed ideal. Since S is a UFD, I is principal. Therefore $F(I)$ is a hypersurface ring. ■

Remark 4.4 One of the surprising results in our investigations has been the following. If $F(I)$ is Gorenstein, then $\mu(I)$ is forced. When $r(I) = 1$, this is done in Proposition 4.1. When $r(I) = 2$ and I has almost minimal multiplicity, we get an upper bound on $\mu(I)$; see Corollary 5.8.

Let $J = (x_1, \dots, x_d)$ be a minimal reduction of I. Since $F(I)$ is Cohen–Macaulay, it easily follows that the reduction number of I with respect to J is the degree of the h-polynomial of $F(I)$. We will use this fact implicitly in all subsequent discussions.

Proposition 4.5 *Set $r = r(I)$, the reduction number of I and let J be a minimal reduction of I. Then*

$$\operatorname{socle} F(I)/JF(I) \cong \bigoplus_{n=1}^{r-1} \frac{(I^{n+1}\mathfrak{m} + JI^n \colon I) \cap I^n}{(I^n\mathfrak{m} + JI^{n-1})} \oplus \frac{I^r}{\mathfrak{m}I^r + JI^{r-1}}.$$

Proof Let $J = (x_1, \dots, x_d)$ be a minimal reduction of I. Since $x_1^\circ, \dots, x_d^\circ$ is a regular sequence for $F(I)$, we have

$$\operatorname{socle} F(I)/JF(I) \cong \operatorname{socle} F(I)/(x_1^\circ, \dots, x_d^\circ)F(I).$$

Since $S := F(I)/JF(I)$ is a standard graded k-algebra with $k = S_0$, a field, we have $\operatorname{socle} S = (0 :_S S_+) = (0 :_S S_1)$, where $S_+ = \bigoplus_{n \geq 1} S_n$. Notice that $S_1 = I/(\mathfrak{m}I + J)$. An easy computation yields the result. ∎

If $r(I) = 2$, then we have the following ideal-theoretic condition to check the Gorenstein property of $F(I)$.

Corollary 4.6 *Let $r(I) = 2$ and let J be a minimal reduction of I. Then $F(I)$ is Gorenstein if and only if*

$$(I^2\mathfrak{m} + JI : I) \cap I = \mathfrak{m}I + J \quad \textit{and} \quad \ell\Big(\frac{I^2}{\mathfrak{m}I^2 + JI}\Big) = 1.$$

By using Proposition 3.2, we get that the fiber cone $F(I)$ of an ideal I, with $r(I) = 2$ having almost minimal mixed multiplicity and $f_0(I) = e_{d-1}(\mathfrak{m}|I)$, is Gorenstein if and only if $(I^2\mathfrak{m} + JI : I) \cap I = \mathfrak{m}I + J$. If $r(I) \geq 3$ and $F(I)$ is Gorenstein, then the symmetry of h-vector yields the following.

Proposition 4.7 *Let $r = r(I) \geq 3$ and J be a minimal reduction of I. If $F(I)$ is Gorenstein, then*

$$\mu(I) = d + \ell\Big(\frac{I^{r-1}}{\mathfrak{m}I^{r-1} + JI^{r-2}}\Big).$$

Proof Note that $S := F(I)/JF(I)$ is a standard graded Gorenstein ring of dimension zero and

$$S_{r-1} = \frac{I^{r-1}}{\mathfrak{m}I^{r-1} + JI^{r-2}} \quad \text{and} \quad S_1 = \frac{I}{\mathfrak{m}I + J}.$$

If $F(I)$ is Gorenstein, then the h-vector of $F(I)$ is symmetric. Since $r \geq 3$ we have $\ell(S_1) = \ell(S_{r-1})$. Since $\ell(S_1) = \mu(I) - d$ we get the result. ∎

As an easy consequence we have:

Corollary 4.8 *Let I be a Sally ideal with $r(I) \geq 3$. If $F(I)$ is Gorenstein, then $\mu(I) = d + 1$.*

Proof Let J be a reduction of I. Then $I^2 \neq JI$. We have $\ell(I^{n+1}/JI^n) \leq 1$ for all $n \geq 1$. Notice that $\mathfrak{m}I^{n+1} \subseteq JI^n$ for all $n \geq 1$. Therefore we have

$$\frac{I^{r-1}}{\mathfrak{m}I^{r-1} + JI^{r-2}} = \frac{I^{r-1}}{JI^{r-2}}.$$

Also $\ell(I^{r-1}/JI^{r-2}) = 1$. The result follows from Proposition 4.7. ∎

The result above does not hold if $r(I) \leq 2$. Consider the following example discussed in [23]. Let $e > 3$ be a positive integer. Set $R = k[[t^e, t^{e+1}, \dots, t^{2e-2}]]$, where k is a field. Since the numerical semigroup generated by $\{e, e+1, \dots, 2e-2\}$ is symmetric with conductor $2e$, R is Gorenstein. Let $\mathfrak{m}$ denote the maximal ideal of R. Then $\mu(\mathfrak{m}) = e(R) + d - 2 = e - 1 > d + 1 = 2$, where $e(R)$ denotes the multiplicity of R. By the proof of [23, Theorem 3.4], $\mathfrak{m}^3 = J\mathfrak{m}^2$ for any minimal reduction J of $\mathfrak{m}$, and it follows from Theorem 3.4 that $G(\mathfrak{m}) = F(\mathfrak{m})$ is Gorenstein.

5 Gorenstein Fiber Cones of Ideals of Almost Minimal Multiplicity

In this section we consider the Gorenstein property of fiber cones of ideals of almost minimal multiplicity. Recall that an $\mathfrak{m}$-primary ideal I in a Cohen–Macaulay local ring $(R, \mathfrak{m})$ is said to have minimal multiplicity (resp., almost minimal multiplicity) if for any minimal reduction J of I, $\mathfrak{m}I = \mathfrak{m}J$ (resp., $\ell(I\mathfrak{m}/J\mathfrak{m}) = 1$). Such ideals have been studied in [8, 9, 14].

In addition to the hypotheses stated in the beginning of the previous section we further assume $G(I)$ is Cohen–Macaulay. Since $I^3 \subseteq J$, we get by the Valabrega–Valla criterion that $I^3 = JI^2$. So $r(I) \leq 2$. Since we have already considered the case $r(I) = 1$, we assume $r(I) = 2$.

Let J be a minimal reduction of I. Set $J = (x_1, \dots, x_d)$. If $G(I)$ is Cohen–Macaulay, then $x_1^*, \dots, x_d^*$ is a $G(I)$-regular sequence. Since $F(I)$ is Cohen–Macaulay, we also have that $x_1^\circ, \dots, x_d^\circ$ is an $F(I)$-regular sequence.

Notation 5.1 Set $(B, \mathfrak{n}) = (A/J, \mathfrak{m}/J)$, $K = I/J$. We have

$$\frac{F(I)}{(x_1^\circ, \dots, x_d^\circ)F(I)} \cong F(K).$$

It follows that $F(I)$ is Gorenstein if and only if $F(K)$ is Gorenstein.

Notice that

(i) $\mathfrak{n}K \cong k$.
(ii) $\mathfrak{n}^2K = 0$ and so $K^3 = 0$ and $\mathfrak{n}K^2 = 0$.
(iii) $0 \neq K^2 \subseteq \mathfrak{n}K$. So $K^2 = \mathfrak{n}K$.

Remark 5.2 If I has almost minimal multiplicity with $r(I) \geq 2$ and $I^2 \cap J = JI$, then I is a Sally ideal. To see this, note that $K^2 \cong I^2/J \cap I^2 = I^2/JI$ and from 5.1(i) and (iii), it follows that $\ell(I^2/JI) = 1$. In particular, if $G(I)$ is Cohen–Macaulay, then I is a Sally ideal.

Remark 5.3 Since $r(I) = 2$, symmetry of the h-vector of Hilbert series of $F(I)$ does not help us in estimating $\mu(I)$. To find conditions on $\mu(I)$, we need the following different criterion.

Proposition 5.4 *Assume that I has almost minimal multiplicity. Set $W = I \cap (\mathfrak{m}J : I)$. Then $F(I)$ is Gorenstein if and only if $W = \mathfrak{m}I + J$.*

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Proof Since $r(I) = 2$ we can use Corollary 4.6. We keep the notation as in 5.1. Note that $K^2/\mathfrak{n}K^2 = (I^2 + J)/(\mathfrak{m}I^2 + J)$. Then we have

$$\frac{I^2 + J}{\mathfrak{m}I^2 + J} \cong \frac{I^2}{(\mathfrak{m}I^2 + J) \cap I^2} = \frac{I^2}{\mathfrak{m}I^2 + JI}.$$

Thus $\ell(I^2/(\mathfrak{m}I^2 + JI)) = 1$. Set $E = (I^2\mathfrak{m} + JI \colon I) \cap I$. Using Corollary 4.6 we have that $F(I)$ is Gorenstein if and only if $E = \mathfrak{m}I + J$.

We now prove that $E = W$. Since $\ell(\mathfrak{m}I/\mathfrak{m}J) = 1$, we have that $\mathfrak{m}I^2 \subseteq J\mathfrak{m}$. It follows that $W \supseteq E$. Conversely, let $t \in W$. Then $tI \in \mathfrak{m}J$. In particular $tI \subseteq J \cap I^2 = JI$. So $t \in E$. Therefore $E = W$. ∎

Remark 5.5 The hypothesis $G(I)$ Cohen–Macaulay is essential in the Proposition above. See Example 6.4.

Definition 5.6 We define a bilinear form,

$$\phi\colon \frac{K}{\mathfrak{n}K} \times \frac{\mathfrak{n}}{\mathfrak{n}^2K \colon K} \longrightarrow \frac{\mathfrak{n}K}{\mathfrak{n}^2K}$$
$$(a + \mathfrak{n}K, b + \mathfrak{n}^2K \colon K) \mapsto ab + \mathfrak{n}^2K.$$

Notice that, with our hypothesis, $\mathfrak{n}^2K = 0$. It is straightforward to check that ϕ is well defined. It is also clear that ϕ is non-degenerate with respect to $\mathfrak{n}/(\mathfrak{n}^2K \colon K)$.

Lemma 5.7 *Assume that I has almost minimal multiplicity. If $W = \mathfrak{m}I + J$, then ϕ is non-degenerate with respect to $K/\mathfrak{n}K$.*

Proof Suppose there exists $a \in I$ (so $\overline{a} \in K$) such that

$$\phi\left(\overline{a} + \mathfrak{n}K, b + \mathfrak{n}^2K \colon K\right) = ab + \mathfrak{n}^2K = \overline{0} \quad \text{for each } b + \mathfrak{n}^2 \in \mathfrak{n}/\mathfrak{n}^2K : K.$$

Thus $\overline{a}\mathfrak{n} \subseteq \mathfrak{n}^2K = \{\overline{0}\}$. So $a\mathfrak{m} \subseteq J$. In particular $aI \subseteq J$. So $aI \subseteq J \cap I^2 \subseteq J\mathfrak{m}$. Thus $a \in (\mathfrak{m}J \colon I) \cap I$. Thus $a \in W = \mathfrak{m}I + J$. So $\overline{a} = 0$. Therefore, ϕ is non-degenerate with respect to $K/\mathfrak{n}K$. ∎

Corollary 5.8 *Assume that I has almost minimal multiplicity and let J be a minimal reduction of I. If $F(I)$ is Gorenstein, then $\ell(J : I/J) = \ell(R/I)$ and $\mu(I) \leq \mu(\mathfrak{m}) + d$.*

Proof Let $x_1, \dots, x_d$ be a superficial sequence in R with respect to I. Set $J = (x_1, \dots, x_d)$ and $(B, \mathfrak{n}) = (R/J, \mathfrak{m}/J)$ and $K = I/J$. Since $F(I)$ is Gorenstein, by Proposition 5.4, $W = \mathfrak{m}I + J$. Therefore, as in the Lemma 5.7, ϕ is non-degenerate both on the left and the right. It follows that $\ell(K/\mathfrak{n}K) = \ell(\mathfrak{n}/(0 : K))$ (notice that we used $\mathfrak{n}^2K = 0$). This immediately yields

$$\mu(I) - d = \mu(K) = \ell(\mathfrak{n}/(0 : K)) \leq \mu(\mathfrak{n}) \leq \mu(\mathfrak{m}).$$

Notice that $0 : K \cong (J : I)/J$. We have

$$\mu(I) - d = \mu(K) = \ell(\mathfrak{n}/(0 : K)) = \ell(\mathfrak{m}/(J : I)),$$

and

$$\ell(\mathfrak{m}/(J : I)) = \ell(\mathfrak{m}/J) - \ell(J : I/J) = e(I) - 1 - \ell(J : I/J).$$

So $\mu(I) = d + e(I) - 1 - \ell(J : I/J)$. Since I has almost minimal mixed multiplicity we have $\mu(I) = d + e(I) - 1 - \ell(R/I)$. So we get $\ell(J : I/J) = \ell(R/I)$. ∎

Remark 5.9 The corollary above has an interesting connection with the Koszul homologies $H_i(I, R)$ of I. Let $\mu(I) = n$ and let $J = (x_1, \dots, x_d)$ be a minimal reduction of I. Since R is Cohen–Macaulay and I is $\mathfrak{m}$-primary we have (see [2, 1.6.16, 1.6.17])

$$\begin{gathered} H_i(I,R) = 0 \quad \text{for } i > n - d. \\ H_{n-d}(I,R) \cong \frac{J : I}{J}, \quad H_0(I,R) = \frac{R}{I}. \end{gathered} \tag{4}$$

Thus (with the hypotheses as in Corollary 5.8), we get that the zeroth Koszul homology and the last non-vanishing Koszul homology of I have the same length. Note that this property is true if R is Gorenstein and I is any $\mathfrak{m}$-primary ideal. We have not been able to find an example where R is not Gorenstein, $G(I)$ is Cohen–Macaulay and $F(I)$ is Gorenstein and (4) is satisfied. However, we do not believe that if for an $\mathfrak{m}$-primary ideal I in a Cohen–Macaulay local ring R such that $G(I)$ is Cohen–Macaulay and $F(I)$ is Gorenstein, then R is Gorenstein.

6 Examples

We end this article by presenting a few examples to illustrate our results. Let k denote a field. The computations have been performed in CoCoA [4].

Example 6.1 Let $A = k[[t^6, t^{11}, t^{15}, t^{31}]]$, $I = (t^6, t^{11}, t^{31})$ and $J = (t^6)$. Then it can easily be verified that $\ell(I^2/JI) = 1$ and $I^3 = JI^2$. Since $I^2 \cap J = JI$, $G(I)$ is Cohen–Macaulay. It can also be seen that $t^{37} \in \mathfrak{m}I^2$, but $t^{37} \notin \mathfrak{m}JI$. Therefore $F(I)$ is not Cohen–Macaulay. This example shows that the length condition $\ell(I^2/JI) = 1$ by itself need not force the fiber cone to be Cohen–Macaulay, even if $G(I)$ is Cohen–Macaulay.

Example 6.2 Here we give an example of an ideal I such that $F(I)$ and $G(I)$ are Cohen–Macaulay, the numerator of the Hilbert series is symmetric, but $F(I)$ is not Gorenstein. Consider $A = k[[t^7, t^{15}, t^{17}, t^{33}]]$, $I = (t^7, t^{17}, t^{33})$ and $J = (t^7)$. Then $\ell(I^2/JI) = 1$, $I^3 = JI^2$, $I^2 \cap J = JI$ and $\mathfrak{m}I^2 = \mathfrak{m}JI$. Therefore, in this case $G(I)$ and $F(I)$ are Cohen–Macaulay. Hence, the Hilbert series of $F(I)$ is

$$HS(F(I), t) = \frac{1 + (\mu(I) - d)t + t^2}{(1 - t)} = \frac{1 + 2t + t^2}{(1 - t)}.$$

It can also be seen that $t^{33} \in (\mathfrak{m}I^2 + JI : I) \cap I$, but $t^{33} \notin J + \mathfrak{m}I$. Therefore, the numerator of the Hilbert series is symmetric, but $F(I)$ is not Gorenstein.

Example 6.3 Now we give an example of an ideal with a Gorenstein fiber cone which is not a hypersurface. Let $A = k[[t^4, t^5, t^6, t^7]]$, $I = (t^4, t^5, y^6)$ and $J = (t^4)$. Then $\ell(I^2/JI) = 1$, $I^3 = JI^2$ and $\mathfrak{m}I = \mathfrak{m}J$. Therefore, $F(I)$ is Cohen–Macaulay. Since $t^{11} \in I^2 \cap J$ and $t^{11} \notin JI$, $G(I)$ is not Cohen–Macaulay. It can also be easily checked that $(\mathfrak{m}I^2 + JI : I) \cap I = J + \mathfrak{m}I$. Hence $F(I)$ is Gorenstein.

Example 6.4 Let $A = k[[x, y]]$, $I = (x^3, x^2y, y^3)$ and $J = (x^3, y^3)$. Then $\ell(I^2/JI) = 1$, $I^3 = JI$, $\ell(\mathfrak{m}I/\mathfrak{m}J) = 1$ and $\mathfrak{m}I^2 = \mathfrak{m}JI$. Since $\mu(I) = d+1$, $F(I)$ is a hypersurface. It can easily be seen that $x^4y^2 \in I^2 \cap J$, but $x^4y^2 \notin JI$. Hence, $G(I)$ is not Cohen–Macaulay. It is easily checked that $(\mathfrak{m}J : I) \cap I \neq J + \mathfrak{m}I$, even if $F(I)$ is Gorenstein. This shows that the assumption on the Cohen–Macaulayness of $G(I)$ in Theorem 5.4 is necessary.

Example 6.5 Let $A = k[[x, y, z]]$, $I = (x^3, y^3, z^3, xy, yz, zx)$ and $J = (x^3 + yz, y^3 + z^3 + xz, xz + xy)$. It is been shown in [6] that I has minimal mixed multiplicity. It can be seen that $I^2 = JI$. Therefore both $G(I)$ and $F(I)$ are Cohen–Macaulay. It can also be seen that $\ell(\mathfrak{m}I/\mathfrak{m}J) = 1$. Since $z^3 \in \mathfrak{m}J : I \cap I$ and $z^3 \notin \mathfrak{m}I + J$, $F(I)$ is not Gorenstein by Proposition 5.4. The Hilbert series of the fiber cone is $HS(F(I), t) = 1 + 3t/(1-t)^3$, which also shows that the fiber cone is not Gorenstein. But, we have the equalities, $\ell(J : I/J) = \ell(A/I) = 7$ and $\mu(I) = 6 = \mu(\mathfrak{m}) + d$. This shows that the converse of Corollary 5.8 is not true.

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