THE RADICAL EQUATION $P(A_n) = (P(A))_n$

R. E. PROPES

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The purpose of this paper is to impose conditions on a radical class P so that the P-radical of the ring of $n \times n$-matrices over a ring A is equal to the ring of $n \times n$-matrices over the ring $P(A)$. In (**1**), Amitsur gave such conditions, but with the stipulation that the radical class P contained all zero-rings (rings in which all products are zero). In what follows, we shall be working within the class of associative rings.

We show that if P is a radical class which is (right or left)-hereditary and (right or left)-strong, then P has the property that the P-radical of the ring of $n \times n$-matrices over a ring A is equal to the ring of $n \times n$-matrices over the ring $P(A)$.

Definition 1. Let P be a radical class. A left ideal I of a ring A is called a *P-left ideal* of A if I is a P-ring, i.e. if $I \in P$. We define *P-right ideals* of A analogously.

Definition 2. As defined in (**2**), a radical class P is said to be *left-strong* in case $P(A)$ contains all P-left ideals of A for each ring A. The concept *right-strong* is defined analogously. A *strong* radical class is one which is both left-strong and right-strong.

Definition 3. A radical class P is said to be *left-hereditary* if each left ideal of a P-ring is also a P-ring. *Right-hereditary* radicals are defined analogously. An *hereditary* radical class is one for which each ideal of a P-ring is also a P-ring.

Remark. If P is the Brown-McCoy radical, then P is hereditary and satisfies the equation $P(A_n) = (P(A))_n$. However, from (**2**, Example 3), P is neither left-strong nor right-strong and is neither left-hereditary nor right-hereditary.

We shall employ the following notation throughout.

If A is a ring and n is a positive integer, A_n denotes the ring of $n \times n$-matrices over A. For $i, j \in \{1, 2, \ldots, n\}$, A_{ij} denotes the subring of A_n consisting of all matrices with elements from A in the (i, j)-position and with 0's elsewhere. For $i \in \{1, 2, \ldots, n\}$, we define R_i as the right ideal $\sum_{j=1}^{n} A_{ij}$ of A_n, and we define L_i as the left ideal $\sum_{k=1}^{n} A_{ki}$ of A_n. If $x \in A$ and J is a non-empty subset of $\{1, 2, \ldots, n\}$ with $i \in J$, then $B_J(i, x)$ denotes the $n \times n$-matrix with x in the (i, j)-position for all $j \in J$ and with 0's elsewhere. Then $B_{J(i)} = \bigcup_{x \in A} B_J(i, x)$ is a left-ideal of the ring R_i. Moreover, $A \cong B_{J(i)}$ under the obvious mapping.

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Theorem 1. *Let P be a radical class, let A be a ring, and let n be a positive integer. The following statements are equivalent.*

(i) *If $A \in P$, then $A_n \in P$.*
(ii) $(P(A))_n \subseteq P(A_n)$.

Proof. Assume (i). Now $P(A) \in P$ so that by (i), $(P(A))_n \in P$. Hence $(P(A))_n \subseteq P(A_n)$. Next assume (ii). Now $A \in P$ implies $P(A) = A$ so that $A_n = (P(A))_n$. By (ii), $(P(A))_n \subseteq P(A_n)$. Whence $A_n = P(A_n)$ and $A_n \in P$.

Theorem 2. *Let P be a radical class, let A be a ring, and let n be a positive integer. The following statements are equivalent.*

(i) *If $A_n \in P$, then $A \in P$.*
(ii) $P(A_n) \subseteq (P(A))_n$.

Proof. Assume (i). By Lemma 7 of Snider (**4**), $P(A_n) = I_n$ for some ideal I of A. From (i), we have $I \in P$. Hence $I \subseteq P(A)$ and so $P(A_n) = I_n \subseteq (P(A))_n$. Assume (ii). Now $A_n \in P$ implies $P(A_n) = A_n$. Thus by (ii), $A_n = P(A_n) \subseteq (P(A))_n$ and so $A_n = (P(A))_n$. Whence $P(A) = A$ and $A \in P$.

Theorem 3. *Let P be a strong radical class. Then $A \in P$ implies $A_n \in P$.*

Proof. The theorem is evident for $n = 1$. Thus, let $n > 1$. Let $i \in \{1, 2, ..., n\}$ be fixed, and let $j \in \{1, 2, ..., n\}$ with $j \neq i$. Set $J = \{i, j\}$. Then since $A \in P$ and $A \cong B_{J(i)}$, we have $B_{J(i)} \in P$. Since P is strong and since $B_{J(i)}$ is a left ideal of the ring R_i, we have that $B_{J(i)} \subseteq P(R_i)$. Setting $K = \{i\}$ we likewise obtain $B_{K(i)} \subseteq P(R_i)$. Hence $B_{J(i)} + B_{K(i)} \subseteq P(R_i)$. Since $j \neq i$, and j was otherwise arbitrary, then $R_i \subseteq P(R_i)$, i.e. $R_i \in P$. Now R_i is a P-right ideal of A_n so that, since P is strong, we must have $R_i \subseteq P(A_n)$. This being true for $i = 1, 2, ..., n$, we obtain $\sum_{i=1}^{n} R_i \subseteq P(A_n)$. Hence $A_n = P(A_n)$ and $A_n \in P$.

Theorem 4. *If P is a left-hereditary (or a right-hereditary) radical class, then $A_n \in P$ implies $A \in P$.*

Proof. Let P be a left-hereditary radical class, and let $A_n \in P$. Since L_1 is a left ideal of A_n, and since P is left-hereditary, then $L_1 \in P$. Now A is a homomorphic image of L_1 under the mapping

$$\begin{bmatrix} a_{11} & 0 \ldots\ldots 0 \\ a_{12} & 0 \ldots\ldots 0 \\ \vdots & \vdots \qquad \vdots \\ a_{1n} & 0 \ldots\ldots 0 \end{bmatrix} \mapsto a_{11}.$$

Thus $A \in P$. The proof for right-hereditary radicals is dual.

The proof of the next theorem is facilitated by a proposition which is due to M. Jaegermann (**3**). If A is a ring, A^+ denotes the zero-ring on A, i.e. the additive group of A with all products being 0.

Proposition. *If P is a hereditary and left-strong (right-strong) radical class, then $A \in P$ implies $A^+ \in P$.*

Theorem 5. *If P is a hereditary and left-strong (right-strong) radical class, then $A \in P$ implies $A_n \in P$.*

Proof. Let P be hereditary and left-strong, and let $A \in P$. From the proof of Theorem 3, the right ideal R_i of A_n belongs to P. By the Proposition, $R_i^+ \in P$. Since the zero-rings R_i^+ and L_i^+ are isomorphic by the matrix transpose function, we have $L_i^+ \in P$. Now $\sum_{j \neq i} A_{ji}$ is an ideal of L_i^+ and so belongs to P, since P is hereditary. But $\sum_{j \neq i} A_{ji}$ is also an ideal of the ring L_i, and $L_i / \sum_{j \neq i} A_{ji} \cong A$. Since $A \in P$ and $\sum_{j \neq i} A_{ji} \in P$, then $L_i \in P$. Since i was arbitrary, and P is left-strong, then $\sum_{i=1}^{n} L_i = A_n \in P$.

The proof for P hereditary and right-strong is dual.

Theorem 6. *If P is a radical class which is (right or left)-hereditary and (right or left)-strong, then $P(A_n) = (P(A))_n$ for each ring A and for each positive integer n.*

Proof. Since P is (right or left)-hereditary, then P is hereditary. The proof now follows from Theorem 4 and Theorem 5.

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The University of Wisconsin-Milwaukee
Milwaukee, Wisconsin