MINKOWSKI'S FUNDAMENTAL INEQUALITY FOR REDUCED POSITIVE QUADRATIC FORMS (II)

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Abstract

A convex polytope $\mathcal{D}(\alpha)$ was defined in Barnes (1978) as the set of Minkowski-reduced forms with prescribed diagonal coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$. A local minimum of the determinant $D(f)$ over $\mathcal{D}(\alpha)$ must occur at a vertex of $\mathcal{D}(\alpha)$. Here a criterion is obtained for a given vertex to provide a local minimum, completely analogous to Voronoi's criterion for a perfect form to be extreme.


1. Introduction

We use the definitions introduced in Part I (Barnes, 1978): $\mathcal{M}$ is the polyhedral cone of Minkowski-reduced forms in $n$ variables; $\mathcal{D}(\alpha)$ is the subset of $\mathcal{M}$ consisting of those positive forms $f(x) = \sum a_{ij} x_i x_j$ for which

$$a_{ij} = \alpha_i \quad (i = 1, \ldots, n),$$

where necessarily

$$0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n.$$ 

$\mathcal{D}(\alpha)$ is in fact a convex polytope.

The convexity of the determinantal surface $D(f) =$ constant implies that the minimum value of $D$ over $\mathcal{D}(\alpha)$, or indeed any local minimum, is attained only at a vertex of $\mathcal{D}(\alpha)$. However, it is not necessarily true that a vertex $v$ of $\mathcal{D}(\alpha)$ provides a local minimum of $D(f)$ for $f \in \mathcal{D}(\alpha)$; a vertex for which this is true we call extreme with respect to $\mathcal{D}(\alpha)$, or, for brevity, $D$-extreme.

The main purpose of this article is to establish an analogue of Voronoi's (1907) well-known criterion for a form to be extreme in the classical sense, namely that it be perfect and eutactic. The analogue of a perfect form is clearly a vertex of $\mathcal{D}(\alpha)$. 

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We construct the analogue of a eutactic form as follows. Recall first that $f$ is Minkowski-reduced if and only if, for all $i = 1, \ldots, n$ and all integral $x$,

$$f(x) \geq a_{ii}$$

if g.c.d. $(x_i, x_{i+1}, \ldots, x_n) = 1$.

If $f \in \mathcal{D}(\alpha)$, denote by $m_k (k = 1, \ldots, t)$ those $x$, other than unit vectors, for which equality holds in (1.3) (as usual, we identify $x$ and $-x$ in such statements). Then we say that $f$ is $\mathcal{D}$-eutactic if its adjoint $F$ is expressible in the form

$$F(x) = \sum_{i=1}^{n} A_{ij} x_i x_j = \sum_{i=1}^{t} \rho_k (m_k x)^2 + \sum_{i=1}^{n} \sigma_i x_i^2,$$

where the $\rho_k, \sigma_i$ are real numbers and

$$\rho_k > 0 \quad (k = 1, \ldots, t).$$

**Theorem 1.** A form $f \in \mathcal{D}(\alpha)$ is $\mathcal{D}$-extreme if and only if it is a vertex of $\mathcal{D}(\alpha)$ and is $\mathcal{D}$-eutactic.

The proof of this theorem will be based on the ideas used in the proof of Voronoï's Theorem given in Barnes (1957); as there, we need

**Theorem 2 (Stiemke, 1915).** The system of linear inequalities

$$\sum_{j=1}^{p} t_{ij} u_j = 0, \quad u_j > 0 \quad (i = 1, \ldots, m; \ j = 1, \ldots, p)$$

has a solution $u$ if and only if every solution $v$ of the dual system

$$\sum_{i=1}^{m} t_{ij} v_i \geq 0 \quad (j = 1, \ldots, p)$$

satisfies

$$\sum_{i=1}^{m} t_{ij} v_i = 0 \quad (j = 1, \ldots, p).$$

We shall also need the following simple geometrical result:

**Lemma 1.** Let $K$ be an $N$-dimensional convex set and $p$ a point of $bd K$, in a neighbourhood of which $K$ is strictly convex and $bd K$ is smooth. Let $h$ be the tangent plane to $K$ at $p$ and $h^+$ the open half-space determined by $h$ and containing $int K$. Let $P$ be a convex polytope with a vertex at $p$, and suppose that the whole of $P$ in some punctured neighbourhood of $p$ lies in $h^+$. Then there exists a punctured neighbourhood $B$ of $p$ such that if $q \in B \cap P$ then $q \in int K$.

**2. Proof of Theorem 1**

We begin with the appropriate analogue of the Lemma of Barnes (1957):

**Lemma 2.** If $f \in \mathcal{D}(\alpha)$, then $f$ is $\mathcal{D}$-extreme if and only if there exists no non-trivial quadratic form $g(x) = \sum_{i=1}^{n} b_{ij} x_i x_j$ satisfying
Minkowski's reduced positive quadratic forms

(2.1) \[ g(e_i) = 0 \quad (i = 1, \ldots, n), \]
(2.2) \[ g(m_k) \geq 0 \quad (k = 1, \ldots, t), \]
(2.3) \[ (F, g) = \sum A_{ij} b_{ij} \leq 0. \]

**Proof.** (i) Suppose that (2.1), (2.2), (2.3) have a non-trivial solution \( g \). Choose one with, say \( \max |b_{ij}| = 1 \), and set \( f' = f + \epsilon g \) where \( \epsilon > 0 \) is small.

Then \( f' \in \mathcal{D}(\alpha) \). For firstly

\[ f'(e_i) = f(e_i) = \alpha_i \quad (i = 1, \ldots, n); \]
and, for any \( k = 1, \ldots, t \) for which \( m_k \) satisfies (1.3) with equality for some \( i \),

\[ f'(m_k) = f(m_k) + \epsilon g(m_k) = \alpha_i + \epsilon g(m_k) \geq \alpha_i. \]

Next, there exist only finitely many other \( x \) in (1.3) which are necessary to specify \( \mathcal{M} \) and, for all of these, \( f(x) > a_{ii} = \alpha_i \), whence also \( f'(x) > \alpha_i \) if \( \epsilon \) is sufficiently small.

Now the tangent plane \( h \) to the determinantal surface (using \( \phi \) as current co-ordinates in the coefficient space) at \( f \) is

\[ (F, \phi) = (F, f) = n \det f. \]

Since

\[ (F, f') = (F, f) + \epsilon (F, g) \leq (F, f), \]

it follows that \( f' \) lies in the closed half-space opposite to that containing the surface \( \det \phi = \det f \); hence, since this surface is strictly convex and \( f' \neq f \), \( \det f' < \det f \).

It follows from these results that \( f \) is not \( \mathcal{D} \)-extreme.

(ii) Suppose that (2.1), (2.2), (2.3) have only the trivial solution; let \( f' = f + g \) \((g \neq 0)\) be any form in \( \mathcal{D}(\alpha) \) close to \( f \). Then

\[ f'(e_i) = \alpha_i = f(e_i), \quad \text{so that} \quad g(e_i) = 0 \quad (i = 1, \ldots, n); \]
\[ f'(m_k) > \alpha_i = f(m_k), \quad \text{so that} \quad g(m_k) \geq 0 \quad (k = 1, \ldots, t). \]

Since \( g \) is non-trivial, our hypothesis implies that (2.3) is false, so that \( (F, g) > 0 \) and

(2.4) \[ (F, f') = (F, f) + (F, g) > (F, f). \]

We now apply Lemma 1, taking \( K \) to be the determinantal body \( \det \phi \geq \det f \), \( h \) the tangent plane \( (F, \phi) = (F, f) \), and \( P \) the polytope \( \mathcal{D}(\alpha) \). Using in particular (2.4), we see that the hypotheses of the lemma hold and it follows that, if \( f' \) is sufficiently close to \( f \) in \( \mathcal{D}(\alpha) \), but distinct from \( f \), then \( \det f' > \det f \). Thus \( f \) is \( \mathcal{D} \)-extreme.
PROOF OF THEOREM 1. Using the coefficients $b_{ij}$ of the form $g$, write (2.1)-(2.3) as

\[
\begin{align*}
    b_{ii} & \geq 0 \quad (i = 1, \ldots, n), \\
    -b_{ii} & \geq 0 \\
    \sum_{k=1}^{n} b_{ij} m_{ik} m_{jk} & \geq 0 \quad (k = 1, \ldots, t), \\
    -\sum_{i=1}^{n} A_{ij} b_{ij} & \geq 0.
\end{align*}
\]

(2.5)

We identify this with the system (1.7), with $b_{11}, b_{12}, \ldots, b_{nn}$ playing the part of the variables $v_1, v_2, \ldots, v_m$. Then (1.6) becomes, using variables $\lambda_i, \mu_i$ $(i = 1, \ldots, n)$, $p_k$ $(k = 1, \ldots, t), v$,

\[
\begin{align*}
    \lambda_i \delta_{ij} - \mu_i & = 0 \quad (i, j = 1, \ldots, n), \\
    \lambda_i > 0, \quad \mu_i > 0, \quad p_k > 0, \quad v > 0.
\end{align*}
\]

(2.6)

(a) Suppose that $f$ is $\mathcal{D}$-extreme. By Lemma 2, every solution of (2.5) is trivial and so certainly has equality throughout; it follows at once that $f$ is a vertex of $\mathcal{D}(a)$. Also, by Stiemke’s Theorem, (2.6) has a solution; dividing through by $v$, we may suppose that the solution has $v = 1$; multiplying by $x_i x_j$ and summing, we obtain

\[
F(x) = \sum_{i=1}^{n} A_{ij} x_i x_j = \sum_{i=1}^{t} p_k \left( m_k x \right)^2 + \sum_{i=1}^{n} \left( \lambda_i - \mu_i \right) x_i^2
\]

which gives (1.4), with (1.5), noting that $\sigma_i = \lambda_i - \mu_i$ is unrestricted in sign.

(b) Suppose next that $f$ is a vertex of $\mathcal{D}(a)$ and is $\mathcal{D}$-eutactic. Then (2.6) has a solution and so, by Stiemke’s Theorem, any solution $g$ of (2.5) satisfies (2.5) with equality throughout; since $f$ is a vertex of $\mathcal{D}(a)$ it then follows that $g \equiv 0$. It now follows from Lemma 2 that $f$ is $\mathcal{D}$-extreme.

3. An example

As noted in Part I, the quaternary form

\[
f(x) = ax_1^2 + ax_1 x_2 - ax_1 x_3 - ax_1 x_4 + bx_2^2 - bx_2 x_4 + cx_3^2 + cx_3 x_4 + dx_4^2
\]

is a vertex of $\mathcal{D}(a) = \mathcal{D}(a, b, c, d)$ which is however not $\mathcal{D}$-extreme for some values of $a, b, c, d$ (where we still of course assume (1.2), that is

\[
0 < a \leq b \leq c \leq d).
\]

THEOREM 2. Suppose that $a < b$. Then the form (3.1), subject to (3.2), is $\mathcal{D}$-extreme if and only if

\[
ad < bc.
\]
PROOF. It is easily verified that for the form (3.1), neglecting unit vectors, there are just seven relations (1.3) that hold with equality, namely

\[ f(-1,1,0,0) = b, \quad f(1,0,1,0) = c, \]
\[ f(1,0,0,1) = f(0,1,0,1) = f(0,0,-1,1) = f(0,1,-1,1) = f(-1,1,-1,1) = d. \]

These suffice to establish that \( f \) is a vertex of \( \mathcal{D}(\alpha) \).

[Note: If \( a = b \), there is one further relation \( f(1, -1, 1, 0) = c \) and the following analysis therefore does not apply.]

The identity (1.4) is now

\begin{equation}
\sum_{i=1}^{4} A_{ij} x_i x_j = \rho_1 (-x_1 + x_2)^2 + \rho_2 (x_1 + x_3)^2 + \rho_3 (x_1 + x_4)^2 + \rho_4 (x_2 + x_4)^2 + \rho_5 (-x_3 + x_4)^2 + \rho_6 (x_2 - x_3 + x_4)^2 + \rho_7 (-x_1 + x_2 - x_3 + x_4)^2 + \sum_{i=1}^{4} \sigma_i x_i^2,
\end{equation}

and this yields

\[ \rho_4 = A_{23} + A_{24} = \frac{1}{4}a(ab + ac - 2ad - 2bc) + \frac{1}{4}a(-ab - ac + 4bc) \]
\[ = \frac{1}{4}a(bc - ad). \]

If now \( ad \geq bc \), then \( \rho_4 \leq 0 \) and so, by Theorem 2, \( f \) is not \( \mathcal{D} \)-extreme. If however \( ad < bc \), it is easy to verify that (3.4) is soluble with all \( \rho_k > 0 \); one may take \( \rho_7 \) sufficiently small and positive and then determine \( \rho_1, \ldots, \rho_6 \) from the relations

\[ \rho_1 = -A_{12} - \rho_7, \quad \rho_2 = A_{13} - \rho_7, \quad \rho_3 = A_{14} + \rho_7, \quad \rho_4 = \rho_5 = \frac{1}{4}a(bc - ad), \quad \rho_6 = -A_{23} - \rho_7. \]

4. A refinement of Theorem 1

Among the inequalities (1.3) there is a finite set which implies all remaining inequalities; that is, there is a finite set \( S \) of vectors \( x \) for which the corresponding equations \( f(x) = a_{ii} \) define the facets of \( \mathcal{M} \). In determining \( \mathcal{D}(\alpha) \) and its vertices, it suffices of course to consider only this minimal set of inequalities. However, the application of Theorem 1 is complicated by the need to know all the examples of equality in (1.3) for a given form \( f \). Fortunately it turns out to be necessary, in testing whether a vertex \( f \) of \( \mathcal{D}(\alpha) \) is \( \mathcal{D} \)-extreme, to consider only vectors in \( S \):

**Theorem 3.** Let \( f \) be a vertex of \( \mathcal{D}(\alpha) \) and \( m_k \) (\( k = 1, \ldots, r \)) \( (r \leq t) \) be the vectors of \( S \), other than unit vectors, for which equality holds in (1.3) Then \( f \) is \( \mathcal{D} \)-extreme if and only if \( f \) is eutactic with respect to this set of vectors, that is, if

\begin{equation}
F(x) = \sum_{i=1}^{n} A_{ij} x_i x_j = \sum_{k=1}^{r} \rho_k (m_k' x)^2 + \sum_{i=1}^{n} \sigma_i x_i^2
\end{equation}

for some real \( \rho_k, \sigma_i \) satisfying \( \rho_k > 0 \) (\( k = 1, \ldots, r \)).
For the proof of Theorem 4 it clearly suffices to establish the corresponding strengthening of Lemma 2, namely

**Lemma 3.** Let \( f \in \mathcal{D}(\mathbf{a}) \) and let \( m_k \) \((k = 1, \ldots, r)\) be those \( x \in S \), other than unit vectors, for which equality holds in (1.3). Then \( f \) is \( \mathcal{D} \)-extreme if and only if there exists no non-trivial quadratic form \( g \) satisfying

\[
g(e_i) = 0 \quad (i = 1, \ldots, n), \quad g(m_k) \geq 0 \quad (k = 1, \ldots, r), \quad (F,g) \leq 0.
\]

**Proof.** Only one modification is needed in the proof given for Lemma 2, namely for the assertion in part (i) that \( f' \in \mathcal{D}(\mathbf{a}) \). We now have

\[
f'(e_i) = f(e_i) = a_i \quad (i = 1, \ldots, n)
\]

and, for each \( k = 1, \ldots, r \) and the corresponding \( i \),

\[
f'(m_k) = f(m_k) + \varepsilon g(m_k) \geq f(m_k) = a_i.
\]

Next, if \( x \in S \) but is not a unit vector or one of \( m_1, \ldots, m_r \), our hypothesis implies that \( f(x) > a_{ii} \), whence also \( f'(x) > a_{ii} = a_i \) if \( \varepsilon \) is sufficiently small. Since \( S \) is finite, it follows that \( f'(x) \geq a_i \) for all \( x \in S \) (and corresponding \( i \)), and hence, by definition of \( S \), for all integral \( x \).

**References**


