Teaching Notes

On the bounds for the perimeter of an ellipse

Let $L(a, b)$ denote the perimeter of an ellipse with semi-axes a, b. It is well known that $\pi(a + b) \leqslant L(a, b) \leqslant 2\pi\sqrt{\frac{a^2 + b^2}{2}}$, with equality if, and only if, $b = a$. There are several proofs of this remarkable fact in the literature; see, for example, [1], [2] and [3]. The aim of this note is to compile a proof suitable for high school students as much as possible. We start with the standard parametrisation $x = a\cos t$, $y = b\sin t$ which gives rise to

$$L(a, b) = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2 t + b^2\sin^2 t}\, dt.$$

Using the substitution $u = \frac{\pi}{2} - t$ we get $L(a, b) = L(b, a)$, hence

$$L(a, b) = 2\int_0^{\frac{\pi}{2}} \left(\sqrt{a^2\cos^2 t + b^2\sin^2 t} + \sqrt{a^2\sin^2 t + b^2\cos^2 t}\right) dt.$$

Since $\left(a^2\cos^2 t + b^2\sin^2 t\right) + \left(a^2\sin^2 t + b^2\cos^2 t\right) = a^2 + b^2$ and $\frac{A + B}{2} \leqslant \sqrt{\frac{A^2 + B^2}{2}}$ for all non-negative real A, B with equality if, and only if, $B = A$, we have

$$\sqrt{a^2\cos^2 t + b^2\sin^2 t} + \sqrt{a^2\sin^2 t + b^2\cos^2 t} \leqslant 2\sqrt{\frac{a^2 + b^2}{2}}$$

for each t, with equality for each t if, and only if, $b = a$. Therefore

$$L(a,b) = 2\int_0^{\frac{\pi}{2}} \left(\sqrt{a^2\cos^2 t + b^2\sin^2 t} + \sqrt{a^2\sin^2 t + b^2\cos^2 t}\right) dt \leqslant 2\pi\sqrt{\frac{a^2 + b^2}{2}}$$

with equality if, and only if, $b = a$.

Also since $\left(\cos^2 t + \sin^2 t\right)^2 = 1$ we get $\cos^4 t + \sin^4 t = 1 - 2\cos^2 t\sin^2 t$, whence

$$\begin{aligned} &\left(a^2\cos^2 t + b^2\sin^2 t\right)\left(a^2\sin^2 t + b^2\cos^2 t\right) \\ &= a^2b^2\left(1 - 2\cos^2 t\sin^2 t\right) + \left(a^4 + b^4\right)\cos^2 t\sin^2 t \\ &= a^2b^2 + \left(a^2 - b^2\right)^2\cos^2 t\sin^2 t, \end{aligned}$$

and hence

$$\begin{aligned} &\left(\sqrt{a^2\cos^2 t + b^2\sin^2 t} + \sqrt{a^2\sin^2 t + b^2\cos^2 t}\right)^2 \\ &= a^2 + b^2 + 2\sqrt{a^2b^2 + (a^2 - b^2)^2\cos^2 t\sin^2 t} \geqslant (a + b)^2 \end{aligned}$$

for each t, with equality for each t if, and only if, $b = a$, giving

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$$L(a,b) = 2\int_0^{\frac{\pi}{2}} \left(\sqrt{a^2\cos^2 t + b^2\sin^2 t} + \sqrt{a^2\sin^2 t + b^2\cos^2 t}\right)dt \geqslant \pi(a+b)$$

with equality if, and only if, $b = a$. Thus

$$\pi(a + b) \leqslant L(a, b) \leqslant 2\pi\sqrt{\frac{a^2 + b^2}{2}},$$

with equality if, and only if, $b = a$ as required.

References

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IVICA GUSIĆ

Faculty of Chemical Engineering and Technology, University of Zagreb, Marulićev trg 19, 10000 Zagreb, Croatia

e-mail: *igusic@fkit.hr*

Cauchy-Schwarz via collisions

Consider a line of n railway trucks with masses $m_1, m_2, \ldots, m_n$ moving on a smooth straight track with velocities $v_1 > v_2 > \ldots > v_n$ (with negative velocities allowed) and spaced so that they successively couple together in the order Truck $n - 1$ to Truck n, Truck $n - 2$ to Trucks $n - 1$ and n, Truck $n - 3$ to Trucks $n - 2$ and $n - 1$ and n, etc. If, when all n trucks are coupled together, their common velocity is V, conservation of momentum shows that $V = \dfrac{\sum m_i v_i}{\sum m_i}$. But kinetic energy cannot be gained in the collisions, so

$$\frac{1}{2}\sum m_i v_i^2 \geqslant \frac{1}{2}\left(\sum m_i\right)V^2 = \frac{1}{2}\frac{\left(\sum m_i v_i\right)^2}{\sum m_i}$$

and thus

$$\left(\sum m_i v_i\right)^2 \leqslant \left(\sum m_i\right)\left(\sum m_i v_i^2\right). \qquad (*)$$

Moreover, physical intuition suggests that no kinetic energy will be lost only in the special case in which $v_1 = v_2 = \ldots = v_n$ where there are no collisions and the total kinetic energy is the same whether the trucks are viewed individually or *en masse*.