LIMITS OF UNBOUNDED SEQUENCES OF CONTINUED FRACTIONS

JINGCHENG TONG

Let $X = \{x_k\}_{k \geq 1}$ be a sequence of positive integers. Let $Q_k = [0; x_k, x_{k-1}, \ldots, x_1]$ be the finite continued fraction with partial quotients $x_i (1 \leq i \leq k)$. Denote the set of the limit points of the sequence $\{Q_k\}_{k \geq 1}$ by $\Lambda(X)$. In this note a necessary and sufficient condition is given for $\Lambda(X)$ to contain no rational numbers other than zero.

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It is easily seen that $0 \in \Lambda(X)$ if and only if $X$ is unbounded. In this note, using the idea in [3], we prove that Angell’s result holds for a large family of unbounded sequences if 0 is excluded from $\Lambda(X)$.

We first introduce some new notions.

**Definition 1:** Let $X = \{x_k\}_{k \geq 1}$ be a sequence of positive integers and $N$ be a positive integer. An infinite subsequence $\{x_{ki}\}_{i \geq 1}$ is said to be an $N$-subsequence if $x_{ki} = N$ for all sufficiently large $i$.

**Definition 2:** Let $X = \{x_k\}_{k \geq 1}$ be a sequence of positive integers. Then $X$ is said to be an $N$-sequence if for each $N$-subsequence $\{x_{ki}\}_{i \geq 1}$, the subsequence $\{x_{ki-1}\}_{i \geq 1}$ is bounded, that is, there is a positive number $I(N)$ such that $x_{ki-1} \leq I(N)$ for $i = 1, 2, \ldots$.

Obviously a bounded sequence is an $N$-sequence. The converse is not true. The following example is an unbounded $N$-sequence.

**Example 1:** $X = \{x_k\}_{k \geq 1} = \{1, 1, 2, 1, 8, 4, 2, 1, 512, 256, 128, 64, 8, 4, 2, 1, \ldots\}$, where $x_k = 1$ for $k = 2^n$, $(n = 0, 1, 2, \ldots)$ and $x_k = 2^i$ for $k = 2^n - i$, $(n = 2, 3, \ldots$ and $1 \leq i < 2^{n-1})$. Because for each $2^i$-subsequence $\{x_{ki}\}$, the subsequence $\{x_{ki-1}\}$ is bounded by $I(2^i) = 2^{i+1}$, $X$ is an $N$-sequence.

Now we give the main result.

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THEOREM 1. \( \Lambda(X) \) contains no rational numbers other than zero if and only if \( X \) is an \( \mathcal{N} \)-sequence.

PROOF: Necessity. Suppose \( X \) is not an \( \mathcal{N} \)-sequence. Then there is an \( \mathcal{N} \)-subsequence \( \{x_{k_i}\}_{i \geq 1} \) such that \( \{x_{k_i-1}\}_{i \geq 1} \) is not bounded. Hence there is a subsequence \( \{x_{k_{m_i}-1}\}_{m \geq 1} \) of \( \{x_{k_i-1}\}_{i \geq 1} \), satisfying \( x_{k_{m_i}-1} \rightarrow \infty \) as \( m \rightarrow \infty \). Since \( \{x_{k_{m_i}}\}_{m \geq 1} \) is a subsequence of the \( \mathcal{N} \)-subsequence \( \{x_{k_i}\}_{i \geq 1} \), we have \( x_{k_{m_i}} = N \) for all sufficiently large \( m \). Hence \( Q_{k_{m_i}} \rightarrow 1/N \) as \( m \rightarrow \infty \). Thus \( \Lambda(X) \) contains a rational number other than zero.

Sufficiency. Let \( a \) be an arbitrary rational number other than 0. Since \( 0 < Q_k < 1 \), without loss of generality, we may assume \( 0 < a < 1 \).

We first prove \( 1 \notin \Lambda(X) \). Suppose there is a subsequence \( Q_{k_i} \rightarrow 1 \) as \( i \rightarrow \infty \). Then since \( Q_{k_i} = [0; x_{k_i}, \ldots, x_{1}] < 1/x_{k_i} \), \( \{x_{k_i}\}_{i \geq 1} \) must be a 1-subsequence. Hence \( \{x_{k_i-1}\}_{i \geq 1} \) is bounded by \( I(1) \) and \( Q_{k_i} < [0; 1, I(1)] = 1/(1 + 1/I(1)) < 1 \), a contradiction to the assumption \( Q_{k_i} \rightarrow 1 \). Therefore \( 1 \notin \Lambda(X) \).

Suppose there is a rational number \( a \neq 0 \) and \( a \in \Lambda(X) \). Since \( 0 < a < 1 \), \( a \) can be expanded as a finite continued fraction: \( a = [0; a_1, \ldots, a_r] \). Let \( Q_{k_i} \rightarrow a \). If \( \{x_{k_i}\}_{i \geq 1} \) is not an \( a_1 \)-subsequence, there are infinitely many \( i \) such that \( x_{k_i} \neq a_1 \). We discuss the following possible cases.

(1) There are infinitely many \( i \) with \( x_{k_i} \geq a_1 + 2 \). For these \( i \), we have
\[
a - Q_{k_i} [0; a_1, a_2, \ldots, a_r - [0; x_{k_i}, \ldots, x_1] > [0; a_1, 1] - [0; a_1 + 2] > 1/(a_1 + 2)^2.
\]
Hence \( Q_{k_i} \not\rightarrow a \).

(2) There are infinitely many \( i \) with \( x_{k_i} \leq a_1 - 2 \). For these \( i \), we have
\[
Q_{k_i} - a > [0; a_1 - 2, 1] - [0; a_1] > 1/a_1^2.
\]
Hence \( Q_{k_i} \not\rightarrow a \).

(3) There are infinitely many \( i \) with \( x_{k_i} = a_1 + 1 \), that is, there is an \( (a_1 + 1) \)-subsequence \( \{x_{k_{m_i}}\}_{m \geq 1} \). Then \( \{x_{k_{m_i}-1}\}_{m \geq 1} \) is bounded by \( I(a_1 + 1) \) and
\[
a - Q_{k_{m_i}} [0; a_1, 1] - [0; a_1 + 1, I(a_1 + 1)] > 1/(1 + I(a_1 + 1))(a_1 + 1)^2.
\]
Hence \( Q_{k_i} \not\rightarrow a \).

(4) There are infinitely many \( i \) with \( x_{k_i} = a_1 - 1 \), that is, there is an \( (a_1 - 1) \)-subsequence \( \{x_{k_{m_i}}\}_{m \geq 1} \). Then there are two possibilities:

(i) There are infinitely many \( n \) with \( x_{k_{m_i}-1} \geq 2 \). For these \( n \), we have
\[
Q_{k_{m_i} - a > [0; a_1 - 1, 2] - [0; a_1] > 1/2a_1^2.
\]
Hence $Q_{k_i} \not\leadsto a$.

(ii) The subsequence $\{x_{k_n}\}_{n \geq 1}$ is a 1-subsequence. Then $\{x_{k_n-2}\}_{n \geq 1}$ is bounded by $I(1)$ and

$$Q_{k_n} - a > [0; a_1 - 1, 1, I(1)] - [0; a_1] > 1/a_1^2(I(1) + 1).$$

Hence $Q_{k_i} \not\leadsto a$.

From the discussion above, we know that $\{x_{k_i}\}_{i \geq 1}$ must be an $a_1$-subsequence. Now we prove $x_{k_i - (j - 1)} = a_j$ ($1 \leq j \leq r$) for all sufficiently large $i$.

Suppose $j_0$ is the smallest index $j$ such that for each $j$ with $1 \leq j \leq j_0$, $\{x_{k_i - (j - 1)}\}_{i \geq 1}$ is an $a_j$-subsequence, but $\{x_{k_i - j_0}\}_{i \geq 1}$ is not an $a_{j_0 + 1}$-subsequence. Then for sufficiently large $i$, we have

$$x_{k_i - (j - 1)} = a_j \quad (1 \leq j \leq j_0),$$

$$a = [0; a_1, \ldots, a_{j_0}, a_{j_0 + 1}, \ldots, a_r],$$

$$Q_{k_i} = [0; a_1, \ldots, a_{j_0}, x_{k_i - j_0}, \ldots, x_1].$$

Let

$$\alpha_{j_0 + 1} = [a_{j_0 + 1}; a_r],$$

$$\beta_{j_0 + 1}(i) = [x_{k_i - j_0}; \ldots, x_1],$$

$$p_{j_0}/q_{j_0} = [0; a_1, \ldots, a_{j_0}].$$

By a well known fact ([2, Theorem 7.3] or [3, Lemma 1]), we have

$$a = \frac{\alpha_{j_0 + 1}p_{j_0} + p_{j_0 - 1}}{\alpha_{j_0 + 1}q_{j_0} + q_{j_0 - 1}},$$

$$Q_{k_i} = \frac{\beta_{j_0 + 1}(i)p_{j_0} + p_{j_0 - 1}}{\beta_{j_0 + 1}(i)q_{j_0} + q_{j_0 - 1}},$$

$$|Q_{k_i} - a| = \frac{|\alpha_{j_0 + 1} - \beta_{j_0 + 1}(i)|}{(\alpha_{j_0 + 1}q_{j_0} + q_{j_0 - 1})(\beta_{j_0 + 1}(i)q_{j_0} + q_{j_0 - 1})}.$$

Consider $D = |\alpha_{j_0 + 1}^{-1} - \beta_{j_0 + 1}(i)^{-1}| = |[0; a_{j_0 + 1}, \ldots, a_r] - [0; x_{k_i - j_0}, \ldots, x_1]|$. Since $\{x_{k_i - j_0}\}_{i \geq 1}$ is not an $a_{j_0 + 1}$-subsequence, there are infinitely many $i$ such that $x_{k_i - j_0} \neq a_{j_0 + 1}$. Similar to the discussion of $x_{k_i}$ being an $a_1$-subsequence, we may discuss the four cases for infinitely many $i$: (1) $x_{k_i - j_0} \geq a_{j_0 + 1} + 2$, (2) $x_{k_i - j_0} \leq a_{j_0 + 1} - 2$, (3) $x_{k_i - j_0} = a_{j_0 + 1} + 1$, (4) $x_{k_i - j_0} = a_{j_0 + 1} - 1$, and obtain the conclusion $Q_{k_i} \not\leadsto a$. Hence $\{x_{k_i - j_0}\}_{i \geq 1}$ must be an $a_{j_0 + 1}$-subsequence, a contradiction to the assumption. Therefore $x_{k_i - (j - 1)} = a_j$ ($1 \leq j \leq r$) for sufficiently large $i$, and

$$Q_{k_i} = [0; a_1, \ldots, a_r, x_{k_i - r}, \ldots, x_1].$$
Again we show that $Q_{k_i} \not\rightarrow a$. We discuss two cases:

1. $r$ is odd. Then $Q_{k_i} > [0; a_1, \ldots, a_r, 1] = (p_r + p_{r-1})/(q_r + q_{r-1})$, and

$$|Q_{k_i} - a| > \frac{p_r}{q_r} - \frac{p_r + p_{r-1}}{q_r + q_{r-1}} > \frac{1}{2q_r^2}.$$

2. $r$ is even. Then $Q_{k_i} < [0; a_1, \ldots, a_r, 1]$ and

$$|Q_{k_i} - a| > \frac{p_r + q_{r-1}}{p_r + q_{r-1} - \frac{p_r}{q_r}} > \frac{1}{2q_r^2}.$$

In both cases, $Q_{k_i} \not\rightarrow a$. Therefore $A(X)$ contains no rational number other than 0. The proof is completed.

References

