SOME REMARKS ON ANGULAR DERIVATIVES AND JULIA'S LEMMA

H. L. Jackson

(received October 19, 1965)

1. Introduction. Let $w = f(z)$ be holomorphic on the unit disk $D = \{z : |z| < 1\}$, with the additional restrictions that $|f(z)| < 1$ and $A. \lim_{z \to 1} f(z) = 1$, where $A. \lim_{z \to 1} f(z)$ denotes the (outer) angular limit of $f(z)$ at $z = 1$. Let us now define $g(z) \equiv \frac{f(z)-1}{z-1}$, and then focus our attention on the behaviour of $g(z)$ in an arbitrary angular neighbourhood of $z = 1$. Whenever $A. \lim_{z \to 1} g(z)$ exists, this limit is commonly referred to as the angular derivative of $f(z)$ at $z = 1$.

We now cite some of the fundamental results, by now classical, pertaining to the theory of angular derivatives for functions restricted as above.

(i) $A. \lim_{z \to 1} |g(z)|$ always exists either as a real number α greater than zero or as $+\infty$.

(ii) If $A. \lim_{z \to 1} |g(z)| = \alpha > 0$, then this same property holds for the functions $g(z)$, $\frac{1-|f(z)|}{1-|z|}$, and $f'(z)$ respectively. By this we mean that each of these functions actually possesses an angular limit at $z = 1$ which in each case turns out to be α.

(iii) $A. \lim_{z \to 1} |g(z)| = \alpha > 0$ if and only if $\varliminf_{z \to 1} \left(\frac{1-|f(z)|}{1-|z|}\right) = \alpha$, where $\varliminf$ means limit inferior.

Canad. Math. Bull. vol. 9, no. 2, 1966

https://doi.org/10.4153/CMB-1966-032-3 Published online by Cambridge University Press

(iv) If $A.\lim_{z\to 1}|g(z)| = \alpha > 0$, then $K_1(f(z)) \geq \frac{1}{\alpha} K_1(z)$, where $K_1(z) \equiv \frac{1-|z|^2}{|1-z|^2}$ and $K_1(f(z)) \equiv \frac{1-|f(z)|^2}{|1-f(z)|^2}$.

(v) If there exists $\lambda > 0$ such that $K_1(f(z)) \geq \lambda K_1(z)$, then $A.\lim_{z\to 1}|g(z)| \leq \frac{1}{\lambda}$. For details regarding the above assertions, the interested reader is referred to ([1], pp. 112-121; [4], pp. 23-32; [5], pp. 236-239; [6], pp. 15-18).

That $\underline{\lim}_{z\to 1} \left(\frac{1-|f(z)|}{1-|z|}\right) = \alpha > 0$ implies $K_1(f(z)) \geq \frac{1}{\alpha} K_1(z)$ is a generalized form of the lemma of Julia. The original Julia lemma essentially states that if $f(z)$ is holomorphic at $z = 1$ also, then $K_1(f(z)) \geq \frac{1}{f'(1)} K_1(z)$.

The generalized Julia lemma and its converse (assertion (v)) reduces the question of existence of a finite angular derivative of $f(z)$ at $z = 1$ to the problem of finding a positive minimal harmonic function with pole at $z = 1$ (i.e., of the form $\lambda K_1(z)$, $\lambda > 0$) which can be dominated by $K_1(f(z))$. Since this condition gives little or no direct information about the mapping properties of $f(z)$, we therefore concern ourselves with the problem of finding some other necessary and sufficiency condition that will ensure the existence of a finite angular derivative of $f(z)$ at $z = 1$ and, at the same time, give direct information about the mapping properties of $f(z)$. Results of this kind have been established by Carathéodory, Valiron, Warschawski and others, and have been summed up in Tsuji's book ([9], Chapt. IX, theorems IX.9, IX.10 and IX.11).

It is the author's opinion that certain aspects of the theory of angular derivatives can be greatly simplified by making use of some of the concepts of modern potential theory, especially the fine topology of Cartan, Brelot and Naïm. In fact, we shall show that the existence of a finite angular derivative of $f(z)$ at $z = 1$ is equivalent to the condition that certain fine neighbourhoods of $z = 1$ are preserved by $f(z)$ in a sense to be made precise.

2. <u>Some Definitions and Results that are Fundamental in Potential Theory.</u> In this section, we introduce some potential theoretic concepts in a form that is adequate for our purposes.

Let R be an open connected subset of the complex plane which tolerates a Green's function $G(z, a)$, with pole at $a \in R$. A set $E \subset R$ is defined to be <u>polar</u>, or of <u>outer capacity zero</u> (with relation to the Green's function as kernel), if there exists a <u>Green potential</u> $U^{\mu}(z) \equiv \int_R G(z, w) d\mu_w \not\equiv +\infty$ on R with respect to the mass distribution μ (a non-negative Borel measure), such that $U^{\mu}(p) = +\infty$ for any $p \in E$.

Now let v be a positive (>0) superharmonic function on R, $E \subset R$, and $\{s\}$ the set of non-negative superharmonic functions on R such that $s \geq v$ on $R-E$ for any $s \in \{s\}$. Then the lower envelope of $\{s\}$, denoted by Σ_v^E with ordering taken to be pointwise, is a non-negative superharmonic function on R <u>quasi-everywhere</u> (except on a polar subset of R). Let us now regularize Σ_v^E by defining a new function

$$\mathcal{E}_v^E \equiv \begin{cases} \Sigma_v^E & \text{wherever } \Sigma_v^E \text{ is lower semi-continuous,} \\ \underline{\lim}_{z \to p} \Sigma_v^E & \text{at any } p \text{ (necessarily in a polar set) where } \Sigma_v^E \text{ fails to be lower semi-continuous.} \end{cases}$$

Then $\mathcal{E}_v^E$ is a non-negative superharmonic function on R; we shall refer to this function as the (outer) <u>extremalized</u> function of v on E, and to $\mathcal{E}$ as the <u>extremalization operator of Brelot</u> (see [8], pp. 127-131 for details). In the particular case where E is open, then

$$\mathcal{E}_v^E \equiv \begin{cases} v & \text{quasi-everywhere on } R-E, \\ H_{v_*}^E & \text{on } E. \end{cases}$$

The function $H_{v_*}^E$ is defined to be the generalized solution of the Dirichlet problem on E with boundary function

$$v_* = \begin{cases} v \text{ on } \partial E \cap R, \\ 0 \text{ at } \infty \text{ (the Alexandroff compactification point of } R). \end{cases}$$

The fine topology of Cartan-Brelot on R is defined to be the least topology on R making continuous the superharmonic functions on R, or equivalently the least topology making continuous the Green potentials on R. A fundamental notion associated with the filter of fine neighbourhoods at a point $a \in R$ is the concept of a thin set introduced by Brelot. A set $E \subset R$ is defined to be thin at $a \in R$ if and only if either

(i) a is an isolated point of $E \cup \{a\}$ in the usual topology on R,

or

(ii) $\exists$ a positive superharmonic function v on R such that

$$v(a) < \varliminf_{\substack{z \to a \\ z \in E}} v(z).$$

A set $N \subset R$ is a fine neighbourhood of a if and only if $a \in N$ and $R - N$ is thin at a.

Let us now choose $a \in R$ as a reference point, and hold it fixed throughout the remainder of this section. The set $R - \{a\}$ shall be denoted by R_a, and the function $\dfrac{G(z, w)}{G(z, a)}$ on $R_a \times R_a$ shall be designated by $K(z, w)$. We shall refer to $K(z, w)$ as the Martin kernel function, although the term normalized Green's function is often used, and note that there exists a compact metric space $\hat{R} = R \cup \Delta$, where R is dense in $\hat{R}$; such that

$$\lim_{\substack{z \to \hat{z} \\ z \in R \\ \hat{z} \in \Delta}} K(z, w) = K_{\hat{z}}(w) \quad \text{or} \quad K(\hat{z}, w)$$

is a positive harmonic function on R where limit is taken with respect to the topology of $\hat{R}$. Δ is called the Martin boundary of R, and if $K_{\hat{z}}(w)$ is minimal (i.e., dominates only non-negative harmonic functions of the form $\lambda K_{\hat{z}}(w)$ where $0 \leq \lambda \leq 1$) then we say that $\hat{z}$ is a minimal Martin boundary point and denote the set of these points by Δ_1. In general $\Delta_1 \subset \Delta$, and Martin showed that for each positive harmonic function h in R there exists a unique mass distribution μ on Δ_1 such that $h(w) = \int_{\Delta_1} K(\hat{z}, w) d\mu_{\hat{z}}$. The measure μ is often referred to as the canonical measure on Δ_1 associated with h. The function $K(z, w)$ originally defined on $R_a \times R_a$, and then extended continuously onto $\hat{R}_a \times R_a$, shall be called the Martin kernel function on $\hat{R}_a \times R_a$.

Let μ be a mass distribution on R_a and $\hat{z} \in \hat{R}_a$. Then the function $V^{\mu}(\hat{z}) = \int_{R_a} K(\hat{z}, w) d\mu_w$, $(\not\equiv +\infty)$, shall be referred to as the Martin potential of μ ([7], p. 21). If $\hat{z} \in R_a$, then $V^{\mu}(\hat{z}) = \dfrac{U^{\mu}(\hat{z})}{G(\hat{z}, a)}$ where $U(\hat{z})$ is the Green potential of μ. In her thesis, Naïm ([7], p.23) introduced a definition of thinness for a set $E \subset R_a$ relative to any point $\hat{z} \in \hat{R}_a$.

According to Naïm's definition, E is thin at $\hat{z}$ if and only if either:

(i) $\hat{z}$ is an isolated point of $E \cup \{\hat{z}\}$ in the Martin topology

or

(ii) there exists a Martin potential V^{μ} on $\hat{R}_a$ such that

$$V^{\mu}(\hat{z}) < \varliminf_{\substack{z \to \hat{z} \\ z \in E}} V^{\mu}(z).$$

Naïm showed that R_a is thin at $\hat{z} \in \Delta - \Delta_1$ ([7], p. 25) and that for any $E \subset R_a$, it follows that E is thin at $\hat{z} \in \Delta_1$ if and only if $\mathcal{E}^{R_a - E}_{K_{\hat{z}}(w)} \not\equiv K_{\hat{z}}(w)$ ([7], p. 27, theorem 5). By making use of a kernel function which she designated by θ, Naim was able to define potentials for mass distributions on $\hat{R}_a$ rather than just on R_a. The function

https://doi.org/10.4153/CMB-1966-032-3 Published online by Cambridge University Press

$$\theta(z, w) = \frac{G(z, w)}{G(z, a)\, G(w, a)} \quad \text{on } R_a \times R_a$$

$$\theta(\hat{z}, w) = \frac{K(\hat{z}, w)}{G(w, a)} \quad \text{on } \hat{R}_a \times R_a$$

$$\theta(\hat{z}, \hat{w}) = \lim_{\substack{w \to \hat{w} \\ w \in R_a}} \theta(\hat{z}, w) \quad \text{on } \Delta \times \Delta_1$$

$$\theta(\hat{z}, \hat{w}) = \int_{\Delta_1} \theta(\hat{z}, \hat{w}')\, d\mu_{\hat{w}'}, \quad \text{on } \Delta \times (\Delta - \Delta_1) \text{ where } \mu \text{ is}$$

the canonical measure (on Δ_1) associated with $K_{\hat{w}}$.

The fine topology of Naïm on $\hat{R}_a$ is defined to be the least topology on $\hat{R}_a$ which makes the θ-potentials continuous ([7], p. 40). Under this topology the boundary of R_a turns out to be the minimal Martin boundary Δ_1. The fine topology of Naïm on $\hat{R}_a$ relativized to R_a coincides with the fine topology of Cartan and Brelot. It is also worth noting that if $z \in R_a$ then any θ-potential $U^{\mu}(z) = \int_{\hat{R}_a} \theta(z, \hat{w})\, d\mu_{\hat{w}}$ is of the form $\dfrac{v(z)}{G(z, a)}$ where $v(z)$ is a non-negative superharmonic function on R_a, and that if $\hat{z} \in \Delta_1$ then $U^{\mu}(\hat{z}) = \lim\limits_{\substack{z \to \hat{z} \\ z \in R_a}} \left(\dfrac{v(z)}{G(z, a)}\right) > 0$ or $+\infty$, ([7], p. 48, theorem 7'-16).

We close this section with a remark on the reference point a. As Parreau ([8], p. 151) has remarked, the Martin boundary Δ of R (as well as Δ_1) has only an apparent dependence on the point a. If a were replaced by $b \in R$, then the spaces $\hat{R}_a$ and $\hat{R}_b$ would be equivalent both as topological spaces and uniform spaces.

3. <u>A Theorem on Angular Derivatives.</u> We shall now state and prove our main theorem.

https://doi.org/10.4153/CMB-1966-032-3 Published online by Cambridge University Press

THEOREM. Let $f(z)$ be defined as above, and let $N \subset D = \{z: |z| < 1\}$ be an open subset relative to the usual topology and at the same time a deleted fine neighbourhood of $z = 1$ relative to the space $\bar{D} = \{z: |z| \leq 1\}$ endowed with the Naïm topology. Then $f(z)$ possesses a finite angular derivative at $z = 1$ if and only if the image of N under f (i.e., $f(N)$) is a deleted fine neighbourhood of $w = 1$ relative to the space $\bar{D}' = \{w: |w| \leq 1\}$ endowed also with the fine topology of Naïm.

Proof. Necessity Part. Suppose that $f(N)$ fails to be a deleted fine neighbourhood of $w = 1$. Then $\mathcal{E}^{f(N)}_{K_1(w)} \equiv K_1(w)$ ([7], p. 27, theorem 5) on $f(N)$, where $\mathcal{E}$ is the extremalisation operator of Brelot.

Since $K_1(w)$ is a solution of a generalized Dirichlet problem on $f(N)$, or quasi-bounded according to Parreau's terminology ([8], p. 164), therefore $K_1(f(z))$ is quasi-bounded on N. This follows immediately from the fact that any quasi-bounded harmonic function on $f(N)$ can be expressed as the limit of a monotone non-decreasing sequence of bounded harmonic functions on $f(N)$.

We now define $K^N_1(z) \equiv K_1(z) - \mathcal{E}^N_{K_1(z)}$ on N, and note that $K^N_1(z)$ is a positive minimal harmonic function on N ([7], p. 42, theorem 12). Since $K_1(f(z))$ is quasi-bounded on N, therefore $K_1(f(z))$ cannot dominate any minimal function of the form $\lambda K^N_1(z)$, $\lambda > 0$. This is a consequence of the fact that

$$F.\lim_{\substack{z \in N \\ z \to 1}} \frac{K_1(f(z))}{K^N_1(z)} = 0$$

([7], p. 49, theorem 8'-17), where $F.\lim$ denotes the fine limit or pseudo limit of Naïm, combined with the result that $z = 1$ may be regarded as a minimal Martin boundary point of both N and D ([7], p. 46, theorem 15).

Since $K^N_1(z) \leq K_1(z)$, it is not possible to choose $\lambda > 0$ such that $K_1 f(z) \geq \lambda K_1(z)$ on N, and hence on D. Combining this result with the generalized lemma of Julia, it follows that

$\lim\limits_{z \to 1} \dfrac{(1-|f(z)|)}{1-|z|} = +\infty$. The necessity of our theorem follows.

<u>Sufficiency Part.</u> We let $N = \{z: K_1(z) > 1\}$ and suppose that $f(N)$ is a deleted fine neighbourhood of $w = 1$. Then $H(w) \equiv K_1(w) - \mathcal{E}^{f(N)}_{K_1(w)}$ is a positive minimal harmonic function on $f(N)$ ([7], p. 42, theorem 12). Furthermore, $\lim\limits_{\substack{w \to w_o \\ w \in f(N)}} H(w) = 0$ where w_o is a boundary point of $f(N)$ in D', keeping in mind that any such point must be a regular boundary point of $f(N)$ ([8], pp. 111-113). Now let z_o be any boundary point of N in D. Since $f(z_o)$ is either a regular boundary point of $f(N)$ in D', or possibly an interior point of $f(N)$, therefore $\lim\limits_{\substack{z \to z_o \\ z \in N}} S(z) = 0$ where $S(z) \not\equiv 0$ in the singular part of $H(f(z))$.

Since the principle of positive singularities of Bouligand and Brelot is applicable to N, $S(z)$ is a positive minimal harmonic function with pole at $z = 1$ ([2], p. 120, theorem 4), and therefore of the form $\lambda_o K_1^N(z)$, $\lambda_o > 0$, where $K_1^N(z) \equiv K_1(z) - 1$ on N. Since $K_1(f(z)) \geq S(z)$ on N and hence $K_1(f(z)) \geq \lambda_o K_1^N(z)$ on N, it follows that $F.\lim\limits_{z \to 1} \dfrac{K_1(f(z))}{K_1(z)} \geq \lambda_o > 0$, because $F.\lim\limits_{z \to 1} \dfrac{K_1^N(z)}{K_1(z)} = 1$. Thus the canonical measure of $K_1(f(z))$ associated with $\{1\}$ is greater than or equal to λ_o ([7], p. 49, theorem 8'-17), and therefore $K_1 f(z) \geq \lambda_o K_1(z)$ on D. Combining this result with the converse of the generalized Julia lemma, it follows that $f(z)$ possesses a finite angular derivative at $z = 1$. This proves the sufficiency.

Since the concept of fine neighbourhood is conformally invariant, our theorem holds true if phrased in terms of the half plane, taking into account the minor modifications which must be made. For holomorphic functions on D, with less restrictive

https://doi.org/10.4153/CMB-1966-032-3 Published online by Cambridge University Press

conditions than those just considered, the theory of angular derivatives can, in certain instances, be associated with the Phragmén-Lindelöf principle. It is the writer's intention to consider this principle in a future work.

REFERENCES

1. L. Bieberbach, Lehrbuch der Funktionentheorie, Vol II, Chelsea, New York, 1945.

2. M. Brelot, Sur le principe des singularités positives et la topologie de R. S. Martin, Annales Univ. Grenoble, Math. Phys., 23 (1948), 113-138.

3. M. Brelot and J. L. Doob, Limites angulaires et limites fines, Annales de l'Institut Fourier de l'Univ. Grenoble, 13 (1963), Fasc. II, 395-415.

4. C. Carathéodory, Theory of Functions, Vol. II, Chelsea, New York, 1954.

5. A. Dinghas, Vorlesungen über Funktionentheorie, Springer, Berlin, 1961.

6. C. Gattengo and A. Ostrowski, Mém. des Sciences Math. Fasc. CIX, Gauthier-Villars, Paris, 1949.

7. L. Naïm, Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel, Thesis, Paris, 1957.

8. M. Parreau, Sur les moyennes des fonctions harmoniques et analytiques, Annales de l'Inst. Fourier de l'Univ. Grenoble, 3 (1951), 103-197.

9. M. Tsuji, Potential theory in modern function theory, Maruzen, Tokyo, 1959.

McMaster University

https://doi.org/10.4153/CMB-1966-032-3 Published online by Cambridge University Press