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Complex Monge-Ampère Measures of Plurisubharmonic Functions with Bounded Values Near the Boundary

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Abstract. We give a characterization of bounded plurisubharmonic functions by using their complex Monge-Ampère measures. This implies a both necessary and sufficient condition for a positive measure to be complex Monge-Ampère measure of some bounded plurisubharmonic function.

0 Introduction

We denote by $\mathrm{PSH}(\Omega)$ the set of all plurisubharmonic (psh) functions in a bounded, strictly pseudoconvex subset Ω of $\mathbb{C}^n$. We use the notations $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$. The complex Monge-Ampère operator $(dd^c)^n$ is well defined for all locally bounded psh functions, see [B-T2], and it plays a great role in pluripotential theory as the Laplace operator in classical potential theory. However, unlike the Laplace operator, the complex Monge-Ampère operator is nonlinear and cannot be defined without problem for all unbounded psh functions, see [K]. Several authors have therefore extended the domain of definition of the complex Monge-Ampère operator to some important classes of unbounded psh functions, see [B], [D], [C1], [C2] and [S]. Among these results, we like to mention that $(dd^c u)^n$ will be a positive Borel measure if the function $u \in \mathrm{PSH}(\Omega)$ is bounded near the boundary $\partial\Omega$.

In this paper we study characterization of Monge-Ampère measures of bounded psh functions in Ω. To handle this problem we consider the class $\mathcal{B}$ of psh functions u, which are bounded near the boundary and $(dd^c u)^n$ are absolutely continuous with respect to the capacity C_n introduced by Bedford and Taylor in [B-T2]. In Section 1 we obtain a comparison theorem for functions in $\mathcal{B}$. This theorem serves as a main tool in the proofs of this paper. In fact, the class $\mathcal{B}$ is natural in the sense that the proofs of comparison theorems in [B-T2] and [X] work without practically any change for functions in $\mathcal{B}$. In Section 2 we prove that any positive measure can be written as a Monge-Ampère measure of some function in $\mathcal{B}$ provided the measure is dominated by a Monge-Ampère measure of functions in $\mathcal{B}$. In Section 3 we characterize bounded psh functions by using their Monge-Ampère measures. As an application we prove a characterization of bounded radial psh functions given in [P]. Finally, in Section 4 we give a both necessary and sufficient condition for a positive measure to be complex Monge-Ampère measure of some bounded psh function. This implies a characterization of the positive measure μ such that each positive measure $f\,d\mu$ with $\int_\Omega f^p\,d\mu \le 1$ and $p > 1$ can be written as a complex Monge-Ampère measure of

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some bounded psh function, whose supremum norm is uniformly bounded by a constant depending on p.

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1 Continuity of $(dd^c)^n$ and a Comparison Theorem

We begin by studying continuity of the complex Monge-Ampère operator. Let C_n be the inner capacity given by Bedford and Taylor in [B-T2], as defined by $C_n(E) = C_n(E,\Omega) = \sup\{\int_E (dd^c u)^n ; u \in \mathrm{PSH}(\Omega), 0 < u < 1\}$ for any Borel subset E of Ω. A sequence of functions u_j is said to converge to a function u in C_n-capacity on a set E if for each constant $\delta > 0$ we have $C_n\{z \in E ; |u_j(z) - u(z)| > \delta\} \to 0$ as $j \to \infty$. In [X] we obtain that if locally uniformly bounded psh functions u_j converge to a psh function u in C_n-capacity on each $E \subset\subset \Omega$, then $(dd^c u_j)^n \to (dd^c u)^n$ weakly in Ω. We generalize now this result to psh functions which are bounded near the boundary $\partial\Omega$ and whose Monge-Ampère measures have small mass on any set of small C_n-capacity. Recall that positive measures μ_j are said to be *uniformly absolutely continuous* with respect to C_n in a set E if for any constant $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for each Borel subset $E' \subset E$ with $C_n(E') < \delta$ the inequality $\mu_j(E') < \varepsilon$ holds for all j. Now we can prove

Theorem 1 *Let $u \in \mathrm{PSH}(\Omega)$. Suppose that there exists a sequence of bounded* psh *functions u_j in Ω such that u_j are uniformly bounded near $\partial\Omega$ for all j, $(dd^c u_j)^n \ll C_n$ uniformly on each subset $E \subset\subset \Omega$ and $u_j \to u$ in C_n on each $E \subset\subset \Omega$. Then $(dd^c u_j)^n$ is weakly convergent to $(dd^c u)^n$ in Ω and $(dd^c u)^n \ll C_n$ on each $E \subset\subset \Omega$.*

Proof Since functions u_j are uniformly bounded near $\partial\Omega$ for all j then the limit function u is bounded near $\partial\Omega$ and hence $(dd^c u)^n$ is well defined as a positive Borel measure, see [B]. To see that $(dd^c u_j)^n \to (dd^c u)^n$ weakly in Ω, for a given smooth function ϕ with compact support in Ω, we write

$$\begin{aligned}\int_\Omega \phi[(dd^c u_j)^n - (dd^c u)^n] &= \int_\Omega \phi\big[(dd^c u_j)^n - \big(dd^c \max(u_j,-c)\big)^n\big] \\ &\quad + \int_\Omega \phi\big[\big(dd^c \max(u_j,-c)\big)^n - \big(dd^c \max(u,-c)\big)^n\big] \\ &\quad + \int_\Omega \phi\big[\big(dd^c \max(u,-c)\big)^n - (dd^c u)^n\big] \\ &\stackrel{\text{def}}{=} A_1 + A_2 + A_3.\end{aligned}$$

It turns out from Proposition 4.2 in [B-T3] that for each sufficiently large constant $c > 0$

$$\begin{aligned}|A_1| &= \Big|\int_{u_j \le -c} \phi\big[(dd^c u_j)^n - \big(dd^c \max(u_j,-c)\big)^n\big]\Big| \\ &\le \max_\Omega |\phi| \Big(\int_{u_j\le -c} (dd^c u_j)^n + \int_{u_j \le -c} \big(dd^c \max(u_j,-c)\big)^n\Big).\end{aligned}$$

Using Lemma 1 in [X] we have

$$\int_{u_j \le -c} \big(dd^c \max(u_j, -c)\big)^n \le \int_{u_j \le -c} \Big(-1 - \frac{2u_j}{c}\Big)^n \big(dd^c \max(u_j, -c)\big)^n$$
$$\le 2^n \int_{u_j < -c/2} \Big(-\frac{c}{2} - u_j\Big)^n \Big(dd^c \max\Big(\frac{u_j}{c}, -1\Big)\Big)^n$$
$$\le 2^n (n!)^2 \int_{u_j < -c/2} (dd^c u_j)^n.$$

Hence for each c large enough and all j we have proved the following estimation

$$|A_1| \le \big(1 + 2^n (n!)^2\big) \max_{\Omega} |\phi| \int_{u_j < -c/2} (dd^c u_j)^n.$$

Since $C_n\{u < -c/2\} \to 0$ as $c \to \infty$ and $u_j \to u$ in C_n we have that $C_n\{u_j < -c/2\}$ uniformly converge to zero for all j as $c \to \infty$. Hence the uniformly absolute continuity of $(dd^c u_j)^n$ implies that the last integral converges to zero uniformly for all j as $c \to \infty$. Thus, for any $\varepsilon > 0$ we can take a constant $c \ge 0$ such that $|A_1| \le \varepsilon$ for all j, and by Corollary 2.3 in [D] we can also require that $|A_3| \le \varepsilon$. However, for such a fixed constant c the convergence assumption implies that functions $\max(u_j, -c)$ converge to $\max(u, -c)$ in C_n on each $E \subset\subset \Omega$ as $j \to \infty$ and hence we conclude by Theorem 1 in [X] that $A_2 \to 0$ as $j \to \infty$. Therefore, we have shown that $(dd^c u_j)^n$ converges weakly to $(dd^c u)^n$.

It remains to show $(dd^c u)^n \ll C_n$ on any open set $E \subset\subset \Omega$. For any $\varepsilon > 0$ we choose $\delta > 0$ such that inequalities $(dd^c u_j)^n(E') \le \varepsilon$ hold for all j and all Borel sets $E' \subset E$ with $C_n(E') < \delta$. For such a subset E' we take an open set G with $E' \subset G \subset E$ and $C_n(G) < \delta$ and then choose a sequence of non-negative smooth functions ψ_k, which increase to the characteristic function of G in Ω. Then $\int_{E'} (dd^c u)^n \le \int_G (dd^c u)^n = \lim_{k\to\infty} \int_\Omega \psi_k (dd^c u)^n = \lim_{k\to\infty} \lim_{j\to\infty} \int_\Omega \psi_k (dd^c u_j)^n \le \overline{\lim}_{j\to\infty} \int_G (dd^c u_j)^n \le \varepsilon$. Hence $(dd^c u)^n \ll C_n$ on E and we have completed the proof of Theorem 1.

In this paper we denote by $\mathcal{B}$ the class of all psh functions u in Ω, which are bounded near the boundary $\partial\Omega$ and have absolutely continuous Monge-Ampère measures with respect to C_n on each $E \subset\subset \Omega$. The class $\mathcal{B}$ includes all limit functions u of Theorem 1. On the other hand, each function u in $\mathcal{B}$ is a decreasing limit of bounded functions $u_j = \max(u, -j)$. Applying the quasicontinuity of psh functions with respect to C_n, see [B-T2], and Dini's theorem, we obtain that $u_j \to u$ in C_n on each $E \subset\subset \Omega$. Hence the class $\mathcal{B}$ consists precisely of all functions u given in Theorem 1 as shown by the weak convergence $(dd^c u_j)^n \to (dd^c u)^n$ and the following fact.

Lemma 1 *Suppose that a sequence of bounded* psh *functions u_j in Ω decreases to a* psh *function u, which is bounded near the boundary $\partial\Omega$. If $(dd^c u)^n \ll C_n$ on any relatively compact subset of Ω then we have $(dd^c u_j)^n \ll C_n$ uniformly for all j on each $E \subset\subset \Omega$.*

Proof By the proof of Theorem 2.7 in [D] we have that $v(dd^c u_j)^n \to v(dd^c u)^n$ weakly in Ω for any locally bounded psh function v on Ω. Thus, Lemma 1 follows directly from Theorem 3.2 in [B-T3].

Bedford and Taylor in [B-T2] proved the comparison theorem for bounded psh function, which has wide application on the Dirichlet problem. In [X] we have obtained a stronger inequality than the comparison theorem. Now we generalize it to functions in $\mathcal{B}$.

Lemma 2 *If $u, v \in \mathcal{B}$ satisfy $\underline{\lim}_{z\to\partial\Omega}\big(u(z) - v(z)\big) \geq 0$, then for any constant $r \geq 1$ and all $w_j \in \mathrm{PSH}(\Omega)$ with $0 \leq w_j \leq 1$, $j = 1, 2, \dots, n$, we have*

$$\frac{1}{(n!)^2}\int_{u<v}(v-u)^n\, dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{u<v}(r-w_1)(dd^c v)^n \leq \int_{u<v}(r-w_1)(dd^c u)^n.$$

Therefore, under the additional assumption $(dd^c v)^n \geq (dd^c u)^n$ in Ω, we obtain that the set $\{u < v\}$ is empty.

Proof We may assume that there exists a subset $E \subset\subset \Omega$ such that $\{u < v\} \subset E$. Otherwise, replace u by $u + 2\delta$ and then let $\delta \searrow 0$. Write $u_k = \max(u, -k)$ and $v_j = \max(v, -j)$. Then $\{u_k < v_j\} \subset E$ for sufficiently large k and j. By Lemma 1 in [X] we have that for any constant $r \geq 1$ and all $w_j \in \mathrm{PSH}(\Omega)$ with $0 \leq w_j \leq 1$, $j = 1, 2, \dots, n$

$$\frac{1}{(n!)^2}\int_{u_k<v_j}(v_j-u_k)^n\, dd^c w_1\wedge\cdots\wedge dd^c w_n+\int_{u_k<v_j}(r-w_1)(dd^c v_j)^n \leq \int_{u_k<v_j}(r-w_1)(dd^c u_k)^n,$$

where k and j are large enough. Since $u_k \searrow u$ then $(dd^c u_k)^n \to (dd^c u)^n$ weakly and by Lemma 1 we have that $(dd^c u_k)^n \ll C_n$ uniformly for all k in the set E. Similarly, $(dd^c v_j)^n \ll C_n$ uniformly for all j in E. Letting $j \to \infty$ and then $k \to \infty$, we can easily get the required inequality by the same argument as in the proof of Lemma 1 of [X]. Thus the proof is complete.

2 Range of $(dd^c)^n$

Now we begin to discuss the range of the complex Monge-Ampère operator. We need a lemma.

Lemma 3 *If $v \in \mathcal{B}$ and f is a non-negative continuous function with compact support in Ω, then there exists a function u in $\mathcal{B}$ such that $(dd^c u)^n = f(dd^c v)^n$ and $\lim_{z\to\partial\Omega} u(z) = 0$.*

Proof Suppose that $\rho(z)$ be a defining function of Ω and that $|v(z)| \leq a$ in a neighborhood of $\Omega \setminus \Omega'$, where $\operatorname{supp} f \subset\subset \Omega' \subset\subset \Omega$. For a sufficiently large constant b we define

$$\bar{v}(z) = \begin{cases} \max\big(v(z) - a - 1, b\rho(z)\big) & \text{in } \Omega \setminus \Omega' \,; \\ v(z) - a - 1 & \text{in } \Omega'. \end{cases}$$

Then it is easy to see that $\bar{v} \in \mathcal{B}$, $\lim_{z\to\partial\Omega} \bar{v}(z) = 0$ and $f(dd^c\bar{v})^n = f(dd^c v)^n$. So without loss of generality we may assume that $\lim_{z\to\partial\Omega} v(z) = 0$ and $0 \leq f \leq 1$. Choose a decreasing sequence of smooth psh functions v_j which vanish on $\partial\Omega$ and decrease to the v in Ω. So $f(dd^c v_j)^n \to f(dd^c v)^n$ weakly and $v_j \to v$ in C_n on any relatively compact subset of Ω, see [B-T2]. Since every $f(dd^c v_j)^n$ can be considered as a bounded continuous function times Lebesgue measure in Ω it follows from [B-T1] that there exists $u_j \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $(dd^c u_j)^n = f(dd^c v_j)^n$, and $u_j(z) = 0$ on $\partial\Omega$. Since the comparison theorem in [B-T2] gives the inequality $0 \geq u_j \geq v_j \geq v$ with $v(z) = 0$ on $\partial\Omega$, then by passing to a subsequence we may assume that u_j converge to a psh function u in Ω almost everywhere with respect to Lebesgue measure, where u vanishes on $\partial\Omega$. On the other hand, $(dd^c u_j)^n \to f(dd^c v)^n$ weakly and by Lemma 1 we have that $(dd^c u_j)^n \ll C_n$ uniformly for all j on any relatively compact subset of Ω. Therefore, to see $(dd^c u)^n = f(dd^c v)^n$ it is enough to show that $u_j \to u$ in C_n on Ω. Now for any given $\delta > 0$ we choose a strictly pseudoconvex set E with $\operatorname{supp} f \subset\subset E \subset\subset \Omega$ such that $|u(z) - u_j(z)| < \delta$ for all $z \in \Omega \setminus E$ and all j. It follows from the quasi-continuity of psh functions, see [B-T2] that for each positive constant $\varepsilon < \delta$ there exists an open set $U \subset E$ with $C_n(U) < \varepsilon$ such that both u and v are continuous in $E \setminus U$ and hence they are bounded, say $u > -c$ and $v > -c$ on $E \setminus U$. Since $u = (\overline{\lim}_{j\to\infty} u_j)^*$, it turns out from Hartog's Lemma that

$$u(z) + \delta > u(z) + \varepsilon \geq u_j(z)$$

holds for all $z \in E \setminus U$ and $j \geq j_0$. So for such $j \geq j_0$ we have

$$\begin{aligned}
&C_n\{z \in \Omega\,;\, |u(z) - u_j(z)| > 4\delta\} \\
&\leq C_n\{z \in E\,;\, |u(z) + \delta - u_j(z)| > 3\delta\} \\
&\leq C_n\{z \in E\,;\, u(z) + \delta - u_j(z) > 3\delta\} + C_n(U) \\
&\leq \sup\Big\{\int_{u-u_j>2\delta} \Big(\frac{u-u_j-\delta}{\delta}\Big)^n (dd^c w)^n\,;\, w \in \mathrm{PSH}(\Omega), 0 < w < 1\Big\} + \varepsilon \\
&\leq \sup\Big\{\frac{1}{\delta^n}\int_{u>u_j+\delta} (u-u_j-\delta)^n (dd^c w)^n\,;\, w \in \mathrm{PSH}(\Omega), 0 < w < 1\Big\} + \varepsilon \\
&\leq \sup\Big\{\frac{1}{\delta^n}\,\underline{\lim}_{k\to\infty}\int_{\max(u,-k)>u_j+\delta} \big(\max(u,-k) - u_j - \delta\big)^n (dd^c w)^n\,;\, w \in \mathrm{PSH}(\Omega), 0 < w < 1\Big\} \\
&\qquad + \varepsilon.
\end{aligned}$$

The last inequality follows from Fatou Lemma. Hence, by Lemma 2 we have

$$\begin{aligned}
C_n\{z \in \Omega\,;\, |u(z) - u_j(z)| > 4\delta\} &\leq \frac{(n!)^2}{\delta^n}\,\underline{\lim}_{k\to\infty}\int_{\max(u,-k)>u_j+\delta} (dd^c u_j)^n + \varepsilon \\
&= \frac{(n!)^2}{\delta^n}\int_{u>u_j+\delta} (dd^c u_j)^n + \varepsilon \\
&\leq \frac{(n!)^2}{\delta^{n+1}}\int_{\{u>u_j+\delta\}\setminus U} (u-u_j) f(dd^c v_j)^n + O\Big(\int_U (dd^c u_j)^n\Big) + \varepsilon \\
&\leq \frac{(n!)^2}{\delta^{n+1}}\int_{\{u>u_j+\delta\}\setminus U} (\varepsilon + u - u_j) f(dd^c v_j)^n \\
&\qquad + O\Big(\int_U (dd^c v_j)^n\Big) + \varepsilon.
\end{aligned}$$

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Let $\rho_1(z)$ be a defining function of the strictly pseudoconvex set E. We define $\bar{u} = \max\bigl(u, a\rho_1(z)\bigr)$ and $\bar{u}_j = \max\bigl(u_j, a\rho_1(z)\bigr)$ in a neighborhood E' of E, which contains the set $\{u > u_j + \delta\}$. Since $u > -c$ and $u_j \geq v_j \geq v > -c$ on $E \setminus U$, then for sufficiently large constant a we have (i) $\bar{u}_j = u_j$ and $\bar{u} = u$ on an open neighborhood of supp f but outside U; (ii) all $\bar{u}_j = \bar{u} = a\rho_1(z)$ in $E' \setminus E$; (iii) $\{\bar{u}_j\}$ is uniformly bounded in E'; (iv) $\bar{u}_j \to \bar{u}$ in $L(E')$. Since the uniformly bounded functions $\bar{u}_j$ converge to $\bar{u}$ in $L(E')$ and $(dd^c v_j)^n \ll C_n$ uniformly for all j on E', it follows from Hartog's Lemma that there exits a subset U_1 of E and an integer $j_1 \geq j_0$ such that $\int_{U_1} |\varepsilon + \bar{u} - \bar{u}_j| (dd^c v_j)^n < \varepsilon$ and $\bar{u} + \varepsilon > \bar{u}_j$ on $E \setminus U_1$ for $j \geq j_1$. Hence for $j \geq j_1$ the last sum does not exceed the following

$$
\begin{aligned}
&\frac{(n!)^2}{\delta^{n+1}} \int_{\{u>u_j+\delta\}\setminus U_1} (\varepsilon + \bar{u} - \bar{u}_j)(dd^c v_j)^n + O\Bigl(\int_U (dd^c v_j)^n + \varepsilon\Bigr) \\
&\qquad \leq \frac{(n!)^2}{\delta^{n+1}} \int_{E\setminus U_1} (\varepsilon + \bar{u} - \bar{u}_j)(dd^c v_j)^n + O\Bigl(\int_U (dd^c v_j)^n + \varepsilon\Bigr) \\
&\qquad = \frac{(n!)^2}{\delta^{n+1}} \int_{E} (\bar{u} - \bar{u}_j)(dd^c v_j)^n + O\Bigl(\int_U (dd^c v_j)^n + \varepsilon\Bigr).
\end{aligned}
$$

By Proposition 4.2 in [B-T3] for each constant $d > 0$ and any integer $k > 0$ we have

$$
\begin{aligned}
\int_E (\bar{u} - \bar{u}_j)(dd^c v_k)^n &= \int_{E\cap\{v_k>-d\}} (\bar{u} - \bar{u}_j)\bigl(dd^c \max(v_k, -d)\bigr)^n \\
&\quad + \int_{E\cap\{v_k\leq -d\}} (\bar{u} - \bar{u}_j)(dd^c v_k)^n \\
&= \int_E (\bar{u} - \bar{u}_j)\bigl(dd^c \max(v_k, -d)\bigr)^n \\
&\quad - \int_{E\cap\{v_k\leq -d\}} (\bar{u} - \bar{u}_j)\bigl(dd^c \max(v_k, -d)\bigr)^n \\
&\quad + \int_{E\cap\{v_k\leq -d\}} (\bar{u} - \bar{u}_j)(dd^c v_k)^n.
\end{aligned}
$$

Applying the uniformly absolute continuity of $(dd^c v_k)^n$ on E and the proof of Theorem 1, the last two integrals converge to zero uniformly for all j and k as $d \to \infty$. Hence

$$
\int_E (\bar{u} - \bar{u}_j)(dd^c v_k)^n = \int_E (\bar{u} - \bar{u}_j)\bigl(dd^c \max(v_k, -d)\bigr)^n + o(1)
$$

uniformly for all j and k as $d \to \infty$. Therefore, we get

$$
\begin{aligned}
&C_n\{z \in \Omega\,;\, |u(z) - u_j(z)| > 4\delta\} \\
&\qquad \leq \frac{(n!)^2}{\delta^{n+1}} \int_E (\bar{u} - \bar{u}_j)\bigl(dd^c \max(v_j, -d)\bigr)^n + O\Bigl(\int_U (dd^c v_j)^n + \varepsilon\Bigr)
\end{aligned}
$$

$$= \frac{(n!)^2}{\delta^{n+1}} \int_E (\bar{u} - \bar{u}_j)\big[\big(dd^c \max(v_j, -d)\big)^n - \big(dd^c \max(v_k, -d)\big)^n\big]$$

$$+ \frac{(n!)^2}{\delta^{n+1}} \int_E (\bar{u} - \bar{u}_j)(dd^c v_k)^n + O\Big(\int_U (dd^c v_j)^n + \varepsilon\Big)$$

$$= A_1 + A_2 + O\Big(\int_U (dd^c v_j)^n + \varepsilon\Big)$$

uniformly for all $j \geq j_1$ and all k as $d \to \infty$. Using an integration by parts we have

$$A_1 = \frac{(n!)^2}{\delta^{n+1}} \int_{E'} \big(\max(v_j, -d) - \max(v_k, -d)\big)(dd^c \bar{u} - dd^c \bar{u}_j)$$

$$\wedge \sum_{l=0}^{n-1} \big(dd^c \max(v_j, -d)\big)^{n-1-l} \wedge \big(dd^c \max(v_k, -d)\big)^l,$$

where for each fixed d the measure has a relatively compact support in E' and is absolutely continuous with respect to C_n, and the integrand $\max(v_j, -d) - \max(v_k, -d) \to 0$ in C_n on each relatively compact subset of E' as $j, k \to \infty$. Hence $A_1 \to 0$ as $j, k \to \infty$. On the other hand, it follows from $\bar{u}_j \to \bar{u}$ in $L(E')$ that for any fixed k we have $A_2 \to 0$ as $j \to \infty$. Finally, letting $\varepsilon \to 0$ and applying the fact that $(dd^c v_j)^n \ll C_n$ uniformly on E we conclude that $u_j \to u$ in C_n on Ω and thus the proof of Lemma 3 is complete.

Theorem 2 *If $v \in \mathcal{B}$ and a positive measure $\mu \leq (dd^c v)^n$ on Ω, then there exists a function u in $\mathcal{B}$ such that $(dd^c u)^n = \mu$ in Ω. Furthermore, if $\lim_{z \to \partial\Omega} v(z) = 0$ then there exists a unique function u in $\mathcal{B}$ such that $(dd^c u)^n = \mu$ and $\lim_{z \to \partial\Omega} u(z) = 0$.*

Proof By Lebesgue-Radon-Nikodym theorem we can write $\mu = f(dd^c v)^n$, where $0 \leq f \leq 1$ in Ω. Choose a sequence of non-negative, bounded functions f_k with compact support in Ω which increase to f in Ω. Then for each f_k there exists a sequence of continuous functions $f_{k,j}$ such that $0 \leq f_{k,j} \leq g_k$ and

$$\int_\Omega |f_{k,j} - f_k|(dd^c v)^n \to 0 \quad \text{as } j \to \infty,$$

where g_k is a non-negative, bounded function with compact support in Ω. Therefore, by Lemma 3 there exist functions $u_{k,j}$ in $\mathcal{B}$ with $(dd^c u_{k,j})^n = f_{k,j}(dd^c v)^n$ and $\lim_{z \to \partial\Omega} u_{k,j}(z) = 0$. Take a function $v_k \in \mathcal{B}$ such that $\lim_{z \to \partial\Omega} v_k(z) = 0$ and $g_k(dd^c v_k)^n = g_k(dd^c v)^n \geq (dd^c u_{k,j})^n$. Then by Lemma 2 we have $(\sup_\Omega g_k)^{1/n} v_k \leq u_{k,j} \leq 0$ in Ω for all j. Now applying Lemma 2 and repeating the proof of Theorem 4 in [X] we can find functions $u_k \in \mathcal{B}$ such that $(dd^c u_k)^n = f_k(dd^c v)^n$ and $\lim_{z \to \partial\Omega} u_k(z) = 0$. Therefore, Lemma 2 yields that u_k decrease to a psh function u in Ω which is clearly the desired function in $\mathcal{B}$. If the $v = 0$ on $\partial\Omega$, by Lemma 2 we have that $0 \geq u_k \geq v$ in Ω for all k. Hence the u vanishes on $\partial\Omega$. The uniqueness of such a solution u follows directly from Lemma 2. So the proof of Theorem 2 is complete.

As a consequence of Theorem 2 and Lemma 2 we obtain the following result in [KO1].

Corollary 1 *Assume that a positive measure $\mu \leq (dd^c v)^n$ on Ω, where v is a bounded* psh *function in Ω. Then there exists a bounded* psh *function u in Ω such that $(dd^c u)^n = \mu$.*

It is probably worth remarking that for a bounded psh function v in Ω the proof of Lemma 3 can be simplicized. This gives a simple proof of Corollary 1. On the other hand, the assumption $\mu \leq (dd^c v)^n$ in Theorem 2 can not be weaken by $\mu \ll (dd^c v)^n$, as shown by the following example.

Example 1 Let $\{z_j\}$ be a sequence of distinguished points which converges to a point $\zeta \in \partial\Omega$. By Theorem 8 in [C-P], for each z_j there exists a function $f_{j,r} \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$ which vanishes on the boundary $\partial\Omega$ and satisfies $(dd^c f_{j,r})^n = d_n^{-1} r^{-2n} j^{-2} \chi_{B(z_j,r)}\, d\lambda$, where the constant d_n denotes the volume of the unit ball in $\mathbb{C}^n$, λ is the Lebesgue measure and $\chi_{B(z_j,r)}$ is the characteristic function of the open ball $B(z_j,r) = \{z \in \mathbb{C}^n ; |z - z_j| < r\}$. It then follows from the definition of C_n-capacity that

$$\frac{1}{j^2} = \int_\Omega (dd^c f_{j,r})^n = \int_{B(z_j,r)} (dd^c f_{j,r})^n \leq C_n\big(B(z_j,r), B(z_j,k)\big) \max_{z \in B(z_j,k)} \big(-f_{j,r}(z)\big)^n,$$

where the constant $k > r > 0$. Since for each fixed $k > 0$ we have that the relative capacity $C_n\big(B(z_j,r), B(z_j,k)\big) \to 0$ as $r \to 0$, then $\max_{z \in B(z_j,k)} \big(-f_{j,r}(z)\big) \to \infty$ as $r \to 0$. Take two sequences $\{k_j\}$ and $\{r_j\}$ such that $B(z_j,k_j)$ for $j = 1, 2, \ldots$ are pairwise disjoint balls in Ω and $\max_{z \in B(z_j,k_j)} \big(-f_{j,r_j}(z)\big) \to \infty$ as $j \to \infty$. Hence the locally bounded function $f \stackrel{\text{def}}{=} \sum_{j=1}^\infty d_n^{-1} r_j^{-2n} j^{-2} \chi_{B(z_j,r_j)}$ is integrable in Ω with respect to the Lebesgue measure λ. It is now easy to see that there exists no function $u \in \mathrm{PSH}(\Omega)$ which is bounded near $\partial\Omega$ and satisfies $(dd^c u)^n = f\, d\lambda$. In fact, if there exists such a function u, by subtracting a constant if necessary, we may assume $u < -1$ in Ω. So for every j we have that $u \leq f_{j,r_j}$ near the boundary $\partial\Omega$ and $(dd^c u)^n = f\, d\lambda \geq (dd^c f_{j,r_j})^n$. Hence Lemma 2 yields $u(z) \leq f_{j,r_j}(z)$ for all $z \in \Omega$. In particular, we get $\max_{z \in B(z_j,k_j)} \big(-u(z)\big) \geq \max_{z \in B(z_j,k_j)} \big(-f_{j,r_j}(z)\big) \to \infty$ as $j \to \infty$, which contradicts that u is bounded near $\partial\Omega$. Therefore, we have proved that there exists no function $u \in \mathrm{PSH}(\Omega)$, which is bounded near $\partial\Omega$ and satisfies $(dd^c u)^n = f\, d\lambda$.

3 Bounded Plurisubharmonic Functions

In this section we discuss characterization of bounded psh functions in terms of Monge-Ampère measures.

Theorem 3 *Suppose that u is a* psh *function in Ω and satisfies $u(z) \geq B$ near the boundary $\partial\Omega$, where B is a constant. Then u is bounded below in the whole domain Ω if and only if there exists a constant $A_u > 0$ such that for any constant $k < B$ with $C_n(u < k) \neq 0$ we can find an increasing sequence $k \leq k_1 < \cdots < k_{s-1} < k_s = B$ with $k_1 < k + 1$ and*

$$\sum_{j=2}^{s} \left(\frac{\|(dd^c u)^n\|_{\{u<k_j\}}}{C_n(u < k_{j-1} + 0)} \right)^{\frac{1}{n}} < A_u,$$

where $C_n(u < k_{j-1} + 0) = \lim_{k \to k_{j-1}+0} C_n(u < k)$.

Proof The necessity is trivial because for each bounded function u, with $u > B$ near $\partial\Omega$, one can choose two constants $k_1 < k_2 = B$ such that the condition $C_n(u < k) \neq 0$ implies

$k_1 < k + 1$. To see the sufficiency, we assume that $C_n(u < k) \neq 0$ for all $k < B$. Otherwise, we have $u \geq k$ for some constant k and the proof is finished. We notice that the assumption of Theorem 3 gives

$$\frac{\|(dd^c u)^n\|_{\{u<k\}}}{C_n(u < k + 1)} \leq \frac{\|(dd^c u)^n\|_{\{u<k_2\}}}{C_n(u < k_1 + 0)} \leq A_u^n.$$

So

$$\|(dd^c u)^n\|_{\{u<k\}} \to 0 \quad \text{as } k \to -\infty,$$

and together with the inequality

$$\|(dd^c u)^n\|_E \leq \|(dd^c u)^n\|_{\{u\leq k\}} + \|\big(dd^c \max(u, k)\big)^n\|_E$$

for each subset $E \subset\subset \Omega$ we get that $(dd^c u)^n$ is absolutely continuous with respect to C_n. Hence $u \in \mathcal{B}$ and it then follows from Lemma 2 that for all $k < k_j$ and each $w \in \mathrm{PSH}(\Omega)$ with $0 < w < 1$ we have

$$(k_j - k)^n \int_{u<k} (dd^c w)^n \leq \int_{u<k_j} (k_j - u)^n (dd^c w)^n \leq \int_{u<k_j} (1 - w)(dd^c u)^n.$$

Let $k \to k_{j-1} + 0$ and we have

$$(k_j - k_{j-1})^n C_n(u < k_{j-1} + 0) \leq \|(dd^c u)^n\|_{\{u<k_j\}}.$$

Therefore

$$0 < B - 1 - k < k_s - k_1 = \sum_{j=2}^{s} (k_j - k_{j-1}) \leq \sum_{j=2}^{s} \left(\frac{\|(dd^c u)^n\|_{\{u<k_j\}}}{C_n(u < k_{j-1} + 0)} \right)^{\frac{1}{n}} < A_u.$$

This implies $C_n\{u < B - 1 - A_u\} = 0$ which contradicts the assumption that $C_n(u < k) \neq 0$ for all $k < B$. The proof of Theorem 3 is complete.

As a consequence we have

Corollary 2 *Let $u \in \mathrm{PSH}(\Omega)$ be bounded near the boundary $\partial\Omega$. If there exist constants $\delta > 1$ and $A > 0$ such that the inequality*

$$\|(dd^c u)^n\|_{\{u<k\}} \leq A\big(C_n(u < k)\big)^{\delta}$$

holds for any constant $k < 0$, then u is bounded in Ω.

Proof We assume without loss of generality that $u > -1$ near $\partial\Omega$. For each $k < -1$ with $C_n\{u < k\} \neq 0$ it is clear that there exists at most a finite numbers of constants $k = k_1 < k_2 < \cdots < k_s = -1$ such that

$$k_j = \inf\Big\{ r\,;\, F(k_{j-1} + 0) < \frac{1}{2} F(r) \Big\} \quad \text{for } j = 2, 3, \ldots, s - 1, \text{ and } \frac{1}{2} F(k_s) \leq F(k_{s-1} + 0),$$

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where the function $F(r) = \|(dd^c u)^n\|_{\{u<r\}}$ is nondecreasing and left continuous for $r \leq 1$, and $F(r+0) = \lim_{t\to r+0} F(t)$. Hence we have

$$\frac{1}{2}F(k_j) \leq F(k_{j-1}+0) < \frac{1}{2}F(k_{j+1}) \quad \text{for } j = 2, 3, \dots, s-1,$$

and

$$\sum_{j=2}^{s}\left(\frac{\|(dd^c u)^n\|_{\{u<k_j\}}}{C_n(u<k_{j-1}+0)}\right)^{\frac{1}{n}} \leq \sum_{j=2}^{s}\left(\frac{2F(k_{j-1}+0)}{C_n(u<k_{j-1}+0)}\right)^{\frac{1}{n}} \leq \sum_{j=2}^{s}\left(2A^{\frac{1}{\delta}}F(k_{j-1}+0)^{\frac{\delta-1}{\delta}}\right)^{\frac{1}{n}}$$
$$\leq 2^{\frac{1}{n}}A^{\frac{1}{\delta n}}\sum_{j=2}^{s-1}\left(\frac{F(-1)}{2^{\frac{s-j-1}{2}}}\right)^{\frac{\delta-1}{\delta n}} \leq 2^{\frac{1}{n}}A^{\frac{1}{\delta n}}F(-1)^{\frac{\delta-1}{\delta n}}\sum_{j=0}^{\infty}2^{\frac{(1-\delta)j}{2\delta n}} < \infty.$$

Therefore, an application of Theorem 3 completes the proof.

By the definition of C_n-capacity we know that the Monge-Ampère measure of a bounded psh function is dominated by a constant multiple of C_n-capacity. However, we can not expect that the Monge-Ampère measure of a bounded psh function is always controlled by C_n-capacity with some power $\delta > 1$, as be shown in the following example.

Example 2 We construct a bounded subharmonic function

$$u(z) = \sum_{k=2}^{\infty}\frac{1}{k^2 2^k}\max\left(-\sqrt{-\ln|z|}, -2^k\right)$$

in the ball $B(0, 1/2)$ of $\mathbb{C}$. For any small $r > 0$ we take an integer j_0 such that $2^{j_0-2} \leq \sqrt{-\ln r} < 2^{j_0-1}$. Since the inequality $j^2 2^j < 100\sqrt{-\ln|z|}\ln^2(-\ln|z|)$ holds for all $z \in E_j = \{2^{j-1} \leq \sqrt{-\ln|z|} < 2^j\}$, we have

$$\|dd^c u\|_{B(0,r)} \geq \sum_{j=j_0}^{\infty}\|dd^c u\|_{E_j} \geq \sum_{j=j_0}^{\infty}\frac{1}{j^2 2^j}\|dd^c\max\left(-\sqrt{-\ln|z|}, -2^j\right)\|_{E_j}$$
$$= \sum_{j=j_0}^{\infty}\frac{1}{j^2 2^j}\|dd^c\sqrt{-\ln|z|}\|_{E_j} \geq \frac{1}{400}\sum_{j=j_0}^{\infty}\left\|\frac{dz\wedge d^c z}{|z|^2\ln^2|z|\ln^2(-\ln|z|)}\right\|_{E_j}$$
$$\geq \frac{1}{400}\left\|\frac{dz\wedge d^c z}{|z|^2\ln^2|z|\ln^2(-\ln|z|)}\right\|_{\{r^8\leq|z|<r^4\}}$$
$$\geq \frac{1}{400\ln^2(-8\ln r)}\left\|\frac{dz\wedge d^c z}{|z|^2\ln^2|z|}\right\|_{\{r^8\leq|z|<r^4\}}$$
$$\geq A\ln^{-2}(-8\ln r)C_1\left(B(0,r)\right),$$

where the last inequality follows from $C_1\{B(0,r)\} = 2\pi/(-\ln 2 - \ln r)$ and the constant A is independent of r. Hence for any $\delta > 1$ there is no constant $A_1 > 0$ such that $\|dd^c u\|_E \leq A_1\left(C_1(E)\right)^{\delta}$ for all subsets E of $B(0, 1/2)$.

Example 2 gives that the inequality assumption of Corollary 2 is not necessary condition. On the other hand, we have a local estimation for the Monge-Ampère measure, see [B-T4, Corollary 2.3] for the case $n = 1$.

Theorem 4 *If the* psh *function u is bounded in Ω then for each $z_0 \in \Omega$*

$$\|(dd^c u)^n\|_{B(z_0,r)} = \mathrm{o}\big(C_n\{B(z_0,r)\}\big) \quad \text{as } r \to 0,$$

where $B(z_0, r)$ denotes the ball with center at z_0 and radius $r > 0$.

Proof Take a positive constant $r_0 < 1$ which satisfies $B(z_0, r_0) \subset\subset \Omega$. By Lemma 2 we have

$$\begin{aligned}
&\int_{B(z_0,r_0)} (\ln r_0 - \ln|z - z_0|)^n (dd^c u)^n \\
&= (\max_\Omega |u|)^n \lim_{k\to\infty} \int_{\max(\ln|z-z_0|,-k)<\ln r_0} \big(\ln r_0 - \max(\ln|z - z_0|, -k)\big)^n \left(dd^c \frac{u}{\max_\Omega |u|}\right)^n \\
&\le (n!)^2 (\max_\Omega |u|)^n \lim_{k\to\infty} \int_{\max(\ln|z-z_0|,-k)<\ln r_0} \big(dd^c \max(\ln|z - z_0|, -k)\big)^n \\
&= \big(n!\big)^2 (2\pi \max_\Omega |u|)^n < \infty.
\end{aligned}$$

So the function $\big(\ln r_0 - \ln|z - z_0|\big)^n$ is integrable in $B(z_0, r_0)$ with respect to the measure $(dd^c u)^n$, and it then follows from $\|(dd^c u)^n\|_{B(z_0,r)} = \mathrm{O}\big(C_n\{B(z_0,r)\}\big) = \mathrm{o}(1)$ as $r \to 0$ that

$$(\ln r_0 - \ln r)^n \|(dd^c u)^n\|_{B(z_0,r)} \le \int_{B(z_0,r)} (\ln r_0 - \ln|z - z_0|)^n (dd^c u)^n \to 0 \quad \text{as } r \to 0$$

which implies the conclusion of Theorem 4 because $(\frac{1}{-\ln r})^n = \mathrm{O}(C_n\{B(z_0, r)\})$.

It is now natural to ask whether or not the inequality assumption in Corollary 2 can be replaced by the weaker condition $\|(dd^c u)^n\|_{\{u<k\}} = \mathrm{o}\big(C_n(u < k)\big)$ as $k \to -\infty$ or $\|(dd^c u)^n\|_{B(z_0,r)} = o\big(C_n\{B(z_0, r)\}\big)$ as $r \to 0$ for all points $z_0 \in \Omega$. The answer is negative, as the following example shows.

Example 3 Let $n = 1$. Since $\phi(x) = -\ln\big(\ln(-x)\big)$ is increasing and convex for $x < -1$, the unbounded function $u(z) = \phi(\ln|z|) = -\ln\big(\ln(-\ln|z|)\big)$ is subharmonic in the ball $B(0, 1/3)$ and bounded near the sphere $|z| = 1/3$. We claim that the measure $dd^c u$ puts no mass at the origin. To see this we assume that ψ is a nonnegative C^∞ function with compact support in $B(0, 1/3)$ and satisfies $\psi(0) = 1$. By Stokes' theorem we have

$$\begin{aligned}
\int_{B(0,\frac{1}{3})} \psi\, dd^c u &= \int_{B(0,\frac{1}{3})} u\, dd^c \psi = \lim_{\varepsilon\to 0} \int_{\varepsilon<|z|<\frac{1}{3}} u\, dd^c \psi \\
&= \lim_{\varepsilon\to 0} \left\{ \int_{\varepsilon<|z|<\frac{1}{3}} \psi\, dd^c u + \int_{|z|=\varepsilon} u\, d^c \psi - \psi\, d^c u \right\} \\
&= \int_{0<|z|<\frac{1}{3}} \psi\, dd^c u + \lim_{\varepsilon\to 0} \int_{|z|=\varepsilon} u\, d^c \psi - \psi\, d^c u,
\end{aligned}$$

where the last term vanishes because

$$\int_{|z|=\varepsilon} u\, d^c\psi - \psi\, d^c u = \mathrm{O}\Big(\varepsilon \ln\big(\ln(-\ln\varepsilon)\big)\Big) + \mathrm{O}\Big(\frac{1}{-\ln\varepsilon \ln(-\ln\varepsilon)}\Big) \quad \text{as } \varepsilon \to 0.$$

Hence $\|dd^c u\|_{\{0\}} = 0$. On the other hand, a direct calculation gives

$$dd^c u = \frac{1 + \ln(-\ln|z|)}{4|z|^2 \ln^2|z| \ln^2(-\ln|z|)}\, dz \wedge d^c z \quad \text{for } z \neq 0.$$

Then

$$\begin{aligned}\int_{B(0,r)} dd^c u &\le \int_{0<|z|<r} \frac{1}{2|z|^2 \ln^2|z| \ln\big(-\ln|z|\big)}\, dz \wedge d^c z \\ &\le \frac{1}{2\ln\big(-\ln r\big)} \int_{B(0,r)} \frac{1}{|z|^2 \ln^2|z|}\, dz \wedge d^c z = \mathrm{o}\Big(C_1\{B(0,r)\}\Big) \quad \text{as } r \to 0,\end{aligned}$$

which implies obviously that both $\|(dd^c u)^n\|_{\{u<k\}} = \mathrm{o}\big(C_1(u<k)\big)$ as $k \to -\infty$ and $\|(dd^c u)^n\|_{B(z_0,r)} = \mathrm{o}\Big(C_1\{B(z_0,r)\}\Big)$ as $r \to 0$ for all points z_0 in $B(0,1/3)$.

Now we give a positive result on this direction.

Theorem 5 *Suppose that $u \in \mathcal{B}$ satisfies $u(z) \ge k_2$ for all z near the boundary $\partial\Omega$. If there exist constants $k_0 < k_1 \le k_2$ and $A_0 < (k_1 - k_0)^n$ such that*

$$\|(dd^c u)^n\|_{\{u<k_1\}} = A_0 C_n(u < k_0),$$

then $u \ge k_0$ in Ω.

Proof It follows from Lemma 2 that for each $w \in \mathrm{PSH}(\Omega)$ with $0 < w < 1$

$$\begin{aligned}(k_1 - k_0)^n \int_{u<k_0} (dd^c w)^n &\le \int_{u<k_0} (k_1 - u)^n\, (dd^c w)^n \le \int_{u<k_1} (k_1 - u)^n\, (dd^c w)^n \\ &\le \int_{u<k_1} (dd^c u)^n = A_0 C_n(u < k_0)\end{aligned}$$

which implies the inequality $(k_1 - k_0)^n C_n(u < k_0) \le A_0 C_n(u < k_0)$, and it then turns out from $A_0 < (k_1 - k_0)^n$ that $C_n(u < k_0) = 0$. Thus $u \ge k_0$ in Ω and the proof of Theorem 5 is complete.

To end this section we prefer to show another application of Theorem 3, which uses a simple integral to characterize bounded radial psh functions, see Corollary 3.4 in [P].

Corollary 3 *Suppose that $\phi(t)$ is increasing and convex on $[-\infty, 0)$, and $\lim_{t\to 0-} \phi(t) = 0$. Then the* psh *function $u(z) = \phi(\ln|z|)$ is bounded on the unit ball $B(0,1)$ if and only if there exists a constant $D_u > 0$ such that for any $k < -1/2$ with $C_n(u < k) \neq 0$ we can find a constant k_1 with $k_1 - 1 < k \le k_1$ and*

$$\int_{r_1}^{\frac{1}{2}} \frac{1}{r} \big(\|(dd^c u)^n\|_{B(0,r)}\big)^{\frac{1}{n}}\, dr < D_u,$$

where r_1 denotes the radius of the ball $\{u < k_1\}$.

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Proof We first show the "only if" part. Since the u is bounded, for any constant $k < -1/2$ with $C_n(u < k) \neq 0$ there exists a sequence $k_1 - 1 < k \leq k_1 < \cdots < k_s = -1/2$ such that the inequality in Theorem 3 holds. Denote by $B(0, r_j)$ the ball $\{u < k_j\}$ for $j = 1, 2, \ldots, s$. It then follows from $C_n(u < k_{j-1} + 0) = (\frac{2\pi}{-\ln r_{j-1}})^n$ that

$$\begin{aligned} A_u > \sum_{j=2}^{s} \left(\frac{\|(dd^c u)^n\|_{\{u<k_j\}}}{C_n(u < k_{j-1} + 0)} \right)^{\frac{1}{n}} &= \frac{1}{2\pi} \sum_{j=2}^{s} \left(\|(dd^c u)^n\|_{\{u<k_j\}}\right)^{\frac{1}{n}} \ln \frac{1}{r_{j-1}} \\ &\geq \frac{1}{2\pi} \sum_{j=2}^{s} \int_{r_{j-1}}^{r_j} \frac{1}{r} \left(\|(dd^c u)^n\|_{B(0,r)}\right)^{\frac{1}{n}} dr \\ &= \frac{1}{2\pi} \int_{r_1}^{r_s} \frac{1}{r} \left(\|(dd^c u)^n\|_{B(0,r)}\right)^{\frac{1}{n}} dr, \end{aligned}$$

which completes the proof of the "only if" part.

To prove the "if" part, for any constant $k < -1/2$ with $C_n(u < k) \neq 0$ and each constant k_1 with $k_1 - 1 < k \leq k_1$, we choose a sequence $k_1 < k_2 < \cdots < k_s < k_{s+1}$ such that $k_{s-1} < -1/2 = k_s$ and $r_j = \sqrt{r_{j-1}}$ for $j = 2, 3, \ldots, s-1, s+1$, where the constants r_j denote radii of balls $B(0, r_j) = \{u < k_j\}$. Hence we have

$$\begin{aligned} \int_{r_1}^{r_{s+1}} \frac{1}{r} \left(\|(dd^c u)^n\|_{B(0,r)}\right)^{\frac{1}{n}} dr &\geq \sum_{j=3}^{s+1} \int_{r_{j-1}}^{r_j} \frac{1}{r} \left(\|(dd^c u)^n\|_{B(0,r)}\right)^{\frac{1}{n}} dr \\ &\geq \sum_{j=3}^{s+1} \left(\|(dd^c u)^n\|_{B(0,r_{j-1})}\right)^{\frac{1}{n}} \int_{r_{j-1}}^{r_j} \frac{1}{r} dr \\ &= \frac{\pi}{2} \sum_{j=2}^{s} \left(\frac{\|(dd^c u)^n\|_{\{u<k_j\}}}{C_n(u < k_{j-1} + 0)} \right)^{\frac{1}{n}}. \end{aligned}$$

It then follows from the assumption and Theorem 3 that the u is bounded in $B(0, 1)$, and the proof of Corollary 3 is complete.

4 Monge-Ampère Measures of Bounded Plurisubharmonic Functions

The complex Monge-Ampère measure of a bounded psh function vanishes on any pluripolar set. So vanishing on all pluripolar sets is a necessary condition for a positive measure to be complex Monge-Ampère measure of some bounded psh function. However, this condition is not sufficient, see Example 3. In the following we prove a characterization of complex Monge-Ampère measures of bounded psh functions.

Theorem 6 *Suppose that μ is a positive measure vanishing on each pluripolar set of Ω. Then $\mu = (dd^c v)^n$ for some bounded* psh *function v in Ω if and only if there exists positive constants A and D such that for any negative $u \in \mathrm{PSH}(\Omega)$, which satisfies $(dd^c u)^n \leq \mu$ and $u(z) \geq -1$ near the boundary $\partial\Omega$, and for each constant $k < -1$ with $C_n(u < k) \neq 0$ we can find a*

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sequence $k \le k_1 < \cdots < k_{s-1} < k_s = -1$ satisfying $k_1 < k + D$ and

$$\sum_{j=2}^{s} \left(\frac{\mu(u < k_j)}{C_n(u < k_{j-1} + 0)} \right)^{\frac{1}{n}} < A.$$

Proof To show the "only if" part, by Lemma 2 any function u in $\mathrm{PSH}(\Omega)$ with $(dd^c u)^n \le \mu = (dd^c v)^n$ and $u(z) \ge -1 \ge v(z) - \sup_\Omega |v| - 1$ near the boundary $\partial\Omega$ satisfies the inequality $u(z) \ge v(z) - \sup_\Omega |v| - 1 \ge -D$ for all $z \in \Omega$, where we take $D = \sup_\Omega |v| - \inf_\Omega |v| + 1$. So for any constant $k < -1$ with $C_n(u < k) \ne 0$ we have that $k \ge -D$. Take a sequence $k \le k_1 < k_2 = -1$ such that the inequality $C_n(u < k_2) \le 2C_n(u < k_1 + 0)$ holds. Hence we obtain the inequality

$$\left(\frac{\mu(u < k_2)}{C_n(u < k_1 + 0)} \right)^{\frac{1}{n}} < A,$$

where the constant $A = 1 + 2^{\frac{1}{n}} \sup_\Omega |v|$. This completes the proof of the "only if" part.

For the proof of "if" part, we assume first that the measure μ has a compact support in Ω. Since μ vanishes on all pluripolar sets, by Theorem 6.3 in [C2] there exists a decreasing sequence of psh functions u_k vanishing on $\partial\Omega$ such that $(dd^c u_k)^n$ increase to μ. It then follows from the assumption on μ and the proof of Theorem 3 that all functions $u_k \ge -A - D - 1$, which gives that the psh function $v = \lim_{k\to\infty} u_k$ is bounded on Ω and by the monotone convergence theorem in [B-T2] we get that $(dd^c u_k)^n \to (dd^c v)^n$. Thus $\mu = (dd^c v)^n$ and we have proved the "if" part for any measure μ with compact support in Ω. In general case, we take a sequence of measures μ_l with compact support which increase to μ as $l \nearrow \infty$. By the above proof there exist psh functions v_l such that $0 \ge v_l \ge -A-D-1$ and $(dd^c v_l)^n = \mu_l$ for all l. Modifying v_l near the $\partial\Omega$, we can assume that $v_l = 1$ on $\partial\Omega$ and $(dd^c v_l)^n \ge \mu_l$. So it follows from Theorem 2 that $\mu_l = (dd^c v_l^\star)^n$ for some bounded psh function $v_l^\star$ with $v_l^\star = 0$ on $\partial\Omega$. Since $\mu \ge \mu_l$ for all l, the functions $v_l^\star$ are uniformly bounded in Ω and hence the monotone limit $v^\star = \lim_{l\to\infty} v_l^\star$ is bounded and satisfies $(dd^c v^\star)^n = \mu$. The proof of Theorem 6 is complete.

Theorem 6 implies that if μ is a Monge-Ampère measure of some bounded psh function in Ω then any positive measure $\mu_1 \le \mu$ is also a Monge-Ampère measure of bounded psh function in Ω. However, there exists a positive measure $\mu \le C_n$ which is not a Monge-Ampère measure of some bounded psh function, see [KO2]. In [KO3] and [KO4], by using a stronger condition Kolodziej obtained a positive result for some classes of measures. Now we have

Corollary 4 *Suppose that μ is a positive measure in Ω and suppose that $\varepsilon > 0$ and $F(x) = x\big(\ln(1 + 1/x)\big)^{-n-\varepsilon}$. If the inequality $\mu(E) \le F\big(C_n(E)\big)$ holds for any set $E \subset \Omega$, then there exists a bounded* psh *function v in Ω such that $\mu = (dd^c v)^n$.*

Proof Repeating the proof of Corollary 2, we get that the measure μ satisfies the inequality assumption in Theorem 6. Hence, it is a Monge-Ampère measure of some bounded psh function in Ω and the proof is complete.

We also record another consequence of Theorem 6.

Corollary 5 *Suppose that μ is a positive measure in Ω and that $p > 1$ and $1/p + 1/q = 1$.*

If there exists $A_p > 0$ such that $\mu(E) \le A_p\big[C_n(E)\big]^p$ for all $E \subset\subset \Omega$, then for any $q_1 > q$ and any nonnegative function f in $L^{q_1}_\mu(\Omega)$ we can find a bounded psh *function v in Ω such that $(dd^c v)^n = f\,d\mu$ and the supremum norm $\sup_\Omega |v|$ are uniformly bounded for all functions f with $\|f\|_{L^{q_1}_\mu(\Omega)} \le 1$.*

Conversely, if for any nonnegative function f in $L^q_\mu(\Omega)$ we can find a bounded psh function v in Ω such that $(dd^c v)^n = f\,d\mu$ and $\sup_\Omega |v|$ are uniformly bounded for all functions f with $\|f\|_{L^q_\mu(\Omega)} \le 1$, then there exists $A_p > 0$ such that $\mu(E) \le A_p\big[C_n(E)\big]^p$ for all $E \subset\subset \Omega$.

Proof Assume that $f \in L^{q_1}_\mu(\Omega)$ be a nonnegative function in Ω. For all $E \subset\subset \Omega$, by Hölder inequality, we have

$$\int_E f\,d\mu \le \|f\|_{L^{q_1}_\mu(\Omega)}\mu(E)^{1-1/q_1} \le \|f\|_{L^{q_1}_\mu(\Omega)} A_p^{1-1/q_1} C_n(E)^{1+p/q-p/q_1},$$

where the exponent $1 + p/q - p/q_1 > 1$. By a similar proof of Corollary 2 we obtain that the positive measure $f\,d\mu$ satisfies the condition in Theorem 6 and hence there exists a bounded psh function v in Ω such that $(dd^c v)^n = f\,d\mu$, where $\sup_\Omega |v|$ are uniformly bounded for all functions f with $\|f\|_{L^{q_1}_\mu(\Omega)} \le 1$.

To prove the converse assertion, we set $f_E = \chi_E/\mu(E)^{\frac{1}{q}}$ for each $E \subset\subset \Omega$, where χ_E denotes the characteristic function of the set E. Then $\|f_E\|_{L^q_\mu(\Omega)} = 1$ and

$$\mu(E)^{\frac{1}{p}} = \int_E f_E\,d\mu = \int_E (dd^c v_E)^n \le (\sup_\Omega |v_E|)^n C_n(E),$$

where, by the assumption, the constants $(\sup_\Omega |v_E|)^n$ are uniformly bounded for all subsets $E \subset\subset \Omega$. Hence there exists $A_p > 0$ such that $\mu(E) \le A_p\big[C_n(E)\big]^p$ for all $E \subset\subset \Omega$. The proof of Corollary 5 is complete.

In [KO3] Kolodziej proved that any positive measure $f\,d\lambda$, where $f \in L^p_\lambda(\Omega)$, $p > 1$ and λ denotes the Lebesgue measure, is the complex Monge-Ampère measure of some bounded psh function. Corollary 5 implies directly

Corollary 6 *Let μ be a positive measure in Ω. Then for any $\delta > 1$ there exists $A_\delta > 0$ such that $\mu(E) \le A_\delta\big[C_n(E)\big]^\delta$ for all $E \subset\subset \Omega$ if and only if for any $p > 1$ there exists $B_p > 0$ such that for all nonnegative functions f in $L^p_\mu(\Omega)$ with $\|f\|_{L^p_\mu(\Omega)} \le 1$ we can find a bounded* psh *function v in Ω such that $(dd^c v)^n = f\,d\mu$ and $\sup_\Omega |v| \le B_p$.*

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