discoverer of the Rubik's Cube, Erno Rubik, "You can learn it. You can learn it from other people, you can learn it from books, you can learn it from different notes, and explanations on the Internet, but the best is if you find your own solution." [3].

References

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Conjecturing a limit
For reasons that will become apparent, teachers and lecturers might find it worthwhile asking first-year undergraduates or sixth-form students familiar with the notion of a limit to explore briefly the following numerical expression:

\[
S = \frac{\frac{1}{1^2} + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{8^4} + \ldots}{\frac{1}{1^2} + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{8^4} + \ldots} = \frac{\sum_{k=1}^{\infty} \frac{1}{F_k}}{\sum_{k=1}^{\infty} \frac{1}{F_k^4}},
\]

where \(F_k\) denotes the \(k\)th Fibonacci number. This does indeed lead to some interesting mathematics, and gives rise to a number of thought-provoking questions.

A first question might be: ‘Does \(S\) exist?’ Well, the numerator and denominator both have finite limits (so \(S\) does indeed exist), but how would the students go about proving these facts? A next step might be to consider the partial sum

\[
S_n = \frac{\sum_{k=1}^{n} \frac{1}{F_k}}{\sum_{k=1}^{n} \frac{1}{F_k^4}},
\]

and to investigate its behaviour as \(n\) increases (noting that \(\lim_{n \to \infty} S_n = S\)). Is \(\{S_n\}\) an increasing sequence for \(n \geq 2\)? What is its limit? The following table was produced using an Excel spreadsheet.
Is it the case that this sequence has the golden ratio $\phi$ as its limit, where $\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618033989$, noting that this number is intimately connected to the Fibonacci numbers? (Binet's formula for $F_n$ is given in terms of $\phi$, [1].) There would appear to be three possibilities here:

(i) The limit is in fact $\phi$.

(ii) The limit is not $\phi$; it is merely coincidental that $S$ would appear to be, in relative terms, so close to it.

(iii) The limit is not $\phi$, but there is some underlying mathematical reason why $S$ and $\phi$ are, in relative terms, so close to each other.

In an attempt to ascertain which of these possibilities is actually the case, students might calculate further terms in the sequence. On doing so, it is found that, to nine decimal places, the spreadsheet gives $1.617872468$ as the value of $S_n$ for $n = 47$ onwards. Could this be a result of rounding errors due to the increments in the numerator and denominator being so small for these values of $n$? Well, it seems we may discount this possibility, since the result was confirmed using Mathematica (which, given enough time and memory, is able to calculate $S_n$ correct to as many decimal places as one pleases). This leads to a relative error of approximately $0.000099825412$, which is less than one hundredth of one percent.

Thus, although on the initial numerical evidence it might at least have appeared plausible that $S = \phi$, on further investigation it seems that such a conjecture would be false. Indeed, on the face of it, (ii) is the most likely to be true. There are many instances of this, particularly with respect to expressions involving well-known irrationals being very close to integers [2]. Nonetheless, it is also possible for (iii) to occur, as is the case for $e^{\pi \sqrt{163}}$, the reason that this is so close to an integer is that 163 is a so-called Heegner number [2, 3].

We argue here that our finding should not be totally unexpected, and provide an explanation which makes it appear somewhat less of a coincidence than it might initially have appeared. To this end, let us consider the following generalisation of $S_n$:

$$S_n(m) = \frac{\sum_{k=1}^{n} \frac{1}{k^m}}{\sum_{k=1}^{n} \frac{1}{k^m}}.$$
For each \( m \in \mathbb{N} \), let \( S(m) = \lim_{n \to \infty} S_n(m) \). Note that \( \{S_m\}_{m \geq 1} \) is a strictly increasing sequence with \( S(1) = 1 \). What then is the limit of this sequence? Well, we have
\[
\sum_{k=1}^{\infty} \frac{1}{F_k^m} \to 2 \text{ as } m \to \infty,
\]
so that
\[
\lim_{m \to \infty} S(m) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{F_k} = \frac{\psi}{2},
\]
where \( \psi \) is known as the reciprocal Fibonacci constant [4, 5]. It is interesting that there is no known closed-form expression for \( \psi \), and its irrationality was not proved until 1989. Anyway, of relevance to our mathematical exploration is the fact that \( \psi = 3.359885666... \), and hence \( \psi/2 = 1.679942883... \).

From this it may be seen that \( \lim_{m \to \infty} S(m) \) is greater than, though reasonably close to, \( \phi \). Thus, as the terms of \( S(m)_{m \geq 1} \) increase from 1, getting ever closer to \( \psi/2 \), perhaps it should not be too unexpected that one of these terms does actually come really rather near, in relative terms, to \( \phi \).

To explain why \( \psi/2 \) is in fact close to \( \phi \), we may perform a simple calculation using a well-known approximation to \( F_k \). It is the case that, for \( k \in \mathbb{N} \), \( F_k \) is equal to the nearest integer to \( \phi^k / \sqrt{5} \) [6]. We therefore have
\[
\frac{\psi}{2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{\phi^k}
\]
\[
= \frac{\sqrt{5}}{2} \sum_{k=1}^{\infty} \frac{1}{\phi^k}
\]
\[
= \frac{\sqrt{5}}{2} \frac{1}{1 - \frac{1}{\phi}}
\]
\[
= \frac{\sqrt{5}}{2} \phi.
\]
Although approximations have been used, the above result, along with the fact that \( \sqrt{5}/2 \) is just a little greater than 1, does provide us with at least some indication that \( \psi/2 \) and \( \phi \) are relatively near to each other, even though it might not necessarily be directly inferred from this that \( \psi/2 \) is indeed larger than \( \phi \).

We might thus cite the reason for a member of \( S(m)_{m \geq 1} \) being very close to \( \phi \) as connected to the fact that \( \psi/2 \) is just a little larger than \( \phi \), which in turn is linked to the fact that, through the properties of \( \phi \) (and its relationship with the Fibonacci numbers), half the sum of the reciprocals of the Fibonacci numbers may be well approximated by the sum of a geometric series, the sum to infinity of which is a little greater than \( \phi \).
There are a number of comments to be made here. First, this investigation does illustrate the need to be careful with respect to making conjectures based on fairly flimsy numerical evidence. Another interesting point relates to whether or not students are willing to accept a numerical result obtained using *Mathematica* as ‘proof’ that the limit is not φ. This has the potential to generate plenty of discussion concerning the nature of proof. Finally, on encountering what might appear to be a mathematical coincidence, it is worth scratching beneath the surface to see if there is any more to things than may initially meet the eye.

**References**


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**Two surprising maximisation problems**

We start with the well-known fact that, if an \(x \times y\) rectangle has fixed perimeter \(P\), then the area of the rectangle has maximum value \(\frac{1}{16}P^2\) when \(x = y = \frac{1}{4}P\).

**Surprise 1:** What is the maximum area of the shape in Figure 1 if its perimeter is fixed?

\[ \text{Figure 1} \]

\[ \theta \]