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EXACT SOLUTIONS FOR THE POISEUILLE FLOW OF A GENERALIZED MAXWELL FLUID INDUCED BY TIME-DEPENDENT SHEAR STRESS

W. AKHTAR ✉ [1], CORINA FETECAU [2] and A. U. AWAN [3]

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Abstract

The Poiseuille flow of a generalized Maxwell fluid is discussed. The velocity field and shear stress corresponding to the flow in an infinite circular cylinder are obtained by means of the Laplace and Hankel transforms. The motion is caused by the infinite cylinder which is under the action of a longitudinal time-dependent shear stress. Both solutions are obtained in the form of infinite series. Similar solutions for ordinary Maxwell and Newtonian fluids are obtained as limiting cases. Finally, the influence of the material and fractional parameters on the fluid motion is brought to light.

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1. Introduction

The study of viscoelastic fluids has many applications in industrial processes. These include the extrusion of polymer fluids, cooling of a metallic plate in a bath, food stuffs, exotic lubricants, and colloidal and suspension solutions. The classical Navier–Stokes theory is inadequate to describe the flows of such fluids, whose non-Newtonian characteristics include stress relaxation, the normal stress difference, shear thinning or thickening, and many others. Due to the complexity of non-Newtonian fluids, various models for viscoelastic fluids have been proposed. Among these, the rate-type models have received much attention during the last few years [1, 6, 7, 19]. There are very few cases in which exact solutions for the motion equations of flows of non-Newtonian fluids can be obtained. However, it is usually necessary to study non-Newtonian

[1]Institute of Space Technology, P. O. Box 2750, Islamabad, Pakistan; e-mail: wasakh75@yahoo.com.
[2]Department of Theoretical Mechanics, Technical University of Iasi, Iasi 700506, Romania; e-mail: cfetecau@yahoo.de.
[3]Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan; e-mail: auawan2003@yahoo.com.

fluid flows in many engineering fields such as oil exploration, the polymer chemical industry, and bio-engineering. The Maxwell fluid is the most common non-Newtonian viscoelastic fluid in industrial fields, such as polymer solutions. It is the simplest rate-type fluid in which the relaxation phenomena are taken into consideration. The first exact solutions for unsteady motions of such fluids seem to have been obtained by Srivastava [18].

Recently, fractional calculus has been very successful in the description of complex dynamics such as relaxation, oscillation, wave behaviour and viscoelastic behaviour. The starting point of the fractional derivative model of non-Newtonian fluids is usually a classical differential equation which is modified by replacing the time derivative of an integer order by a time fractional derivative. This generalization allows us to define precisely noninteger order integrals or derivatives. Many exact solutions corresponding to different motions of non-Newtonian fluids with fractional derivatives have been established, but we mention here only a few in cylindrical domains [2, 5, 11, 12, 16, 17, 20, 21]. Furthermore, the one-dimensional fractional derivative Maxwell model has been very useful in modelling the linear viscoelastic response of some polymers in the glass transition and the glass state [10]. In other cases, it has been shown that the governing equations employing fractional derivatives are also linked to molecular theories [8]. The use of fractional derivatives in the context of viscoelasticity was first proposed by Germant [9]. Later, Bagley and Torvic [3] demonstrated that the theory of viscoelasticity of coiling polymers predicts constitutive relations with fractional derivatives, and Makris *et al.* [14] achieved a very good fit to experimental data when the fractional derivative Maxwell model was used instead of the Maxwell model for the silicon gel fluid. However, it is worth pointing out that almost all of the aforementioned works deal with problems in which the velocity is given on the boundary.

The aim of this paper is to discuss the Poiseuille flow of a generalized Maxwell fluid in an infinite circular cylinder that applies a time-dependent shear to the fluid. More precisely, we establish exact solutions for the velocity field and the adequate shear stress corresponding to the motion of such a fluid induced by an infinite cylinder subject to a time-dependent shear stress of the form (3.1). The solutions that have been obtained, presented as series forms in terms of the generalized $R_{a,b}(\cdot, t)$ and $G_{a,b,c}(\cdot, t)$ functions, can be easily reduced to give similar solutions for ordinary Maxwell and Newtonian fluids undergoing the same motion. Finally, the influence of pertinent parameters on the fluid motion is illustrated via some numerical examples.

2. Governing equations

The motions to be considered here have velocity field $\mathbf{v}$ and shear stress $\mathbf{S}$ of the form

$$\mathbf{v} = \mathbf{v}(r, t) = v(r, t)\mathbf{e}_z, \quad \mathbf{S} = S(r, t), \tag{2.1}$$

where $\mathbf{e}_z$ is the unit vector in the z-direction of the cylindrical coordinate system r, θ and z. For such flows, the constraint of incompressibility is automatically satisfied and the governing equations corresponding to incompressible Maxwell fluids are [6]

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(r,t)=\mu\frac{\partial v(r,t)}{\partial r},$$
$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial v(r,t)}{\partial t}=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t), \tag{2.2}$$

where $\tau(r,t)=S_{rz}(r,t)$ is the shear stress, λ the relaxation time, μ the dynamic viscosity, $\nu=\mu/\rho$ the kinematic viscosity, and ρ the constant density of the fluid.

The governing equations corresponding to the generalized Maxwell fluids (GMF), undergoing the same motion, are

$$(1+\lambda D_t^\alpha)\frac{\partial v(r,t)}{\partial t}=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t), \tag{2.3}$$

$$(1+\lambda D_t^\alpha)\tau(r,t)=\mu\frac{\partial v(r,t)}{\partial r}, \tag{2.4}$$

where the fractional differential operator D_t^α is defined by [15]

$$D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(\tau)}{(t-\tau)^\alpha}\,d\tau,\quad 0<\alpha<1, \tag{2.5}$$

in which $\Gamma(\cdot)$ is the Gamma function. In order to obtain the above governing equations, we assumed that there is no pressure gradient in the flow direction. Furthermore, although we have kept the same notation, the new material constant λ has the dimension of t^α and reduces to the relaxation time if $\alpha\to 1$.

In the following section, the fractional partial differential equations (2.3) and (2.4), with appropriate initial and boundary conditions, will be solved by means of the finite Hankel and Laplace transforms. In order to avoid lengthy calculations of residues and contour integrals, the discrete inverse Laplace method will be used [2, 5, 11, 12, 16, 17, 21].

3. Longitudinal flow through an infinite circular cylinder

Let us consider an incompressible GMF at rest in an infinite circular cylinder of radius R. At time $t=0^+$, the cylinder is suddenly pulled with the time-dependent shear stress

$$\tau(R,t)=\frac{f}{\lambda}R_{\alpha,-2}\left(-\frac{1}{\lambda},t\right), \tag{3.1}$$

where the generalized $R_{a,b}(c,t)$ functions are defined by

$$R_{a,b}(c,t)=\sum_{n=0}^{\infty}\frac{c^n t^{(n+1)a-b-1}}{\Gamma[(n+1)a-b]},\quad \mathrm{Re}(a-b)>0.$$

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Owing to the shear, the fluid is gradually moved. Its velocity is of the form (2.1) while the governing equations are (2.3) and (2.4). The appropriate initial and boundary conditions are

$$v(r, 0) = 0, \quad r \in [0, R), \tag{3.2}$$

$$(1 + \lambda D_t^{\alpha})\tau(R, t) = \mu \frac{\partial v(r, t)}{\partial r}\bigg|_{r=R} = ft, \quad t \geq 0, \tag{3.3}$$

where f is a constant. Of course, as we shall later see, $\tau(R, t)$ given by (3.1) is just the solution of the fractional differential equation (3.3). For $\alpha \to 1$, (3.1) takes the form

$$\tau(R, t) = f[t - \lambda(1 - e^{-t/\lambda})], \tag{3.4}$$

corresponding to an ordinary Maxwell fluid. Taking $\lambda \to 0$, (3.4) reduces to the simple form

$$\tau(R, t) = ft \tag{3.5}$$

and corresponds to a problem with a constantly accelerating shear on the boundary. A similar problem, with f instead of ft in (3.3), has been studied in [6]. For Newtonian and second-grade fluids [2] it corresponds to a constant shear stress on the boundary.

3.1. Calculation of the velocity field Applying the Laplace transform to (2.3) and (3.3) and using the Laplace transform formula for sequential fractional derivatives [15], we obtain

$$(q + \lambda q^{\alpha+1})\overline{v}(r, q) = \nu\left(\frac{\partial^2 \overline{v}(r, q)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}(r, q)}{\partial r}\right), \tag{3.6}$$

$$\frac{\partial \overline{v}(R, q)}{\partial r} = \frac{f}{\mu q^2}, \tag{3.7}$$

where

$$\overline{v}(r, q) = \int_0^{\infty} v(r, t)e^{-qt}\, dt$$

is the Laplace transform of $v(r, t)$. In the following calculations we denote by

$$\overline{v}_H(r_n, q) = \int_0^R r\overline{v}(r, q)J_0(rr_n)\, dr \tag{3.8}$$

the finite Hankel transform of $\overline{v}(r, q)$, where r_n, $n = 1, 2, 3, \ldots$ are the positive roots of the transcendental equation $J_1(Rr) = 0$. Here $J_0(\cdot)$, $J_1(\cdot)$ are Bessel functions of the first kind.

Multiplying both sides of (3.6) by $rJ_0(rr_n)$, integrating with respect to r from 0 to R and taking into account the condition (3.7) and the identity [4]

$$\int_0^R r\left[\frac{\partial^2 \overline{v}(r, q)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}(r, q)}{\partial r}\right]J_0(rr_n)\, dr$$
$$= RJ_0(Rr_n)\frac{\partial \overline{v}(R, q)}{\partial r} - r_n^2 \overline{v}_H(r_n, q), \tag{3.9}$$

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we find that

$$\overline{v}_H(r_n, q) = \frac{Rf}{\rho} J_0(Rr_n) \frac{1}{q^2(q + \lambda q^{\alpha+1} + \nu r_n^2)}. \tag{3.10}$$

Now, for a more suitable presentation of the final results, we rewrite (3.10) in the equivalent form

$$\overline{v}_H(r_n, q) = \overline{v}_{1H}(r_n, q) + \overline{v}_{2H}(r_n, q), \tag{3.11}$$

where

$$\begin{aligned} \overline{v}_{1H}(r_n, q) &= \frac{Rf J_0(Rr_n)}{r_n^2} \frac{1}{\mu q^2}, \\ \overline{v}_{2H}(r_n, q) &= -\frac{Rf J_0(Rr_n)}{\mu r_n^2} \frac{1 + \lambda q^\alpha}{q(q + \lambda q^{\alpha+1} + \nu r_n^2)}. \end{aligned} \tag{3.12}$$

The inverse Hankel transforms of the functions $\overline{v}_{1H}(r_n, q)$ and $\overline{v}_{2H}(r_n, q)$ are [4]

$$\frac{fr^2}{2R} \frac{1}{\mu q^2} \quad \text{and} \quad -\frac{2f}{\mu R} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)} \frac{1 + \lambda q^\alpha}{q(q + \lambda q^{\alpha+1} + \nu r_n^2)}, \tag{3.13}$$

respectively. Using the above results, we obtain

$$\overline{v}(r, q) = \frac{fr^2}{2R} \frac{1}{\mu q^2} - \frac{2f}{\mu R} \sum_{n=1}^{\infty} \frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)} \frac{1 + \lambda q^\alpha}{q(q + \lambda q^{\alpha+1} + \nu r_n^2)}, \tag{3.14}$$

and we write the last factor in the summand of the infinite series as

$$\begin{aligned} \frac{1 + \lambda q^\alpha}{q(q + \lambda q^{\alpha+1} + \nu r_n^2)} &= \frac{1 + \lambda q^\alpha}{\lambda q^2[(q^\alpha + 1/\lambda) + (\nu r_n^2/\lambda) q^{-1}]} \\ &= \sum_{k=0}^{\infty} \frac{(1 + \lambda q^\alpha)(-(\nu r_n^2/\lambda) q^{-1})^k}{\lambda q^2 (q^\alpha + 1/\lambda)^{k+1}} \\ &= \sum_{k=0}^{\infty} \frac{1}{\lambda} \left(-\frac{\nu r_n^2}{\lambda}\right)^k \frac{q^{-k-2} + \lambda q^{\alpha-k-2}}{(q^\alpha + 1/\lambda)^{k+1}}. \end{aligned} \tag{3.15}$$

To obtain the velocity field $v(r, t) = L^{-1}\{\overline{v}(r, q)\}$, we use (3.15) and the formula [13]

$$L^{-1}\left\{\frac{q^b}{(q^a - d)^c}\right\} = G_{a,b,c}(d, t), \quad \operatorname{Re}(ac - b) > 0, \tag{3.16}$$

where

$$G_{a,b,c}(d, t) = \sum_{j=0}^{\infty} \frac{d^j \Gamma(c + j)}{\Gamma(c)\Gamma(j + 1)} \frac{t^{(c+j)a-b-1}}{\Gamma[(c + j)a - b]} \tag{3.17}$$

are the generalized G-functions.

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Finally, we obtain the velocity field in the following form:

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}\sum_{k=0}^{\infty}\frac{1}{\lambda}\left(-\frac{\nu r_n^2}{\lambda}\right)^k \times\left[G_{\alpha,-k-2,k+1}\left(-\frac{1}{\lambda},t\right)+\lambda G_{\alpha,\alpha-k-2,k+1}\left(-\frac{1}{\lambda},t\right)\right]. \tag{3.18}$$

3.2. Calculation of the shear stress By applying the Laplace transform to (2.4), we find that

$$\overline{\tau}(r,q) = \frac{\mu}{1+\lambda q^\alpha}\frac{\partial \overline{v}(r,q)}{\partial r}. \tag{3.19}$$

Differentiating (3.14) with respect to r and using the result in (3.19), we obtain

$$\begin{aligned}\overline{\tau}(r,q) &= \frac{rf}{R}\frac{1}{q^2(1+\lambda q^\alpha)} + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}\frac{1}{q(q+\lambda q^{\alpha+1}+\nu r_n^2)} \\ &= \frac{rf}{\lambda R}\frac{q^{-2}}{q^\alpha+1/\lambda} + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \\ &\quad\times\sum_{k=0}^{\infty}\frac{1}{\lambda}\left(-\frac{\nu r_n^2}{\lambda}\right)^k\frac{q^{-k-2}}{(q^\alpha+1/\lambda)^{k+1}}.\end{aligned} \tag{3.20}$$

Applying the inverse Laplace transform to (3.20) and using the relation [13, (21)]

$$L^{-1}\left\{\frac{q^b}{q^a-c}\right\} = R_{a,b}(c,t), \qquad \mathrm{Re}(a-b)>0,\ \mathrm{Re}(q)>0,$$

where $R_{a,b}(c,t)$ has been defined in Section 3, we find that the shear stress $\tau(r,t)$ takes the simple form

$$\tau(r,t) = \frac{rf}{\lambda R}R_{\alpha,-2}\left(-\frac{1}{\lambda},t\right) + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}\sum_{k=0}^{\infty}\frac{1}{\lambda}\left(-\frac{\nu r_n^2}{\lambda}\right)^k G_{\alpha,-k-2,k+1}\left(-\frac{1}{\lambda},t\right). \tag{3.21}$$

4. Limiting cases

We now consider the following limiting cases.

(1) Taking $\alpha \to 1$ in (3.18) and (3.21), we obtain the velocity field

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}\sum_{k=0}^{\infty}\frac{1}{\lambda}\left(-\frac{\nu r_n^2}{\lambda}\right)^k \times\left[G_{1,-k-2,k+1}\left(-\frac{1}{\lambda},t\right)+\lambda G_{1,-k-1,k+1}\left(-\frac{1}{\lambda},t\right)\right] \tag{4.1}$$

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and shear stress

$$\tau(r,t) = \frac{rf}{\lambda R} R_{1,-2}\left(-\frac{1}{\lambda}, t\right) + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}\sum_{k=0}^{\infty}\frac{1}{\lambda}\left(-\frac{\nu r_n^2}{\lambda}\right)^k G_{1,-k-2,k+1}\left(-\frac{1}{\lambda}, t\right) \tag{4.2}$$

corresponding to an ordinary Maxwell fluid undergoing the same motion.

By using (A.3), (A.4) and (A.6) from the Appendix, the above expressions can be written in the simplified forms

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)} \times \left[1 - \left\{\mathrm{ch}(b_n t) + \frac{1-2\lambda\nu r_n^2}{a_n}\mathrm{sh}(b_n t)\right\}e^{-t/2\lambda}\right], \tag{4.3}$$

$$\tau(r,t) = \frac{rf}{R}[t - \lambda(1-e^{-t/\lambda})] + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}\left[1 - \left\{\mathrm{ch}(b_n t) + \frac{\mathrm{sh}(b_n t)}{a_n}\right\}e^{-t/2\lambda}\right], \tag{4.4}$$

where $a_n = \sqrt{1-4\lambda\nu r_n^2}$, $b_n = a_n/(2\lambda)$.

(2) Taking $\lambda \to 0$ in (4.3) and (4.4) and using

$$\lim_{\lambda\to 0}(a_n) = 1, \quad \lim_{\lambda\to 0}(b_n) = \infty, \quad \lim_{\lambda\to 0} e^{-t/2\lambda}\mathrm{ch}(b_n t) = \lim_{\lambda\to 0} e^{-t/2\lambda}\mathrm{sh}(b_n t) = \frac{1}{2}e^{-\nu r_n^2 t}, \tag{4.5}$$

we find the velocity field

$$v(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}[1 - e^{-\nu r_n^2 t}] \tag{4.6}$$

and shear stress

$$\tau(r,t) = \frac{rft}{R} + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}[1 - e^{-\nu r_n^2 t}] \tag{4.7}$$

for a Newtonian fluid undergoing the same motion.

5. Conclusion and numerical results

The velocity field and the associated shear stress corresponding to the longitudinal flow induced by an infinite circular cylinder in an incompressible generalized Maxwell

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fluid have been determined using Laplace and Hankel transforms. The motion is produced by the infinite circular cylinder that is initially pulled with a time-dependent shear along its axis. The solutions that have been obtained, written in terms of the generalized R and G-functions, satisfy all imposed initial and boundary conditions. In the special case when $\alpha \to 1$, or $\alpha \to 1$ and $\lambda \to 0$, the corresponding solutions for ordinary Maxwell and Newtonian fluids, undergoing the same motion, are obtained.

It is worth pointing out that the flow that has been studied here, unlike that from [6], is unsteady and remains unsteady for all kinds of fluids. Furthermore, the solutions corresponding to ordinary Maxwell and Newtonian fluids, resulting from (4.3), (4.4) and (4.6), (4.7), are written as a sum of the large-time and transient solutions. The large-time solutions for velocity,

$$v_{NL}(r,t) = v_{ML}(r,t) = \frac{fr^2}{2\mu R}t - \frac{2f}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}, \tag{5.1}$$

are identical, while those corresponding to the shear stress,

$$\tau_{NL}(r,t) = \frac{rft}{R} + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)} \tag{5.2}$$

and

$$\tau_{ML}(r,t) = \frac{rf}{R}[t - \lambda(1 - e^{-t/\lambda})] + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}, \tag{5.3}$$

are different. The flow studied in [6] ultimately becomes steady, and the steady solutions corresponding to ordinary Maxwell and Newtonian fluids,

$$v_{NS}(r) = v_{MS}(r) = \frac{R_1 f}{\mu}\ln\left(\frac{r}{R_2}\right) \quad \text{and} \quad \tau_{NS}(r) = \tau_{MS}(r) = \frac{R_1 f}{r}, \tag{5.4}$$

are identical.

Finally, the influence of the pertinent parameters on the velocity field and shear stress is illustrated in Figures 1–6. A series of calculations was performed for different situations with typical values. For example, we chose $\rho = 900$ and $\nu = 0.05$, corresponding to the values of crude oil, and different values of λ and α were chosen to illustrate their effects on the flow. Figures 1 and 2 show the influence of the fractional parameter α and the relaxation time λ on the fluid velocity. Qualitatively, their influence seems to be the same. In the neighbourhood of the cylinder the velocity field $v(r,t)$ is an increasing function of α and λ, while it decreases around its axis. From Figure 3, as was to be expected, we see that for $\lambda \to 0$ and $\alpha \to 1$, the diagrams of the velocity corresponding to Newtonian and generalized Maxwell fluids are almost identical.

The remaining figures give similar representations for the shear stress $\tau(r,t)$. They are in accordance with those corresponding to the velocity field. The shear stress,

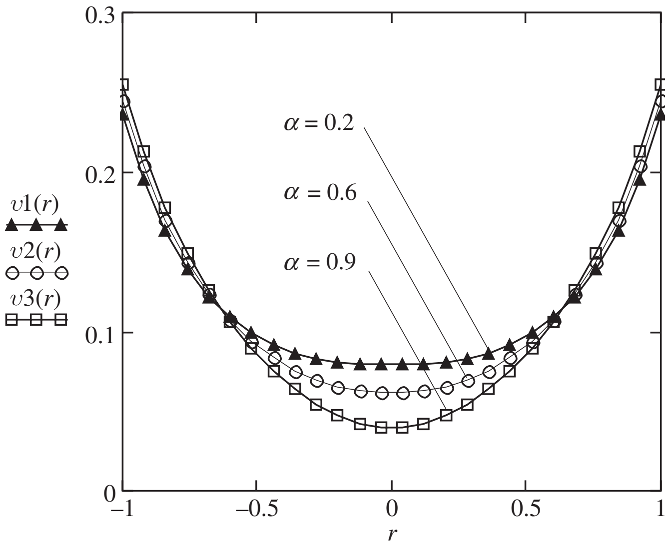

FIGURE 1. Profiles of the velocity field $v(r, t)$ given by (3.18), for $\nu = 0.05$, $\rho = 900$, $R = 1$, $f = 5$, $\lambda = 2$, $t = 5$ s and different values of α.

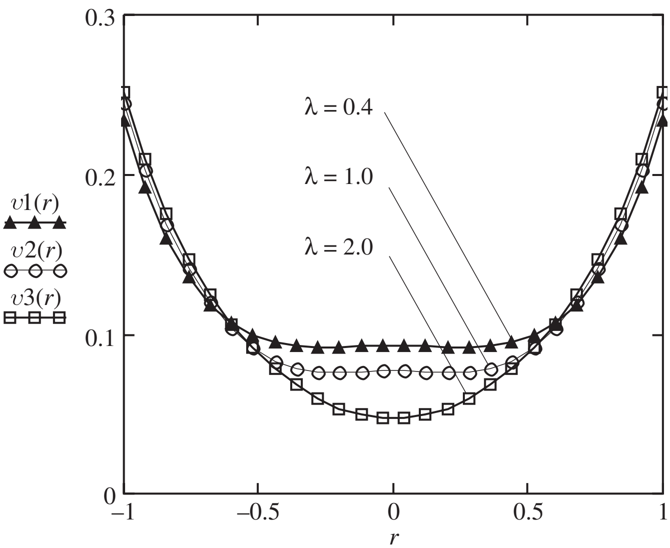

FIGURE 2. Profiles of the velocity field $v(r, t)$ given by (3.18), for $\nu = 0.05$, $\rho = 900$, $R = 1$, $f = 5$, $\alpha = 0.85$, $t = 5$ s and different values of λ.

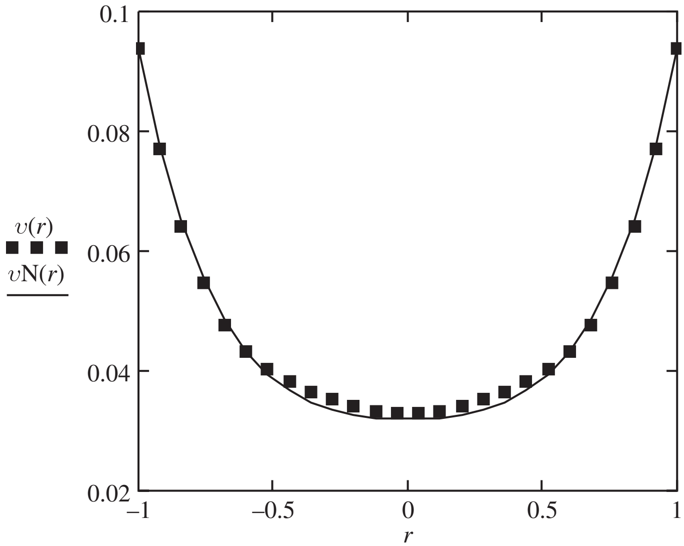

FIGURE 3. Profiles of the velocity field $v(r, t)$ given by (3.18) (curve $v(r)$) and $v_N(r, t)$ given by (4.6) (curve vN(r)) for $\nu = 0.05$, $\rho = 900$, $R = 1$, $f = 5$, $\alpha = 0.7$, $\lambda = 0.1$ and $t = 2$ s.

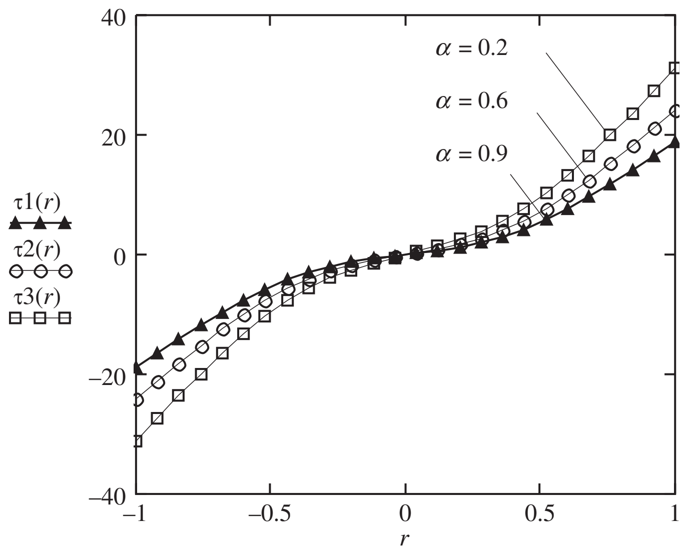

FIGURE 4. Profiles of the shear stress $\tau(r, t)$ given by (3.21) for $\nu = 0.05$, $\rho = 900$, $R = 1$, $f = 5$, $\lambda = 2$, $t = 5$ s and different values of α.

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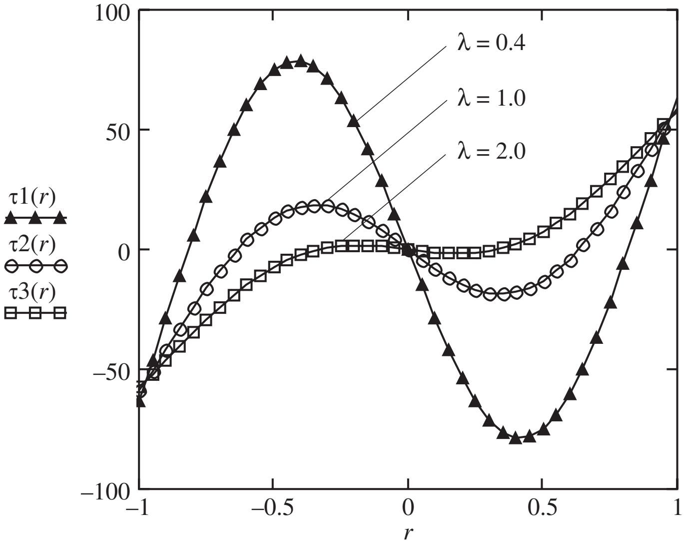

FIGURE 5. Profiles of the shear stress $\tau(r, t)$ given by (3.21) for $\nu = 0.05$, $\rho = 900$, $R = 1$, $f = 5$, $\alpha = 0.85$, $t = 5$ s and different values of λ.

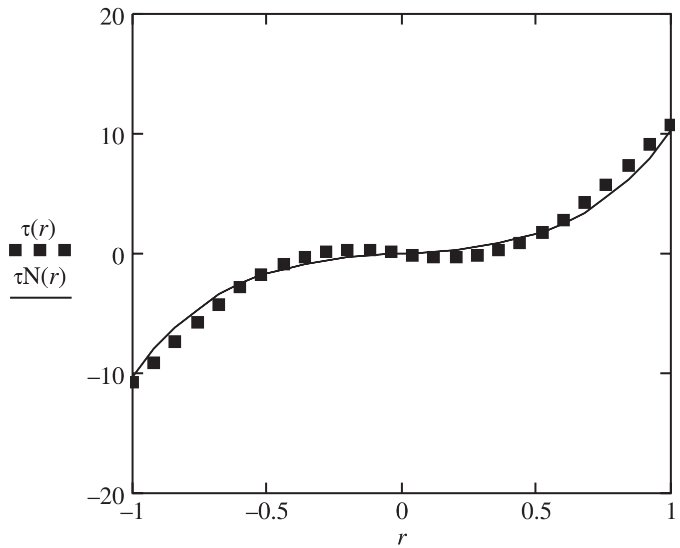

FIGURE 6. Profiles of the shear stress $\tau(r, t)$ given by (3.21) (curve τ(r)) and $\tau_N(r, t)$ given by (4.7) (curve τN(r)) for $\nu = 0.05$, $\rho = 900$, $R = 1$, $f = 5$, $\alpha = 0.7$, $\lambda = 0.1$ and $t = 2$ s.

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as shown in Figures 4 and 5, is zero at the middle of the channel and maximal on the boundary. It is a decreasing function with respect to α and λ on the whole flow domain. From Figure 6 it is also seen that for $\lambda \to 0$ and $\alpha \to 1$, the diagrams of the shear stress corresponding to the two models (Newtonian and generalized Maxwell) are almost identical.

In order to bring to light the effect of fractional derivatives on the fluid motion, special attention has to be given to Figures 1 and 4. From Figure 1, for instance, it is clear that the fractional Maxwell fluid flows more slowly near the boundary and faster in the middle of the channel in comparison with the ordinary Maxwell fluid. From Figure 3 it is also seen that the Newtonian fluid, as expected, is the faster moving. The units of the constants in Figures 1–6 are SI units and the roots r_n have been approximated by $(4n+1)\pi/(4R)$.

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Appendix

We consider the function

$$G(r_n, q) = \frac{1}{\lambda q^2 + q + \nu r_n^2} = \frac{1}{\lambda} \frac{1}{(q + 1/(2\lambda))^2 - (1 - 4\lambda \nu r_n^2)/(4\lambda^2)}.$$

Let us denote $a_n = \sqrt{1 - 4\lambda \nu r_n^2}$, $b_n = a_n/(2\lambda)$, so that

$$\begin{aligned} G(r_n, q) &= \frac{1}{\lambda} \frac{1}{(q + 1/(2\lambda))^2 - (b_n)^2} = \frac{1}{\lambda b_n} \frac{b_n}{(q + 1/(2\lambda))^2 - (b_n)^2} \\ &= \frac{2}{a_n} \frac{b_n}{(q + 1/(2\lambda))^2 - (b_n)^2} \end{aligned}$$

and

$$L^{-1}\{G(r_n, q)\} = \frac{2}{a_n} e^{-t/2\lambda} \mathrm{sh}(b_n t). \tag{A.1}$$

On the other hand, $G(r_n, q)$ can be written in the form

$$\begin{aligned} G(r_n, q) &= \frac{1}{\lambda q} \frac{1}{(q + 1/\lambda) + (\nu r_n^2/\lambda) q^{-1}} = \frac{1}{\lambda} \frac{q^{-1}}{(q + 1/\lambda) + (\nu r_n^2/\lambda) q^{-1}} \\ &= \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(-\nu r_n^2/\lambda)^k q^{-k-1}}{(q + 1/\lambda)^{k+1}}, \end{aligned}$$

and thus

$$L^{-1}\{G(r_n, q)\} = \sum_{k=0}^{\infty} \frac{1}{\lambda}\left(-\frac{\nu r_n^2}{\lambda}\right)^k G_{1,-k-1,k+1}\left(-\frac{1}{\lambda}, t\right). \tag{A.2}$$

From (A.1) and (A.2) we obtain (with a_n, b_n defined as above)

$$\sum_{k=0}^{\infty} \frac{1}{\lambda}\left(-\frac{\nu r_n^2}{\lambda}\right)^k G_{1,-k-1,k+1}\left(-\frac{1}{\lambda}, t\right) = \frac{2}{a_n} e^{-t/2\lambda}\,\mathrm{sh}(b_n t), \tag{A.3}$$

$$\sum_{k=0}^{\infty} \left(-\frac{\nu r_n^2}{\lambda}\right)^k G_{1,-k-2,k+1}\left(-\frac{1}{\lambda}, t\right) = \frac{\lambda}{\nu r_n^2}\left[1 - \left\{\mathrm{ch}(b_n t) + \frac{1}{a_n}\mathrm{sh}(b_n t)\right\} e^{-t/2\lambda}\right], \tag{A.4}$$

$$G_{1,-1,1}\left(-\frac{1}{\lambda}, t\right) = \sum_{j=0}^{\infty}\left(-\frac{1}{\lambda}\right)^j \frac{t^{j+1}}{\Gamma(j+2)} = -\lambda \sum_{j=0}^{\infty} \frac{(-t/\lambda)^{j+1}}{(j+1)!} = \lambda[1 - e^{-t/\lambda}], \tag{A.5}$$

$$G_{1,-2,1}\left(-\frac{1}{\lambda}, t\right) = \lambda[t - \lambda(1 - e^{-t/\lambda})]. \tag{A.6}$$

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